Vacuum state and Schwarzschild solution in ten-dimensional gravity

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(Received 2 March 1987; revised manuscript received 8 May 1987)

Gravity in more than four dimensions may involve terms of higher order in the curvature, as well as the linear terms present in ordinary general relativity. I explore the ten-dimensional vacuum configuration $M^4 \times S^6$. The ten-dimensional spherically symmetric potential is examined, and I determine conditions under which the formation of black holes is forbidden. Consequences for the stability of the vacuum are discussed.

I. INTRODUCTION

The application of the methods of general relativity to space with more than four dimensions began in the 1920's with the observation by Kaluza and Klein that gravity and electromagnetism may be unified by adding a fifth dimension to space-time. A similar unification of gravity with non-Abelian gauge theories can be achieved by the addition of more than one extra dimension.[1] This approach makes use of the Lagrangian formulation of general relativity. The D-dimensional Einstein field equations for a mass-energy tensor equal to zero can be obtained from the variation of the action

$$I = -\frac{1}{16\pi G_D} \int R\sqrt{|g|}\, d^D x \ , \tag{1.1}$$

where R is the scalar curvature and g is the determinant of the metric of D-dimensional space-time. This metric is parametrized in the following suggestive manner:[2]

$$g_{MN} = W(\phi) \begin{bmatrix} g'_{\mu\nu} + A^a_\mu A^b_\nu \xi^i_a \xi^j_b \phi_{ij} & A^c_\mu \xi^i_c \phi_{ij} \\ A^c_\nu \xi^j_c \phi_{ij} & \phi_{ij} \end{bmatrix} . \tag{1.2}$$

Note that $\sqrt{|g|} = \sqrt{|g'|}\sqrt{|\det\phi|}\,[W(\phi)]^{D/2}$ with g' defined as the determinant of $g'_{\mu\nu}$.

The geometry must now be restricted. The matrix ϕ_{ij} is the metric of a $(D-4)$-dimensional space B. The symmetries of B are the gauge symmetries in the effective four-dimensional world. Let T_a, $a = 1, \ldots, N$ generate the symmetry group G of B, and y^i be coordinates in B. The "Killing vector" associated with T_a is defined by the equation $\xi^i_a \equiv T_a y^i$. Killing vectors have the property

$$\xi^j_a \xi^i_{b,j} - \xi^j_b \xi^i_{a,j} = -C^c_{ab} \xi^i_c \ , \tag{1.3}$$

where C^c_{ab} are the structure constants of the group G. It can be shown that under these circumstances the action I can rewritten as

$$I = -\frac{1}{16\pi G_D} \int \left[R' + \tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu b} \xi^i_a \xi^j_b \phi_{ij} + \cdots \right] \times \sqrt{-g'}\sqrt{|\det\phi|}\,[W(\phi)]^{(D/2)-1} d^D x \ . \tag{1.4}$$

R' is the curvature associated with the matrix $g'_{\mu\nu}$, and the Maxwell field strength appropriate to the gauge group is defined by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + C^a_{bc} A^b_\mu A^c_\nu \ . \tag{1.5}$$

$W(\phi)$ is chosen to be $|\det\phi|^{-1/(D-2)}$. The ellipsis stands for debris associated with the metric ϕ_{ij} and the derivative of A^a_μ with respect to y^i.

Since only four dimensions are observed, B must be a compact space with a very small characteristic scale. The space is said to be compactified. At low energies, it is therefore reasonable to neglect the dependence of the macroscopic quantities R' and $F^a_{\mu\nu}$ on coordinates in B. The extra $D-4$ coordinates are integrated out. Using the formula[3]

$$\int_B \phi_{ij} \xi^i_a \xi^j_b d^{D-4} x = \text{const} \times \delta_{ab} \tag{1.6}$$

and rescaling G_D and $F^a_{\mu\nu}$, the low-energy limit of the action I becomes

$$I = -\frac{1}{16\pi G} \int \left[R' - \tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \right] \sqrt{-g'}\, d^4 x + \cdots \ , \tag{1.7}$$

where the ellipsis, the result of the dependence of A^a_μ and ϕ_{ij} on the coordinates in B, may be neglected. The metric $g'_{\mu\nu}$ may now be identified as the metric of ordinary four-dimensional space-time. A variation of $-(1/16\pi G)\int R'\sqrt{-g'} d^4 x$ gives the familiar equations of four-dimensional general relativity, while

$$\frac{1}{16\pi G} \int \tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \sqrt{-g'} d^4 x$$

is the action associated with the gauge fields. The convention $(+ - \ldots -)$ is used throughout.

We return to the original action (1.1). The next step is to apply the methods of general relativity to specific physical situations. This is usually done by specifying a form of the metric of space-time. The ansatz must be compatible with the equations of motion which may further restrict the metric. The simplest example of this is the vacuum solution. Gravity and the gauge fields are absent. The field A^a_μ is zero, and $g'_{\mu\nu}$ is the Minkowski metric $\eta_{\mu\nu}$. The resulting metric is

$$g_{MN}=\begin{bmatrix}\eta_{\mu\nu} & 0\\ 0 & \phi_{ij}\end{bmatrix} . \tag{1.8}$$

To a good approximation, this describes physical reality in deep space, and the metric should be compatible with the equations of motion.

With a cosmological constant included for the sake of completeness, the D-dimensional Einstein equations are

$$R_{MN}-\tfrac{1}{2}g_{MN}(R+\Lambda)=0 . \tag{1.9}$$

R_{MN} is the Ricci tensor. For $M=N\leq 3$, $R_{MN}=0$ and

$$R=-\Lambda . \tag{1.10}$$

Contracting (1.9) gives

$$R=\frac{D\Lambda/2}{1-D/2} . \tag{1.11}$$

The only solution to (1.10) and (1.11) is $R=\Lambda=0$. For $M,N>3$, Eq. (1.9) then gives $R_{MN}=0$. The compactified space B must be Ricci flat.

In the theory originally proposed by Kaluza and Klein, the compactified space is S^1, which is trivially Ricci flat, but in order to unify gravity with the forces corresponding to non-Abelian gauge theories, higher-dimensional compact Ricci-flat spaces are needed. This greatly restricts the choice of spaces. The $(D-4)$-sphere, for example, is not acceptable. Furthermore, the requirements that the symmetries of the compactified space generate the gauge group and that the space by Ricci flat imply that the Killing vectors are not symmetries of the metric ϕ_{ij}[2]. There is, thus, no natural choice for the vacuum field configuration. It is possible[1] to introduce a nonzero energy-momentum tensor $T_{\mu\nu}$ as a source of curvature for the space B. This, however, would not correspond to a true vacuum. Quantum corrections may also provide a source of curvature for the compactified space, but in the absence of a consistent theory of quantum gravity, it is impossible to verify this possibility.[1]

Work to create such a theory has continued. In the early 1980s, higher-dimensional supergravity theories created a flurry of interest in the Kaluza-Kein approach to grand unification. Other problems, such as doubts about the stability of the vacuum $M^4\times B$, arose,[4] and Kaluza-Klen theories again fell from favor. Recently, superstring theories have employed many of the ideas discussed above. Thus, despite its shortcomings, the Kaluza-Klein approach remains a topic of interest.

