Adiabatic regularization in closed Robertson-Walker universes

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Adiabatic regularization is a particularly efficient method of regularization for numerical studies of the dynamics of quantum scalar fields in homogeneous cosmological spacetimes. We show that of the possible ways to apply adiabatic regularization in a closed Robertson-Walker universe, only one yields the accepted trace anomaly and vacuum energy for a conformal scalar field. The method is to use the continuum measure appropriate to a flat space, while otherwise retaining the form of the subtractions appropriate to a closed space. We also show that this procedure is equivalent to point splitting, although technically much simpler.

I. INTRODUCTION

Adiabatic regularization is a method of finding the finite parts of expectation values of products of quantum scalar fields in homogeneous cosmological spacetimes.[1,2] First introduced in studies of the particle number in spatially flat Robertson-Walker universes, the method was generalized and applied to the energy-momentum tensor in Refs. 2–4. It has been found to be particularly efficient for numerical studies of the dynamics of quantum scalar fields in spatially flat Robertson-Walker, Bianchi type-I and Gowdy T^3 universes.[4–9] Birrell[10] has shown that for spatially flat Robertson-Walker spacetimes, it is equivalent to point splitting. Hu[11,12] showed that in the spatially flat Robertson-Walker and Bianchi type-I spacetimes adiabatic regularization gives a trace anomaly in agreement with other methods, and Bunch[13] showed this for the spatially flat and hyperbolic Robertson-Walker universes.

However, if one attempts to follow the approach of Ref. 13 in the spatially closed Robertson-Walker universe, then one obtains zero for the trace anomaly, as we show in the next section. The reason is that the adiabatic method yields the trace anomaly as the result of infrared divergences which appear in certain integrands as the mass of the scalar field approaches zero. However, these infrared divergences are absent when the momentum integrations are replaced by mode sums in the closed space.

It was noted already in Ref. 2 (p. 350) that "the method may need to be slightly modified by including in $\rho_{\rm reg}$ and $P_{\rm reg}$ (the regularized energy density and pressure), even when the metric is static, a nonvanishing vacuum energy density and pressure associated with the curvature of the three-space." Ford,[14] in his work on the Casimir effect in the closed static Einstein universe, showed how the energy associated with the closed spatial topology may be calculated, and pointed out the close relation to adiabatic regularization. Ford's method in conjunction with adiabatic regularization was used by Berger to obtain the vacuum energy in a Gowdy T^3 cosmology.[9,15] However, to our knowledge a definite and systematic prescription for applying adiabatic regularization in spatially closed universes which yields the trace anomaly has not been spelled out. We wish to fill this gap here. The basic result we find is that in calculating the vacuum subtraction, one should use the flat space measure in the mode sum in all cases, while retaining the form of the subtractions appropriate to the actual spatial curvature. This is consistent with what one would expect if the subtraction process is local, since the mode sums depend on the global properties of the spacetime, while the spatial curvature terms are present in the local curvature tensor.

Let us first briefly review the method of adiabatic regularization, in order to understand where possible ambiguity may arise, before proceeding in the following sections to present the solution and prove (by adapting the methods of Birrell[10]) that it must give the same results as the more laborious covariant point-splitting method.

II. REVIEW

The Robertson-Walker metric can be written in the form[16]

$$ds^2=a^2(\eta)\left[d\eta^2-\frac{dr^2}{1-Kr^2}-r^2d\Omega^2\right], \tag{2.1a}$$

where $a(\eta)$ is the scale factor and $K=0,-1,+1$ for spaces with zero, negative, or positive spatial curvature, respectively. The equation satisfied by the scalar field is

$$\Box\phi+m^2\phi+\xi R\phi=0\ , \tag{2.1b}$$

where ξ is a dimensionless constant and R is the scalar curvature. The field is expanded in terms of mode functions

$$u_{\mathbf{k}}=a^{-1}(\eta)\mathcal{Y}_{\mathbf{k}}(\mathbf{x})\psi_k(\eta) \tag{2.2a}$$

such that

$$\phi=\int d\tilde{\mu}(\mathbf{k})(a_{\mathbf{k}}u_{\mathbf{k}}+a_{\mathbf{k}}^{\dagger}u_{\mathbf{k}}^{*}) . \quad (2.2b)$$