Research in the mathematical foundations of general relativity in more than four dimensions progressed independently. It was realized as early as 1970 by Lovelock that in more than four dimensions the action (1.1) is not the most general possibility.[5] Many texts[6] derive the Einstein field equations by requiring that the energy-momentum tensor T^{MN} be proportional to the most general possible tensor A^{MN}, subject to the following conditions: (i) $A^{MN}=A^{NM}$, (ii) A^{MN} is a function of the metric and its first and second derivatives, (iii) $A^{MN}{}_{;N}=0$, and (iv) A^{MN} is linear in the second derivative of g_{AB}. The Einstein field equation is, thus,

$$A^{MN}=-8\pi G_D T^{MN} . \tag{1.12}$$

In four dimensions $A^{\mu\nu}=aG^{\mu\nu}+bg^{\mu\nu}$, where $G^{\mu\nu}$ is the Einstein tensor. Lovelock showed that, in D dimensions, the most general A_N^M is given by

$$A_N^M=\sum_{p=1}^{D-1} a_p\delta_{Nj_1\cdots j_{2p}}^{Mh_1\cdots h_{2p}}R_{h_1h_2}{}^{j_1j_2}\cdots R_{h_{2p-1}h_{2p}}{}^{j_{2p-1}j_{2p}}+a\delta_N^M , \tag{1.13}$$

where a and a_p are arbitrary constants and

$$\delta_{j_1\cdots j_n}^{i_1\cdots i_n}=\det\begin{bmatrix}\delta_{j_1}^{i_1} & \cdots & \delta_{j_n}^{i_1}\\ \vdots & \ddots & \vdots\\ \delta_{j_1}^{i_n} & \cdots & \delta_{j_n}^{i_n}\end{bmatrix} . \tag{1.14}$$

R_{ABCD} is the Riemann curvature tensor. Lovelock also showed that the equations $A^{MN}=0$ are the equations of motion corresponding to the Lagrange density

$$L=\sqrt{|g|}\sum_{p=1}^{D-1}2a_p\delta_{j_1\cdots j_{2p}}^{h_1\cdots h_{2p}}R_{h_1h_2}{}^{j_1j_2}\cdots R_{h_{2p-1}h_{2p}}{}^{j_{2p-1}j_{2p}}+2a\sqrt{|g|} . \tag{1.15}$$

This is not the most general Lagrange density that gives rise to $A^{MN}=0$.

Lagrangians of this form have appeared independently in the study of superstrings. In the low-energy limit, superstring theory gives rise to a ten-dimensional theory of gravity. The action associated with this theory has not been determined, but it is evident that it does not have the simple form of (1.1). Terms quadratic in the Riemann curvature are known to appear.[7] Such terms lead to ghosts and violate unitarity,[7] but Zweibach has shown that the D-dimensional action $(D>4)$

$$\int (R_{ABCD}R^{ABCD}-4R_{AB}R^{AB}+R^2)\sqrt{|g|}\,d^Dx \tag{1.16}$$

avoids such problems.[8] Since such a term has the form of the Euler invariant in four dimensions extended to higher dimension, and the Einstein action $\int R\sqrt{|g|}\,d^Dx$ is the Euler invariant in two dimensions extended to D dimensions, it was suggested[8] that a general Lagrangian in D dimensions would include terms corresponding to the Euler invariant in all even dimensions less than D. The Euler invariant for odd dimensions is zero and does not contribute. It turns out that this is the Lagrangian postulated by Lovelock.

Zumino introduced a more practical formalism for working with such a Lagrangian by constructing it in terms of the vielbein and curvature forms.[7] The vielbein one-form is defined by

$$e^a = e^a{}_M dx^M , \tag{1.17}$$

where

$$g_{MN} = e^a{}_M e^b{}_N \eta_{ab} \tag{1.18}$$

is the defining equation for $e^a{}_M$. For a diagonal metric,

$$e^a{}_M = \sqrt{|g_{aM}|} . \tag{1.19}$$

In keeping with the notation above, upper-case latin indices are raised and lowered by the metric of ten-dimensional curved space-time, and lower-case latin indices by the ten-dimensional Minkowski metric. The greek letters μ and ν indicate four-dimensional space-time, and the latin indices i, j, and k are reserved for the subspace B. Thus,

$$R_{ABCD} = R_{abcd} e^a{}_A e^b{}_B e^c{}_C e^d{}_D . \tag{1.20}$$

The curvature two-form is defined by

$$\mathcal{R}_{ab} = \tfrac{1}{2} R_{ab}{}^{cd} e_c \wedge e_a . \tag{1.21}$$

The Lagrangian for gravity generalized to D dimensions can be written as a linear combination of these terms:

$$L_{0,D} = \underbrace{e_a \wedge \cdots \wedge e_n}_{D \text{ factors}} \epsilon^{a \cdots n} \tag{1.22}$$

$$L_{1,D-2} = \mathcal{R}_{ab} \wedge e_c \wedge \cdots \wedge e_n \epsilon^{a \cdots n} , \tag{1.23}$$

$$L_{2,D-4} = \mathcal{R}_{ab} \wedge \mathcal{R}_{cd} \wedge e_f \wedge \cdots \wedge e_n \epsilon^{a \cdots n} , \tag{1.24}$$

$$\left.\begin{matrix} L_{D/2,0} \\ L_{(D-1)/2,1} \end{matrix}\right\} = \begin{cases} \mathcal{R}_{ab} \wedge \cdots \wedge \mathcal{R}_{mn} \epsilon^{a \cdots n}, & D \text{ even} , \\ \mathcal{R}_{ab} \wedge \cdots \wedge \mathcal{R}_{lm} \wedge e_n \epsilon^{a \cdots n}, & D \text{ odd} . \end{cases} \tag{1.25}$$

The terms $L_{0,D}$ and $L_{1,D-2}$ represent the cosmological constant and Einstein action, respectively. Considering for the moment D even only, $L_{D/2,0}$ is the Euler invariant in D dimensions. It is a perfect derivative and does not contribute to the equations of motion. $L_{1,D-2a}$ ($2a < D$) is the Euler invariant in $2a$ dimensions continued to D dimensions. Similar results can be obtained for odd dimensions.