The measure $d\tilde{\mu}(\mathbf{k})$ and mode functions are described in Ref. 2. Briefly they are as follows. The functions $\mathcal{Y}_{\mathbf{k}}(\mathbf{x})$ are eigenfunctions of the three-space Laplacian $\Delta^{(3)}$ such that $\Delta^{(3)}\mathcal{Y}_{\mathbf{k}}(\mathbf{x})=-(k^2-K)\mathcal{Y}_{\mathbf{k}}(\mathbf{x})$ and their explicit form is

$$\mathcal{Y}_{\mathbf{k}}(\mathbf{x})=(2\pi)^{-3/2}e^{i\mathbf{k}\cdot\mathbf{x}}, \quad K=0 , \quad (2.3a)$$

$$\mathcal{Y}_{\mathbf{k}}(\mathbf{x})=\Pi_{kl}^{(\pm)}(x)Y_{lm}(\theta,\phi), \quad K=\pm 1 . \quad (2.3b)$$

Here the $Y_{lm}(\theta,\phi)$'s are the standard spherical harmonics and the properties of the $\Pi_{kl}^{(\pm)}$ functions can be found in Refs. 2 and 17. The measure $d\tilde{\mu}(\mathbf{k})$ is

$$\int d\tilde{\mu}(\mathbf{k})\equiv\int d^3k, \quad K=0$$
$$\equiv\int_0^{\infty} dk \sum_{l,m}, \quad K=-1$$
$$\equiv\sum_{k,l,m}, \quad K=1 .$$

The time-dependent mode functions $\psi_k(\eta)$ satisfy

$$\psi_k''+[k^2+m^2a^2+(\xi-\tfrac{1}{6})a^2R]\psi_k=0 . \quad (2.4)$$

Here, primes denote derivatives with respect to η; $R=6a^{-2}(a^{-1}a''+K)$ is the scalar curvature and ξ is the coupling to the scalar curvature.

The classical expression for $T_{\mu\nu}$ of a scalar field is

$$T_{\mu\nu}=(1-2\xi)\phi_{;\mu}\phi_{;\nu}+(2\xi-\tfrac{1}{2})g_{\mu\nu}g^{\alpha\beta}\phi_{;\alpha}\phi_{;\beta}$$
$$-2\xi\phi_{;\mu\nu}\phi+\tfrac{1}{2}\xi g_{\mu\nu}\phi\Box\phi-\xi[R_{\mu\nu}+(\tfrac{3}{2}\xi-\tfrac{1}{2})Rg_{\mu\nu}]\phi^2$$
$$+(\tfrac{1}{2}-\tfrac{3}{2}\xi)m^2g_{\mu\nu}\phi^2 . \quad (2.5)$$

Substituting Eq. (2.2) into Eq. (2.5) and taking the vacuum expectation value, one finds, in a Robertson-Walker spacetime[13]

$$\langle 0|T^0{}_0|0\rangle_u=(4\pi^2a^4)^{-1}\int d\mu(k)(|\psi_k'|^2+(k^2+m^2a^2)|\psi_k|^2$$
$$+6(\xi-\tfrac{1}{6})\{(a'/a)(\psi_k\psi_k^{*\prime}+\psi_k^{*}\psi_k')-[(a'/a)^2-K]|\psi_k|^2\}) \quad (2.6a)$$

and

$$\langle 0|T|0\rangle_u=(2\pi^2a^4)^{-1}\int d\mu(k)(m^2a^2|\psi_k|^2+6(\xi-\tfrac{1}{6})[|\psi_k'|^2-(a'/a)(\psi_k\psi_k'^{*}+\psi_k^{*}\psi_k')$$
$$-(k^2+m^2a^2)|\psi_k|^2-(a''/a-a'^2/a^2)|\psi_k|^2$$
$$-(\xi-\tfrac{1}{6})a^2R|\psi_k|^2]) , \quad (2.6b)$$

where the measure $d\mu(k)$ is such that

$$\int d\mu(k)\equiv\int_0^{\infty} dk\, k^2, \quad K=0,-1 \quad (2.7a)$$

$$\equiv\sum_{k=1}^{\infty} k^2, \quad K=1 . \quad (2.7b)$$

Here T denotes the trace $T_\mu{}^\mu$. The other components of $\langle T_{\mu\nu}\rangle$ can be obtained from these by using the spatial isotropy. Note that in the case of conformal coupling, $\xi=\frac{1}{6}$, the factor of m^2 in Eq. (2.6b) reflects the fact that the classical trace of the stress tensor of the conformally invariant field is zero (the field equation with conformal coupling is invariant under conformal transformations of the metric only for $m=0$).