It is straightforward to convert these Lagrangians to a more familiar form. For example,

$$\begin{aligned} L_{0,D} &= e_a \wedge \cdots \wedge e_n \epsilon^{a \cdots n} \\ &= e^a \wedge \cdots \wedge e^n \epsilon_{a \cdots n} \\ &= e^a{}_A \cdots e^n{}_N dx^A \wedge \cdots \wedge dx^N \epsilon_{a \cdots n} . \end{aligned} \tag{1.26}$$

The wedge product is proportional to the volume element:

$$dx^A \wedge \cdots \wedge dx^N = \epsilon_{A \cdots N} d^D x .$$

The upper-lower index conventional must be abandoned. Equation (1.26) now becomes

$$\begin{aligned} L_{0,D} &= e^a{}_A \cdots e^n{}_N \epsilon_{a \cdots n} \epsilon_{A \cdots N} d^D x \\ &= D! e^a{}_1 \cdots e^n{}_D \epsilon_{a \cdots n} d^D x \\ &= D! \det(e^a{}_A) d^D x . \end{aligned} \tag{1.27}$$

For a diagonal matrix, $\det(e^a{}_A) = \sqrt{|g|}$ and

$$L_{0,D} = D! \sqrt{|g|} \, d^D x . \tag{1.28}$$

Similarly,

$$L_{1,D-2} = (D-2)! R \sqrt{|g|} \, d^D x , \tag{1.29}$$

$$\begin{aligned} L_{2,D-4} = (D-4)! (&R_{ABCD} R^{ABCD} \\ &- 4 R_{AB} R^{AB} + R^2) \sqrt{|g|} \, d^D x . \end{aligned} \tag{1.30}$$

In what follows, the equations of motion for a simple example, the vacuum state $M^4 \times S^6$, will be obtained from the Lagrangian

$$L = a_0 L_{0,10} + a_1 L_{1,8} + a_2 L_{2,6} + a_3 L_{3,4} + a_4 L_{4,2} . \tag{1.31}$$

With the cosmological constant assumed to be zero, these equations produce expressions for a_2/a_1 and a_3/a_1 in terms of the radius of S^6. Beginning with the Lagrangian (1.31), I will derive the equations of motion for a static spherically symmetric metric in ten dimensions. When the expressions for a_2/a_1 and a_3/a_1 are substituted into these equations, the condition that no Schwarzschild radius exist restricts the allowable values of a_4/a_1.

II. $M^4 \times S^6$ COMPACTIFICATION

Flat Cartesian coordinates x^μ, $\mu = 0,1,2,3$ label a point in four-dimensional Minkowski space-time. The sphere S^6 is parametrized by angular coordinates θ^i, $i = 1, \ldots, 6$. The metric for $M^4 \times S^6$ is

$$g_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & -\rho^2 \left[\prod_{k=1}^{i-1} \sin\theta^k \right]^2 \delta_{ij} \end{pmatrix} , \tag{2.1}$$

where ρ is the radius of S^6. The vielbeins for this metric are

$$e^a = \begin{cases} dx^\mu, & a = 0,1,2,3 \text{ and } \mu = a , \\ \rho \left[\prod_{k=1}^{i-1} \sin\theta^k \right] d\theta^i, & a = 4, \ldots, 9 \text{ and } i = a-3 . \end{cases} \tag{2.2}$$

The connection one-forms

$$\omega^a{}_b = \omega^a{}_{bM} dx^M \tag{2.3}$$

are defined by the equation

$$\mathcal{R}^a{}_b \equiv \tfrac{1}{2} R^a{}_{bcd} e^c \wedge e^d = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_d \tag{2.4}$$

and the torsion one-forms by

$$T^a \equiv de^a + \omega^a{}_b \wedge e^b \ . \tag{2.5}$$

Equations (2.3) and (2.4) are known as Cartan's structure equations.

It can be shown[9] that the metricity condition $g_{MN;A}=0$ is equivalent to the condition

$$\omega_{ab} = -\omega_{ba} \tag{2.6}$$

and the torsion-free condition $\Gamma^M_{AB} = \Gamma^M_{BA}$ implies $T^a = 0$, and thus

$$de^a = -\omega^a{}_b \wedge e^b \ . \tag{2.7}$$

For the vielbeins (2.2) Eq. (2.7) may be evaluated by the "hypothesis" method.[10] The result is

$$\omega^a{}_b = -\omega^b{}_a = \begin{cases} 0, & a \le 3, \\ \left[\prod_{k=j+1}^{i-1} \sin\theta^k \right] \cos\theta^j d\theta^i, & a > b > 3,\ i = a-3,\ j = b-3. \end{cases} \tag{2.8}$$

These solutions for the connection one-forms are substituted into (2.4), and the resulting expression for the curvature two-form is

$$\mathcal{R}_{ab} = \begin{cases} 0, & a \text{ or } b \le 3, \\ -\dfrac{1}{\rho^2} e_a \wedge e_b, & a,b > 3. \end{cases} \tag{2.9}$$

The expressions for the curvature and vielbeins are now substituted into the Lagrangian (1.31). Equation (1.28) results in

$$L_{0,10} = 10! \sqrt{-g}\, d^{10}x \ . \tag{2.10}$$

The other terms are evaluated using the notation discussed in the introduction. From now on, I omit the wedge sign in the product of differential forms:

$$\begin{aligned} L_{1,8} &= \mathcal{R}_{ab} e_c e_d e_f e_g e_h e_l e_m e_n \epsilon^{abcdfghlmn} \\ &= \left[-\frac{1}{\rho^2} \right] \sum_{a,b>3,c,\ldots,n} e_a e_b e_c e_d e_f e_g e_h e_l e_m e_n \epsilon^{abcdfghlmn} \\ &= \left[-\frac{1}{\rho^2} \right] 30 \times 8! \sqrt{-g}\, d^{10}x \ , \end{aligned} \tag{2.11}$$

$$\begin{aligned} L_{2,6} &= \mathcal{R}_{ab} \mathcal{R}_{cd} e_f e_g e_h e_l e_m e_n \epsilon^{abcdfghlmn} \\ &= \left[-\frac{1}{\rho^2} \right]^2 \sum_{a,b,c,d>3,f,\ldots,n} e_a e_b e_c e_d e_f e_g e_h e_l e_m e_n \epsilon^{abcdfghlmn} \\ &= \left[\frac{1}{\rho^4} \right] 360 \times 6! \sqrt{-g}\, d^{10}x \ , \end{aligned} \tag{2.12}$$

$$\begin{aligned} L_{3,4} &= \mathcal{R}_{ab} \mathcal{R}_{cd} \mathcal{R}_{fg} e_h e_l e_m e_n \epsilon^{abcdfghlmn} \\ &= \left[-\frac{1}{\rho^2} \right]^3 \sum_{a,\ldots,g>3,h,\ldots,n} e_a e_b e_c e_d e_f e_g e_h e_l e_m e_n \epsilon^{abcdfghlmn} \\ &= \left[-\frac{1}{\rho^6} \right] 24 \times 6! \sqrt{-g}\, d^{10}x \ , \end{aligned} \tag{2.13}$$

$$L_{4,2} = \mathcal{R}_{ab} \mathcal{R}_{cd} \mathcal{R}_{fg} \mathcal{R}_{hl} e_m e_n \epsilon^{abcdfghlmn} = 0 \ . \tag{2.14}$$

The last result is due to a saturation of indices. The curvature two-form $\mathcal{R}_{ab}$ is nonzero only for $a > 3$. Because of the presence of the antisymmetric tensor, the eight indices a through l must be different, and at least two of them must be less than four.

The Lagrangian (1.31) evaluated for $M^4 \times S^6$ is

$$L = \mathcal{L} \sqrt{-g}\, d^{10}x \ , \tag{2.15}$$

where

$$\mathcal{L} = 10! a_b - \frac{8! \times 30}{\rho^2} a_1 + \frac{6! \times 360}{\rho^4} a_2 - \frac{6! \times 24}{\rho^6} a_3 \ . \tag{2.16}$$

The cosmological constant is set equal to zero.