The mode sums in Eq. (2.6) have ultraviolet divergences. In adiabatic regularization one subtracts off these divergences thereby obtaining a finite result. To isolate the divergences one considers the case of a slowly varying spacetime. In this case the particle number is an adiabatic invariant, remaining constant in the limit of infinitely slow change in the scale factor $a(\eta)$ (Refs. 1 and 18). Therefore, by using the adiabatic approximation to the mode functions, one can evaluate the vacuum contribution for a slow change in $a(\eta)$. The ultraviolet divergences are included in this unobservable vacuum contribution. Subtracting this contribution gives the finite observable result. Furthermore, if one introduces a parameter (the adiabatic or slowness parameter) such that each time derivative of $a(\eta)$ brings out one more factor of this parameter, then the particle number will be constant to any finite power of the adiabatic parameter.[1,2,18] Therefore, the vacuum contribution, which is to be subtracted mode by mode, can be obtained to any necessary order in the slowness parameter by using the expansion of the mode function (i.e., the adiabatic approximation) in powers of the adiabatic parameter. In order to cancel all ultraviolet divergences in $\langle T_{\mu\nu}\rangle$, while at the same time preserving the condition that $\langle T_\mu{}^\nu\rangle_{;\nu}=0$, it is sufficient to subtract the vacuum contribution to fourth order in the adiabatic parameter. Having done this for a slowly varying spacetime, one then observes that the same counterterms remove the divergences in $\langle T_{\mu\nu}\rangle_u$ for arbitrary variations of the scale factor.

The adiabatic approximation (which is a generalized WKB approximation) to the time-dependent mode function $\psi_k(\eta)$ is obtained by writing

$$\psi_k=(2W)^{-1/2}\exp\left[-i\int^{\eta} W(\eta)d\eta\right] \quad (2.8)$$

and then expanding to fourth order in the adiabatic parameter. The results are calculated in Refs. 2, 4, and 13. The iterative calculation is readily summarized as follows.[13] Substitute Eq. (2.8) into Eq. (2.4), obtaining

$$W^2=\omega^2+(\xi-\tfrac{1}{6})a^2R-\frac{1}{2}\left[\frac{W''}{W}-\frac{3}{2}\frac{W'^2}{W^2}\right], \quad (2.9)$$

where

$$\omega^2\equiv k^2+m^2a^2 .$$

The order of the adiabatic parameter in a term corresponds to the number of time derivatives appearing. Now solve for W in terms of ω and its time derivatives to fourth adiabatic order by iterating Eq. (2.9).

Substitution of the fourth-order expression for W into Eqs. (2.6) using Eq. (2.8) gives the following adiabatic vacuum contributions:[13]