The equations of motion associated with this Lagrangian are

$$\mathcal{L}=0\ , \tag{2.17}$$

$$\frac{\partial(\mathcal{L}\sqrt{-g}\,)}{\partial\rho}=0\ . \tag{2.18}$$

These equations result when the generalized Einstein tensor (1.13) is evaluated for the geometry of $M^4\times S^6$ and then substituted into the equation

$$A_{MN}=0\ . \tag{2.19}$$

It is possible to derive (2.17) and (2.18) without explicitly evaluating all terms in A_{MN}. The variation of the action is

$$\delta I=\int \delta(\mathcal{L}\sqrt{-g}\,)\,d^{10}x$$
$$=\int \delta\mathcal{L}\sqrt{-g}\;d^{10}x+\int \mathcal{L}\delta\sqrt{-g}\;d^{10}x\ . \tag{2.20}$$

The second term may be written as

$$\int (-\tfrac{1}{2}g_{MN}\sqrt{-g}\;\mathcal{L}\delta g^{MN})d^{10}x\ . \tag{2.21}$$

To examine the first term, I consider the part linear in the curvature separately from the quadratic and cubic parts. The procedure for evaluating δR is well known:

$$\delta R=\delta R_{MN}g^{MN}+R_{MN}\delta g^{MN}\ . \tag{2.22}$$

Since[11]

$$\delta R_{MN}=\delta\Gamma^{L}{}_{MN;L}-\delta\Gamma^{L}_{ML;N}\ , \tag{2.23}$$

where $\delta\Gamma^A_{BC}$ is the variation of the Christoffel symbol and g^{MN} commutes with the covariant derivative; $\delta R_{MN}g^{MN}$ is a covariant divergence and does not contribute to the variation.

A similar method is applied to the other parts of $\int \delta\mathcal{L}\sqrt{-g}\;d^{10}x$. This time, three types of terms appear. The first type is the product of the variation of a Riemann curvature tensor, $\delta R^{A}{}_{BCD}$, and other Riemann curvature tensors, as well as components of the metric tensor. The variation of a curvature tensor is given by

$$\delta R^{A}{}_{BCD}=\delta\Gamma^{A}{}_{BD;C}-\delta\Gamma^{A}{}_{BC;D}\ . \tag{2.24}$$

Equation (2.9) implies that R_{EFGH} is zero if any index is less than four and

$$R_{ABCD}=-\frac{1}{\rho^2}(g_{AC}g_{BD}-g_{AD}g_{BC}) \tag{2.25}$$

otherwise. The Riemann curvature commutes with the covariant derivative, as do the components of the metric tensor. The terms involving $\delta R^{A}{}_{BCD}$ are therefore the integrals of a divergence, and do not contribute to (2.19). The other types of terms are the product of $R_{MABC}R_{NDEF}$ or R_{MN} and other Riemann curvatures, components of the metric tensor, and the variation δg^{MN}. Thus,

$$\delta I=\int (R_{MN}\cdots+R_{MABC}R_{NDEF}\cdots$$
$$-\tfrac{1}{2}g_{MN}\mathcal{L})\sqrt{-g}\;\delta g^{MN}d^{10}x\ . \tag{2.26}$$

Equation (2.19) is obtained by setting the coefficient of δg^{MN} equal to zero.

For $M=N\leq 3$, R_{MN} and R_{MABC} are zero and Eq. (2.17) results. Notice that for any maximally symmetric space, the curvature has the form (2.25) and these results hold.

Equation (2.18) comes from the equations of motion for $M=N>3$. Inspection of (1.13) reveals that these equations of motion are identical to that which results for the space S^6 alone. The only equation of motion associated with the six-sphere must come from the variation of the radius. Note that the term cubic in the curvature is the Euler invariant in six dimensions and it cannot contribute to the equation of motion. This is indeed true since the contribution of this term to Lagrangian density goes as $1/\rho^6$, while $\sqrt{-g}$ goes as ρ^6, and the product is independent of ρ.

Similar arguments are used to show that the equations $A_{MN}=0$ are satisfied trivially for $M\neq N$. All terms contain a factor R_{MABC}, which is zero if either M or $N\leq 3$. For $M,N>3$ and $M\neq N$, the equations are again formally the same as those for the subspace S^6. The cubic term cannot contribute, and explicit calculation of the linear and quadratic terms of A_{MN} shows that they are proportional to g_{MN}, which is zero for $M\neq N$.

When the Lagrangian (2.16) is substituted into the equations of motion, expressions for a_2/a_1 and a_3/a_1 result:

$$\frac{a_2}{a_1}=\frac{28}{3}\rho^2\ , \tag{2.27}$$

$$\frac{a_3}{a_1}=70\rho^4\ . \tag{2.28}$$

Notice that Eq. (2.17) for the general Lagrangian (1.15) replaces the previous condition $R=0$. Ricci flatness is no longer required of the compactified space, and $M^4\times S^6$ is an allowable vacuum configuration, even in the absence of an external matter field.

Müller-Hoissen[12] has used a slightly different approach to calculate the equations of motion for the more general vacuum $M^4\times S^N$. He assumes a Lagrangian of the form (1.31) and includes terms up to cubic order in the curvature. The variation of the Lagrangian is determined directly and the torsion is subsequently set equal to zero. The restrictions on the coefficents a_n follow from the resulting equations of motion.

III. THE SCHWARZSCHILD SOLUTION IN TEN DIMENSIONS

The Schwarzschild solution in D dimensions describes the external field of a static spherically symmetric body. It can be shown that the most general metric for such a field is

$$g_{MN}=\begin{bmatrix} B(r) & & 0 \\ & -A(r) & \\ 0 & & -r^2\left[\prod_{k=1}^{P-2}\sin\theta^k\right]^2\delta_{PQ}\end{bmatrix}\ , \tag{3.1}$$

where the D-dimensional space-time is parametrized by angular coordinates θ^k, $k=1,\ldots,D-2$, the radial coordinate r and the time t. In this section, the upper-case latin indices P and Q are reserved for angular coordinates $(P,Q=2,\ldots,D)$. Notice that though similar in appearance, this metric is different from (2.1). The radius r is a coordinate and not a parameter. The equations of motion for this geometry give solutions for the functions $A(r)$ and $B(r)$. An event horizon is associated with a singularity of $A(r)$ at some radius R, called the Schwarzschild radius. If the radius of the body is smaller than the Schwarzschild radius, then it is called a black hole. The existence of such solutions is important in the study of Kaluza-Klein theories.

Witten has pointed out that the Schwarzschild solution is associated with a semiclassical instability of the Kaluza-Klein vacuum.[13] The decay of the vacuum may intuitively be seen to result from the spontaneous formation of black holes with Schwarzschild radius on the order of the compactification scale. After a short time, these holes are expanding to infinity at the speed of light. The state does not decay into a more stable state as would be the case for a classical instability, but rather it decays into nothing.

Hawking has furthermore shown that virtual black holes lead to a loss of quantum coherence.[14] If the space has nontrivial topology, then pure quantum-mechanical states may evolve into mixed states described by density matrices. There is also a breakdown of unitary time evolution. In particular, this can occur with the Kaluza-Klein vacuum. Although it is a matter of controversy whether or not this is physically acceptable,[15] it is clear that the existence of virtual black holes would require a revision of quantum mechanics.

The Schwarzschild solution in ten dimensions is found by substituting the metric (3.1) into the Lagrangian (1.31). The vielbeins for this metric are given by

$$e^a=\begin{cases}\sigma^0 dt=\sqrt{B}\,dt, & a=0,\\ \sigma^1 dr=\sqrt{A}\,dr, & a=1,\\ \sigma^a d\theta^i=r\left[\prod_{k=1}^{i-1}\sin\theta^k\right]d\theta^i, & a=2,\ldots,9 \text{ and } i=a-1\,.\end{cases} \tag{3.2}$$

As before, Eq. (2.7) is used to solve for the connection one-forms:

$$\begin{aligned}\omega^0{}_1&=\omega^1{}_0=\frac{1}{2\sqrt{AB}}\frac{dB}{dr}dt,\quad \omega^p{}_0=\omega^0{}_p=0\,,\\ \omega^p{}_1&=-\omega^1{}_p=\frac{1}{\sqrt{A}}\left[\prod_{k=1}^{i-1}\sin\theta^k\right]d\theta^i\,,\\ \omega^p{}_q&=-\omega^q{}_p=\left[\prod_{k=j+1}^{i-1}\sin\theta^k\right]\cos\theta^j d\theta^i\,,\end{aligned} \tag{3.3}$$

for $p>q>1$, $i=p-1$, and $j=q-1$. Equation (2.4) is used to solve for the Riemann curvature:

$$\mathcal{R}_{ab}=k(a,b)e_a\wedge e_b\,, \tag{3.4}$$

where

$$\begin{aligned}k(0,1)&=k_1=\frac{1}{2\sqrt{AB}}\frac{d}{dr}\left[\frac{1}{\sqrt{AB}}\frac{dB}{dr}\right],\\ k(0,p)&=k_2=\frac{1}{2rAB}\frac{dB}{dr}\,,\\ k(1,p)&=-k_3=-\frac{1}{2rA^2}\frac{dA}{dr}\,,\\ k(p,q)&=-k_4=-\frac{1-1/A}{r^2}\,,\end{aligned} \tag{3.5}$$

for $p,q>1$. I define $k(a,a)=0$ and, from the symmetry of $\mathcal{R}_{ab}$, it follows that $k(a,b)=k(b,a)$.

The variation of the Lagrangian (1.31) is

$$\delta L=a_0\delta L_{0,10}+a_1\delta L_{1,8}+a_2\delta L_{2,6}+a_3\delta L_{3,4}+a_4\delta L_{4,2}\,. \tag{3.6}$$

The first term may be evaluated using Eq. (1.27):

$$\begin{aligned}L_{0,10}&=10!\det(e^a{}_A)d^{10}x\\ &=10!\sigma^0\sigma^1\sigma^2\sigma^3\sigma^4\sigma^5\sigma^6\sigma^7\sigma^8\sigma^9 d^{10}x\,.\end{aligned} \tag{3.7}$$

Since only the functions A and B are subject to variation, $\delta\sigma^a=0$ for $a>1$ and

$$\delta L_{0,10}=10!(\sigma^1\sigma^2\cdots\sigma^9\delta\sigma^0+\sigma^0\sigma^2\sigma^3\cdots\sigma^9\delta\sigma^1)d^{10}x\,. \tag{3.8}$$

The evaluation of the rest of expression (3.6) requires a different approach. From the definitions (1.23)–(1.25), it follows that

$$\begin{aligned}\delta L_{n,10-2n}&=n\mathcal{R}_{ab}\cdots\mathcal{R}_{cd}\delta\mathcal{R}_{fg}e_h\cdots e_m\epsilon^{a\cdots m}\\ &\quad+(10-2n)\mathcal{R}_{ab}\cdots\mathcal{R}_{fg}e_h\cdots e_l\delta e_m\epsilon^{a\cdots m}\,.\end{aligned} \tag{3.9}$$

An expression for the variation of the curvature two-form is found by using Eq. (2.4):

$$\delta\mathcal{R}^a{}_b=d(\delta\omega^a{}_b)+\delta\omega^a{}_c\wedge\omega^c{}_h+\omega^a{}_c\wedge\delta\omega^c{}_b=D(\delta\omega^a{}_b)\,. \tag{3.10}$$

The form Dx is the covariant generalization of the differential form dx. Equation (3.9) is simplified by the use of

$$d(\mathcal{R}_{ab}\cdots\mathcal{R}_{cd}\delta\omega_{fg}e_h\cdots e_m\epsilon^{a\cdots m})=D(\mathcal{R}_{ab}\cdots\mathcal{R}_{cd}\delta\omega_{fg}e_h\cdots e_m\epsilon^{a\cdots m})$$
$$=(n-1)\mathcal{R}_{ab}\cdots D(\mathcal{R}_{cd})\delta\omega_{fg}e_h\cdots e_m\epsilon^{a\cdots m}$$
$$+\mathcal{R}_{ab}\cdots\mathcal{R}_{cd}D(\delta\omega_{fg})e_h\cdots e_m\epsilon^{a\cdots m}$$
$$+(10-2n)\mathcal{R}_{ab}\cdots\mathcal{R}_{cd}\delta\omega_{fg}e_h\cdots e_l D(e_m)\epsilon^{a\cdots m}\ . \tag{3.11}$$

The torsion De_m vanishes. The Bianchi identity may be written in the form

$$D\mathcal{R}^a{}_b=0\ , \tag{3.12}$$

and it follows that

$$\mathcal{R}_{ab}\cdots\mathcal{R}_{cd}\delta\mathcal{R}_{fg}e_h\cdots e_m\epsilon^{a\cdots m}=d(\mathcal{R}_{ab}\cdots\mathcal{R}_{cd}\delta\omega_{fg}e_h\cdots e_m\epsilon^{a\cdots m})\ . \tag{3.13}$$

The first term in (3.7) therefore does not contribute to the equations of motion and

$$\delta L_{n,10-2n}=(10-2n)\mathcal{R}_{ab}\cdots\mathcal{R}_{fg}e_h\cdots e_l\delta e_m\epsilon^{a\cdots m}\ . \tag{3.14}$$

This result is used to calculate the remaining terms in δL:

$$\delta L_{1,8}=8\mathcal{R}_{ab}e_ce_de_fe_ge_he_le_m\delta e_n\epsilon^{abcdfghlmn}$$
$$=8\sum_{a\cdots n}k(a,b)e_ae_be_ce_de_fe_ge_he_le_m\delta e_n\epsilon^{abcdfghlmn}$$
$$=8\sum_{a\cdots n}k(a,b)\sigma^a\sigma^b\sigma^c\sigma^d\sigma^f\sigma^g\sigma^h\sigma^l\sigma^m\delta\sigma^n(\epsilon_{abcdfghlmn})^2d^{10}x$$
$$=[8\times 8!(-7k_4-2k_3)\sigma^1\sigma^2\sigma^3\sigma^4\sigma^5\sigma^6\sigma^7\sigma^8\sigma^9\delta\sigma^0$$
$$+8\times 8!(-7k_4+2k_2)\sigma^0\sigma^2\sigma^3\sigma^4\sigma^5\sigma^6\sigma^7\sigma^8\sigma^9\delta\sigma^1]d^{10}x\ . \tag{3.15}$$