$$\begin{aligned}
{}_A\langle 0|T^0{}_0|0\rangle_A=(4\pi^2a^4)^{-1}\int d\mu(k)\Bigg\{&\omega+\frac{m^4a^4}{8\omega^5}\frac{a'^2}{a^2}-\frac{m^4a^4}{32\omega^7}\left[2\frac{a'''a'}{a^2}-\frac{a''^2}{a^2}+\frac{4a''a'^2}{a^3}-\frac{a'^4}{a^4}\right]\\
&+\frac{7m^6a^6}{16\omega^9}\left[\frac{a''a'^2}{a^3}+\frac{a'^4}{a^4}\right]-\frac{105m^8a^8}{128\omega^{11}}\frac{a'^4}{a^4}\\
&+(\xi-\tfrac{1}{6})\Bigg[-\frac{3}{\omega}\left[\frac{a'^2}{a^2}-K\right]-\frac{3m^2a'^2}{\omega^3}+\frac{3m^2a^2}{4\omega^5}\left[\frac{2a'''a'}{a^2}-\frac{a''^2}{a^2}-\frac{a'^4}{a^4}\right]\\
&\qquad-\frac{15m^4a^4}{8\omega^7}\left[\frac{4a''a'^2}{a^3}+\frac{3a'^4}{a^4}+\frac{a'^2}{a^2}K\right]+\frac{105m^6a^6}{8\omega^9}\frac{a'^4}{a^4}\Bigg]\\
&+(\xi-\tfrac{1}{6})^2\Bigg[-\frac{9}{2\omega^3}\left[\frac{2a'''a'}{a^2}-\frac{a''^2}{a^2}-\frac{4a''a'^2}{a^3}-\frac{2a'^2}{a^2}K+K^2\right]\\
&\qquad+\frac{27m^2a^2}{\omega^5}\left[\frac{a''a'^2}{a^3}+\frac{a'^2}{a^2}K\right]\Bigg]\Bigg\}
\end{aligned} \quad (2.10a)$$

and

$$\begin{aligned}
{}_A\langle 0|T|0\rangle_A=(4\pi^2a^4)^{-1}\int d\mu(k)\Bigg\{&\frac{m^2a^2}{\omega}+\frac{m^4a^4}{4\omega^5}\left[\frac{a''}{a}+\frac{a'^2}{a^2}\right]-\frac{5m^6a^6}{8\omega^7}\frac{a'^2}{a^2}-\frac{m^4a^4}{16\omega^7}\left[\frac{a''''}{a}+\frac{4a'''a'}{a^2}+\frac{3a''^2}{a^2}\right]\\
&+\frac{m^6a^6}{32\omega^9}\left[\frac{28a'''a'}{a^2}+\frac{126a''a'^2}{a^3}+21\frac{a''^2}{a^2}+21\frac{a'^4}{a^4}\right]\\
&-\frac{231m^8a^8}{32\omega^{11}}\left[\frac{a''a'^2}{a^3}+\frac{a'^4}{a^4}\right]+\frac{1155m^{10}a^{10}}{128\omega^{13}}\frac{a'^4}{a^4}\\
&+(\xi-\tfrac{1}{6})\Bigg[-\frac{6}{\omega}\left[\frac{a''}{a}-\frac{a'^2}{a^2}\right]-\frac{3m^2a^2}{\omega^3}\left[\frac{2a''}{a}-\frac{a'^2}{a^2}+K\right]\\
&\qquad+\frac{9m^4a^4}{\omega^5}\frac{a'^2}{a^2}+\frac{3m^2a^2}{2\omega^5}\left[\frac{a''''}{a}-\frac{2a''a'^2}{a^3}+\frac{a'^4}{a^4}\right]\\
&\qquad-\frac{15m^4a^4}{4\omega^7}\left[\frac{4a'''a'}{a^2}+\frac{3a''^2}{a^2}+\frac{8a''a'^2}{a^3}-\frac{a'^4}{a^4}+\frac{a''K}{a}+\frac{a'^2}{a^2}K\right]\\
&\qquad+\frac{105m^6a^6}{8\omega^9}\left[\frac{8a''a'^2}{a^3}+\frac{5a'^4}{a^4}+\frac{a'^2}{a^2}K\right]-\frac{945m^8a^8}{8\omega^{11}}\frac{a'^4}{a^4}\Bigg]\\
&+(\xi-\tfrac{1}{6})^2\Bigg[-\frac{9}{\omega^3}\left[\frac{a''''}{a}-\frac{4a'''a'}{a^2}-\frac{3a''^2}{a^2}+\frac{6a''a'^2}{a^3}-\frac{2a''}{a}K+\frac{2a'^2}{a^2}K\right]\\
&\qquad+\frac{27m^2a^2}{2\omega^5}\left[\frac{4a'''a'}{a^2}+\frac{3a''^2}{a^2}-\frac{6a''a'^2}{a^3}+\frac{4a''}{a}K-\frac{2a'^2}{a^2}K+K^2\right]\\
&\qquad-\frac{135m^4a^4}{\omega^7}\left[\frac{a''a'^2}{a^3}+\frac{a'^2}{a^2}K\right]\Bigg]\Bigg\}.
\end{aligned} \quad (2.10b)$$

These are to be subtracted under the mode sum from the corresponding expressions calculated using the exact mode functions ψ_k. The result is the physically relevant expectation value.