Similarly,

$$\delta L_{2,6}=[24\times 6!k_4(62k_4+43k_3)\sigma^1\sigma^2\sigma^3\sigma^4\sigma^5\sigma^6\sigma^7\sigma^8\sigma^9\delta\sigma^0$$
$$+24\times 6!k_4(62k_4-43k_2)\sigma^0\sigma^2\sigma^3\sigma^4\sigma^5\sigma^6\sigma^7\sigma^8\sigma^9\delta\sigma^1]d^{10}x\ , \tag{3.16}$$

$$\delta L_{3,4}=[24\times 6!(-k_4{}^2)(28k_4+54k_3)\sigma^1\sigma^2\sigma^3\sigma^4\sigma^5\sigma^6\sigma^7\sigma^8\sigma^9\delta\sigma^0$$
$$+24\times 6!k_4{}^2(-28k_4+54k_2)\sigma^0\sigma^2\sigma^3\sigma^4\sigma^5\sigma^6\sigma^7\sigma^8\sigma^9\delta\sigma^1]d^{10}x\ , \tag{3.17}$$

$$\delta L_{4,2}=[2\times 8!k_4{}^3(k_4+8k_3)\sigma^1\sigma^2\sigma^3\sigma^4\sigma^5\sigma^6\sigma^7\sigma^8\sigma^9\delta\sigma^0$$
$$+2\times 8!k_4{}^3(k_4-8k_2)\sigma^0\sigma^2\sigma^3\sigma^4\sigma^5\sigma^6\sigma^7\sigma^8\sigma^9\delta\sigma^1]d^{10}x\ . \tag{3.18}$$

The variation of the action is zero if and only if the coefficients of $\delta\sigma^0$ and $\delta\sigma^1$ are zero. Two equations of motion result:

$$0=b_0-b_1(7k_4+2k_3)+b_2k_4(62k_4+43k_3)$$
$$-b_3k_4{}^2(28k_4+54k_3)+b_4k_4{}^3(k_4+8k_3) \tag{3.19}$$

and

$$0=b_0+b_1(-7k_4+2k_2)+b_2k_4(62k_4-43k_2)$$
$$+b_3k_4{}^2(-28k_4+54k_2)+b_4k_4{}^3(k_4-8k_2)\ , \tag{3.20}$$

where

$$b_0=315a_0,\quad b_1=224a_1,\quad b_2=12a_2\ ,$$
$$b_3=12a_3,\quad b_4=56a_4\ .$$

The difference of (3.19) and (3.20) is

$$(k_3+k_2)(2b_1-43b_2k_4+54b_3k_4{}^2-8b_4k_4{}^3)=0\ . \tag{3.21}$$

One of the factors in this product must equal zero. We first consider the possibility that k_4 is a real root c of the cubic equation

$$2b_1-43b_2k_4+54b_3k_4{}^2-8b_4k_4{}^3=0\ . \tag{3.22}$$

We get

$$A(r)=\frac{1}{1-cr} \tag{3.23}$$

from the definition of k_4. Substitution of this function into Eq. (3.20) yields the equation

$$\left[c-\frac{1}{r}\right]\frac{1}{B}\frac{dB}{dr}+\kappa=0\ , \tag{3.24}$$

where

$$\kappa=-\frac{b_0-7b_1c+62b_2c^2+54b_3c^3+b_4c^4}{b_1-43b_2c+28b_3c^2-8b_4c^3}\ . \tag{3.25}$$

The solution of (3.24) is

$$B(r)=Q_0(cr-1)^{-\kappa/c^2}e^{-\kappa r/c} . \quad (3.26)$$

This type of solution is a new feature of higher-dimensional gravity with quadratic and higher-order terms in the Lagrangian. It does not approach the metric of flat space-time as r goes to infinity, and so does not obey the natural boundary conditions for a Schwarzschild solution.

We now turn our attention to the other possibility allowed by condition (3.21), $k_3=-k_2$. It follows that

$$B(r)=\frac{1}{A(r)} . \quad (3.27)$$

This condition also arises in gravity theories without higher-order terms. It is used to eliminate the function $A(r)$ in Eq. (3.20) and upon integration the equations of motion for a spherically symmetric space reduce to the relation

$$\ln r^{-9}+C=\int\frac{ds}{sD}+\frac{129}{70}q_1\int\frac{ds}{D}+\frac{243}{82}q_2\int\frac{s\,ds}{D}+4q_3\int\frac{s^2ds}{D} , \quad (3.28)$$

where

$$B(r)=1-\left[\frac{r^2}{\rho^2}\right]s , \quad (3.29)$$

$$D=1+q_1s+q_2s^2+q_3s^3 , \quad (3.30)$$

$$q_1=-\frac{35}{3}\left[\frac{b_2}{b_1}\right]\left[\frac{1}{\rho^2}\right] , \quad (3.31)$$

$$q_2=9\left[\frac{B_3}{b_1}\right]\left[\frac{1}{\rho^4}\right] , \quad (3.32)$$

$$q_3=-\left[\frac{b_4}{b_1}\right]\left[\frac{1}{\rho^6}\right] . \quad (3.33)$$

It is natural to fix the constant of integration C by imposing the requirement that the solution approach the Newtonian limit as r goes to infinity. For a D-dimensional space in the Newtonian limit, the potential ϕ associated with a point particle of mass M is given by the equation

$$\nabla_{(D-1)}{}^2\phi=4\pi G_DM\delta(\mathbf{r}) , \quad (3.34)$$

where $\nabla_{(D-1)}{}^2$ is the Laplacian associated with $D-1$ spatial dimensions and $\mathbf{r}$ is the space coordinate.

The solution to this equation is

$$\phi(\mathbf{r})=-4\pi G_DM\int\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2}d^{D-1}k . \quad (3.35)$$

For $D>4$, this integral diverges, and must be regularized. Substitution of the identity

$$\frac{1}{(k^2)^z}=\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}e^{-k^2t}dt \quad (3.36)$$

into integral (3.35) yields in the limit as $z\to1$ an expression for the potential:

$$\phi(\mathbf{r})=-4G_DM\pi^{(D+1)/2}\Gamma\left[-1+\frac{D-1}{2}\right]\left[\frac{2}{r}\right]^{D-3} . \quad (3.37)$$

It can be shown[26] that, in the Newtonian limit,

$$g_{00}=1+2\phi . \quad (3.38)$$

It follows that, for a ten-dimensional spherically symmetric space as r becomes large,

$$s\to\kappa\rho^2r^{-9} , \quad (3.39)$$

where

$$\kappa=15\pi^6G_DM . \quad (3.40)$$

This boundary condition is substituted into (3.28):

$$\ln\kappa\rho^2r^{-9}=\frac{\ln s}{D(s)}+\int_0^s\frac{\ln\bar{s}D'(\bar{s})}{D^2(\bar{s})}d\bar{s}+\frac{129}{70}q_1\int_0^s\frac{d\bar{s}}{D(\bar{s})}+\frac{243}{82}q_2\int_0^s\frac{\bar{s}\,d\bar{s}}{D(\bar{s})}+4q_3\int_0^s\frac{\bar{s}^2d\bar{s}}{D(\bar{s})} . \quad (3.41)$$

I denote the right-hand side of this equation by $F(s)$. This function can only have singularities at $s=0$ and at zeros of the polynomial $D(s)$. I assume for the moment that such zeros exist and denote the smallest one by s_0 (see Fig. 1). If $F(s)$ is indeed singular at $s=s_0$, then the boundary condition implies that the branch of $F(s)$ that is of physical interest lies between $s=0$ and $s=s_0$.