When the subtraction is carried out, the divergent terms will cancel. However, Eqs. (2.10) contain finite terms as well. In particular, consider the trace T. In the limit that $\xi \to \frac{1}{6}$, $m \to 0$, the expression in Eq. (2.6b) with the exact mode functions will make no contribution to the vacuum expectation value $\langle T \rangle$, as it comes directly from the classical expression. For $K=0$ and -1, the integral over k in Eq. (2.10) can be done for the finite terms and the limit as $\xi \to \frac{1}{6}$, $m \to 0$ can then be taken, with the result that[11,13]

$$_A\langle 0 | T | 0 \rangle_A = \frac{6}{2880\pi^2}\left[-\frac{a''''}{a^5}+\frac{4a'''a'}{a^6}+\frac{3a''^2}{a^6} -\frac{8a''a'^2}{a^7}+\frac{2a'^4}{a^8}\right]. \tag{2.11}$$

The limit as m approaches zero is taken only after integration over k because the expectation value of T must be obtained for finite m before the limit can be taken. The result is just the negative of the well known trace anomaly. Since it is the subtraction of this quantity which gives rise to the only contribution to the trace in this case, one obtains the usual trace anomaly for $\langle T \rangle$.

III. THE SITUATION FOR $K=1$

For the $K=1$ universe, the expression in Eq. (2.6b) for $\langle T \rangle_u$ has a discrete summation over modes. If one uses the same discrete summation in the adiabatic vacuum subtraction of Eq. (2.10b), then the summations over the finite terms in (2.10b) give ζ functions in the limit $m \to 0$. Thus, the trace anomaly does not appear.

The trace anomaly does appear in the $K=0$ and -1 cases because the integrals of the finite terms are infrared divergent in the limit $m \to 0$. The positive powers of m in the coefficients of these terms exactly cancel the divergences, leaving a result that is independent of the mass. For $K=1$, the closed spatial topology results in a mode sum beginning at $k=1$, thereby removing the infrared divergences. Thus if adiabatic regularization with measure (2.7b) in the vacuum subtraction is applied to the $K=1$ Robertson-Walker universe, it gives zero trace anomaly, in disagreement with other methods such as dimensional, ζ-function and point-splitting regularization. Examination of (2.6a) and (2.10a) with the measure (2.7b) shows that the Casimir energy also does not appear for the conformally invariant field in this case. In fact, $\langle 0 | T_{\mu\nu} | 0 \rangle \equiv 0$ for the conformally invariant field in the conformal vacuum state in a $K=1$ Robertson-Walker universe if the measure (2.7b) is used in (2.10a).

We argue that the ultraviolet vacuum subtraction should make use only of locally available information. Therefore, we propose that in the adiabatic vacuum subtractions of Eqs. (2.10) the flat space ($K=0$) mode integration of Eq. (2.7a) should be used in all cases, including the closed universe ($K=1$). Terms involving K in the integrand do not violate the locality principle because K arises from local quantities such as the scalar curvature. As we shall see, this method of applying adiabatic regularization to the $K=1$ case gives results consistent with the other regularization schemes. The latter also do not take into account the global spatial topology when computing the regularized terms corresponding to the divergences in $\langle T_{\mu\nu} \rangle_u$. The measure for the exact modes must clearly remain the same as before. This is because the formal expression $\langle T_{\mu\nu} \rangle_u$ is defined with the appropriate measure and exact solutions which result from boundary conditions involving the global topology. The use of only local information in the vacuum subtraction is also consistent with the classic calculation in quantum electrodynamics of the Casimir effect between conducting plates[19,20] (for further discussion see Sec. IV).