We now examine the divergence of $F(s)$ near s_0. It is useful to rewrite Eq. (3.41) in the form

$$F(s)=\text{const}+\int^s\frac{d\bar{s}}{\bar{s}D(\bar{s})}+\frac{129}{70}q_1\int^s\frac{d\bar{s}}{D(\bar{s})}+\frac{243}{82}q^2\int^s\frac{\bar{s}\,d\bar{s}}{D(\bar{s})}+4q_3\int^s\frac{\bar{s}^2d\bar{s}}{D(\bar{s})} . \quad (3.42)$$

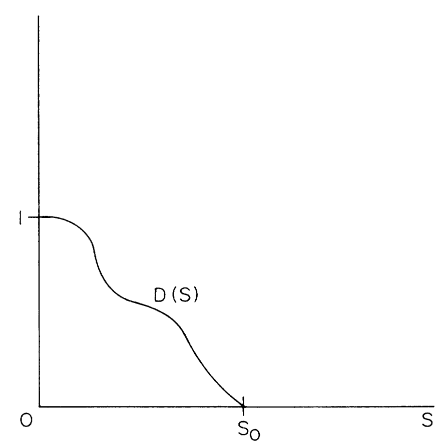

FIG. 1. Let s_0 be the smallest positive zero of $D(s)$. The region of physical interest lies between $s=0$ and $s=s_0$.

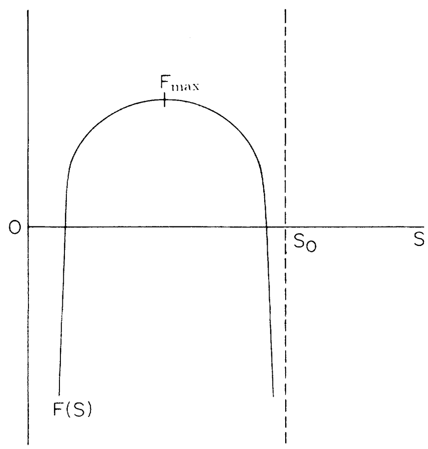

FIG. 2. If $L(s_0)<0$, then $F(s)$ is not defined for $F>F_{\max}$. Such solutions are unphysical.

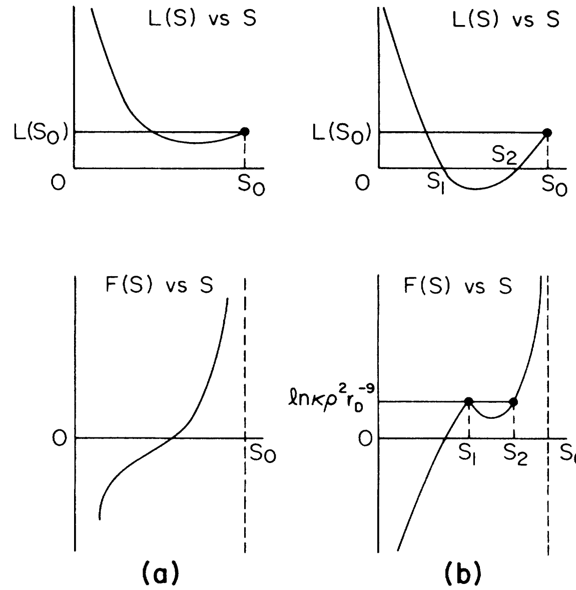

FIG. 3. Solutions such that $L(s_0)>0$ are of physical interest. (a) $L(s)$ is strictly positive in the interval $0<s<s_0$ and it follows that $F(s)$ is a monotonically increasing function of s. The formula $\ln\kappa\rho^2 r^{-9}=F(s)$ may be inverted to give s as a function of r. (b) is similar, but s jumps from s_1 to s_2 at $r=r_D$.

If I define

$$L(s)=\frac{1}{s}+\frac{129}{70}q_1+\frac{243}{82}q_2 s+4q_3 s^2 , \tag{3.43}$$

then

$$F(s)\to L(s_0)\int^s \frac{d\bar{s}}{D(\bar{s})}+\text{const} \tag{3.44}$$

as s approaches s_0.

If $L(s_0)=0$, there is no singularity of $F(s)$ at $s=s_0$, and it is necessary to repeat the analysis for the next larger zero of $D(s)$, if any exist. If $L(s_0)<0$, then $F(s)$ is as is shown in Fig. 2. $F(s)$ is not defined in the region of space $r<r_{\min}$, where

$$r_{\min}=\left[\frac{\kappa\rho^2}{e^{F_{\max}}}\right]^{1/9} . \tag{3.45}$$

Equation (3.41) cannot be solved in this region and this is therefore not a physical solution.

The $L(s_0)>0$ case is of physical interest. In this case, $L(s)$ either has a pair of zeros between 0 and s_0, or is strictly positive in this region. The resulting function $F(s)$ for each case is sketched in Fig. 3. Equation (3.41) has a solution for all r between zero and infinity. In Fig. 3(b), there is a radius r_0 where $A(r)$ and $B(r)$ are discontinuous as s jumps from s_1 to s_2. The metric is well defined everywhere else. If $F(s)$ is finite for positive s, then it is given by Fig. 4. Near $s=0$, $F(s)$ goes as $\ln s$, and as s goes to infinity, the divergence of $F(s)$ is proportional to $\ln s$.