To show that our version of adiabatic regularization for a $K=1$ spacetime really is in agreement with the schemes mentioned above, we shall compute $\langle 0 | T_{\mu\nu} | 0 \rangle$ for the conformally invariant field and show that the trace anomaly is recovered. Then we will show that for a field with arbitrary mass and coupling to the scalar curvature, all of the divergences in $\langle T_{\mu\nu} \rangle_u$ are canceled by the counterterms. Finally, in the Appendix we give a proof which shows that adiabatic regularization is equivalent to point splitting for arbitrary mass and coupling to the scalar curvature. Birrell[10] did the original proof for the case $K=0$ and we have generalized it to include the cases $K=\pm 1$.

For a conformally invariant scalar field in a $K=1$ spacetime the mode equation, (2.4), can be solved exactly for arbitrary functions of the scale factor $a(\eta)$ with the result that

$$\langle 0 | T^0{}_0 | 0 \rangle = \frac{1}{4\pi^2 a^4}\sum_{k=1}^{\infty} k^3 - {}_A\langle 0 | T^0{}_0 | 0 \rangle_A , \tag{3.1a}$$

$$\langle 0 | T | 0 \rangle = -{}_A\langle 0 | T | 0 \rangle_A . \tag{3.1b}$$

Inserting a cutoff of the form $e^{-\alpha k}$ in both terms of (3.1a), and setting $\alpha=0$ at the end of the calculation, as in Ref. (14) one finds

$$\langle 0 | T^0{}_0 | 0 \rangle = \frac{6}{2880\pi^2 a^4}\left[1+\frac{a'''a'}{a^2}-\frac{1}{2}\frac{a''^2}{a^2} -\frac{2a''a'^2}{a^3}+\frac{1}{2}\frac{a'^4}{a^4}\right], \tag{3.2}$$

and $\langle 0 | T | 0 \rangle$ is given by negative of Eq. (2.11). These expressions are in complete agreement with the results of dimensional regularization, ζ-function regularization, and point splitting.[21] Thus one obtains both the trace anomaly and the Casimir energy.

We next show that all of the divergences in $\langle T_{\mu\nu} \rangle_u$ are canceled when $_A\langle 0 | T_{\mu\nu} | 0 \rangle_A$ is computed for a $K=1$ spacetime using the continuum measure (2.7a). First of all, they are certainly canceled when the discrete measure (2.7b) is used since the asymptotic expansion obtained by iterating Eq. (2.9) is valid in the limit $k \to \infty$ and the measure (2.7b) is the same when $K=1$ as that used in computing $\langle T_{\mu\nu} \rangle_u$. From (2.10a) and (2.10b) one sees that each divergent term in $_A\langle 0 | T_{\mu\nu} | 0 \rangle_A$ has the form

$$I_n = f_n(\eta) \int \frac{d\mu(k)}{(k^2+m^2a^2)^{n/2}} , \tag{3.3}$$

with $f_n(\eta)$ some function which is independent of k. Divergences occur for $n=-1,1,3$. Using the Plana summation formula[22-25] one finds

$$\sum_{k=1}^{\infty} \frac{k^2}{(k^2+m^2a^2)^{n/2}} = \int_0^{\infty} \frac{dk\, k^2}{(k^2+m^2a^2)^{n/2}} - \int_0^1 \frac{dk\, k^2}{(k^2+m^2a^2)^{n/2}} + \tfrac{1}{2}(1+m^2a^2)^{-n/2} - 2\int_0^{\infty} \frac{dt\, q(t)}{e^{2\pi t}-1} ,$$
$$q(t) = \frac{1}{2i}\left[\frac{(1+it)^2}{[(1+it)^2+m^2a^2]^{n/2}} - \frac{(1-it)^2}{[(1-it)^2+m^2a^2]^{n/2}} \right] . \tag{3.4}$$

The only divergent term in (3.4) for $n=-1,1,3$ is the first one. This is just the term which occurs if the continuum measure (2.7a) is used so the divergence structure of ${}_A\langle 0 | T_{\mu\nu} | 0 \rangle_A$ is the same regardless of which measure is used to compute it.