We now examine the conditions for the existence of a Schwarzschild radius. This occurs when

$$B(r)=1-\left[\frac{r^2}{\rho^2}\right]s=0 \tag{3.46}$$

and, therefore,

$$r=\frac{\rho}{\sqrt{s}} . \tag{3.47}$$

At the Schwarzschild radius, the left-hand side of Eq. (3.41) equals

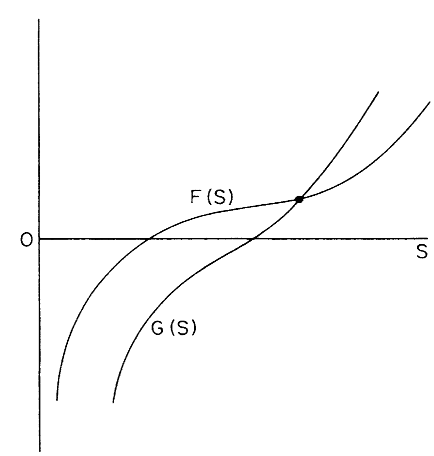

FIG. 4. If $F(s)$ has no singularities, then $F(s)$ and $G(s)$ always intersect and black holes form for any mass.

$$\frac{9}{2}\ln s+\ln\frac{\kappa}{\rho^7}\ . \tag{3.48}$$

I denote this function by $G(s)$. A Schwarzschild radius exists if and only if the curves $G(s)$ and $F(s)$ intersect in the region of physical interest. Consider the case when $F(s)$ is as shown in Fig. 3. For s near zero, $F(s)$ goes as $\ln s$, while $G(s)$ goes as $\frac{9}{2}\ln s$. Thus, $G(s)<F(s)$ near zero. For a proper choice of parameters, no intersection of $F(s)$ and $G(s)$ and, therefore, no Schwarzschild radius, exists (see Fig. 5). A black hole associated with a mass M does not form. Notice that as κ gets smaller, $G(s)$ moves further away from $F(s)$. Since κ is directly proportional to the mass of the black hole, we have a mechanism whereby black holes of small energy can be disallowed. Since virtual black holes of large energy are extremely short lived, their effect is small and violation of quantum coherence may not be observable. The instability of the Kaluza-Klein vacuum proposed by Witten would be expected to occur for κ/ρ^7 on the order of 1 and so, for $s_0<1$, it is reasonable to conclude that this instability is avoided. Macroscopic black holes (i.e., those with large κ) are still allowed.

If $F(s)$ is described by Fig. 4, the result is different. Near zero the situation is as before. As s goes to infinity, the divergence of $F(s)$ is given by

$$F(s)\sim\begin{cases}4\ln s, & q_3\neq 0\\ \frac{243}{82}\ln s, & q_3=0,\ \ q_2\neq 0,\\ \frac{129}{70}\ln s, & q_3=q_2=0,\ \ q_1\neq 0\ .\end{cases} \tag{3.49}$$

In all three cases, $F(s)$ diverges more slowly than $G(s)$, which goes as $\frac{9}{2}\ln s$. As shown in Fig. 4, a Schwarzschild radius exists and black holes form for any mass M.

The problems that black holes have caused in Kaluza-Klein theory are avoided only if there exists an $s_0<1$ such that $D(s_0)=0$ and $L(s_0)>0$. In the next section we examine whether this is possible for the values of q_1 and q_2 found in the calculation for $M^4\times S^6$ above.

IV. CONCLUSION

It is natural to assume that the Lagrangian (1.31) emerges from some fundamental theory such as the superstring theory. The constants a_0 through a_4 would be given once and for all and would be independent of the metric of space-time. We can therefore substitute into the analysis of the Schwarzschild problem the values for a_2/a_1 and a_3/a_1 given by the vacuum equations.

In our example the vacuum is $M^4\times S^6$. Results (2.27) and (2.28) imply

$$D(s)=1-\tfrac{35}{6}s+\tfrac{135}{4}s^2+qs^3\ , \tag{4.1}$$

where the subscript has been dropped from q_3. Equation $D(s_0)=0$ may be inverted to give an equation for q:

$$q=-\frac{1}{{s_0}^3}(1-\tfrac{35}{6}s_0+\tfrac{135}{4}{s_0}^2)\ . \tag{4.2}$$

This function is graphed in Fig. 6. The behavior of $D(s)$ as a function of q is summarized in Fig. 7. In Fig. 7(a) we are interested in the branch between $s=0$ and $s=s_0$. There is a one-to-one correspondence between q and s_0.

We need to find the sign of $L(s_0)$. There are no real solutions to the simultaneous equations:

$$D(s_0)=0,\ \ L(s_0)=0\ .$$

Therefore, either $L(s_0)<0$ or $L(s_0)>0$ for all s_0. It can easily be verified that the former holds. It was shown in Sec. III that the case $L(s_0)<0$ is unphysical.

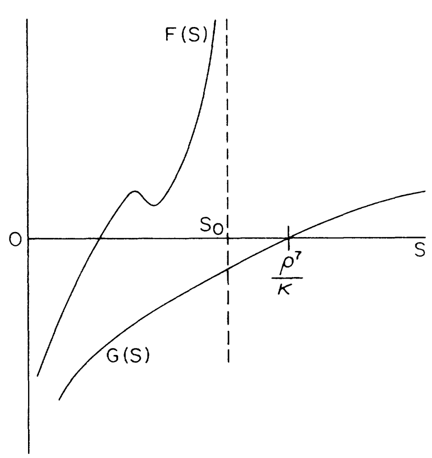

FIG. 5. For $F(s)$ as shown in Fig. 3, and a proper choice of parameters, no intersection of $F(s)$ and $G(s)$ exists. There is no Schwarzschild radius and a black hole does not form.

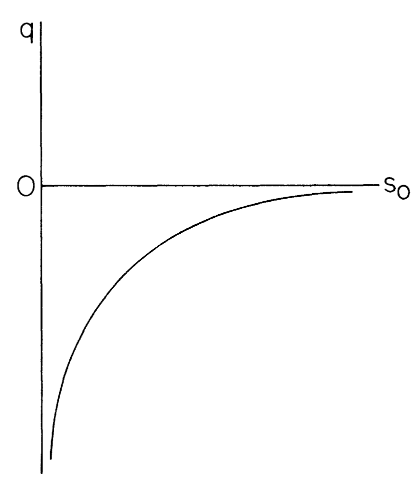

FIG. 6. Equation $D(s_0)=0$ may be inverted to give q as a function of s_0.

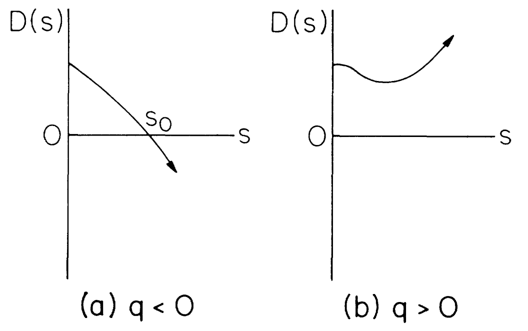

FIG. 7. The behavior of $D(s)$ as q varies is summarized above.

Figure 7(b) corresponds to Fig. 4. The metric is well defined, but black holes of any mass are allowed. The space $M^4 \times S^6$ therefore has no physical solutions.

We have found that if one uses the proper extension of general relativity to higher dimensions, it is in principle possible to avoid many of the problems associated with Kaluza-Klein theories. Physical requirements place strong restrictions on the coefficients of the Lagrangian (1.31). This analysis provides a test for any theory which predicts a gravitational Lagrangian of this form and a vacuum solution of the form $M^4 \times B$.

ACKNOWLEDGMENTS

I would like to thank Dr. G. Domokos and Dr. S. Kovesi-Domokos for the patient advice and guidance that made this work possible. I would also like to thank Cenalo Vaz for numerous stimulating discussions. The symbolic manipulation program MACSYMA was used to simplify some of the algebra. This research was supported in part by the U.S. Department of Energy under Contract No. DOE-BE-FG02-85ER40211.

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