IV. DISCUSSION

We have shown that in a $K=1$ Robertson-Walker universe adiabatic regularization with the discrete measure (2.7b) does not give the trace anomaly for the scalar field. We have also shown that with the continuum measure (2.7a) the trace anomaly is obtained, all divergences in $\langle T_{\mu\nu} \rangle_u$ are still canceled and adiabatic regularization is equivalent to point splitting.

These results warrant some discussion. Adiabatic regularization with discrete measure (2.7b) is a valid regularization scheme in the sense that it does remove the divergences in $\langle T_{\mu\nu} \rangle_u$. Because it makes use of the global topology, it leads to results which differ from other schemes which only make use of locally available information. However, it is a mathematically consistent procedure. Furthermore, other regularization methods could conceivably be altered to take account of the global topology and would then probably give the same results as does the discrete measure. Therefore, one may ask why is adiabatic regularizations with the continuum measure (2.7a) in the closed ($K=1$) Robertson-Walker universe to be preferred over the closely related procedure with discrete measure (2.7b)?

One may argue that since the divergences are ultraviolet divergences and hence local, only local information should be used for computing the counterterms to $\langle T_{\mu\nu} \rangle_u$. A second argument makes use of a theorem proved by Wald.[26] It says that if two candidates for the renormalized stress-energy tensor $\langle T_{\mu\nu} \rangle$ both obey the three reasonable conditions of covariant conservation, causality, and standard results for "off-diagonal" elements, then they can differ at most by a local, conserved tensor. Now in a closed Robertson-Walker universe, adiabatic regularization with either measure results in expressions for $\langle T_{\mu\nu} \rangle$ which satisfy these conditions and the two expressions differ only by a local conserved tensor. However, in a nonconformally flat spacetime such as a Bianchi type-IX universe the situation is different. In such a case a discrete measure analogous to (2.7b) would have to be used to compute $\langle T_{\mu\nu} \rangle_u$. If it was also used to compute ${}_A\langle 0 | T_{\mu\nu} | 0 \rangle_A$ then the trace anomaly would almost certainly not result while if a continuum measure analogous to (2.7a) were used the trace anomaly probably would result. However, the trace anomaly outside of an exactly conformally flat spacetime cannot be derived from a local, conserved tensor[21,27] and thus adiabatic regularization with a discrete measure would probably violate one or more of the three conditions mentioned above. This is particularly true since point splitting obeys the Wald axioms[21,26] and results in the trace anomaly for an arbitrary spacetime.[28] This argument, then favors the continuum measure for a closed ($K=1$) Robertson-Walker space.

There is also one piece of experimental evidence available. This is the Casimir effect in electrodynamics.[19,20] In computing $\langle 0 | T_{\mu\nu} | 0 \rangle$ in the Casimir effect one does not use, when constructing the counterterms to $\langle 0 | T_{\mu\nu} | 0 \rangle_u$, the mode sum appropriate to the vanishing boundary conditions on the uncharged conducting plates, but rather subtracts the Minkowski expression involving the continuum measure. The force caused by the resulting vacuum energy of the electromagnetic field between two uncharged conducting plates has been experimentally verified. Therefore, although we lack direct observational evidence concerning the vacuum energy or pressure in a closed universe, the use of the flat space-mode integration (2.7a) in the vacuum subtraction does appear to be clearly favored by physical considerations.

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APPENDIX

In this Appendix the proof that adiabatic regularization is equivalent to point splitting for an arbitrary Robertson-Walker spacetime is given. In point splitting[21,28,29] one computes the quantity $G_u^{(1)}(X',X'') = \langle \{\phi(X')\phi(X'')\} \rangle_u$ and subtracts from it $G_{\mathrm{DS}}^{(1)}(X',X'')$ which is obtained using the DeWitt-Schwinger expansion.[28] $\langle T_{\mu\nu} \rangle$ is obtained by differentiating the regularized $G^{(1)}(X',X'')$ and then letting the points come together along the shortest geodesic connecting them. Thus

$$\langle T_{\mu\nu}\rangle_{\rm PS}=\lim_{\substack{X'\to X\\ X''\to X}}\mathcal{D}_{\mu\nu}(X',X'')G^{(1)}(X',X'')=\lim_{\substack{X'\to X\\ X''\to X}}\mathcal{D}_{\mu\nu}(X',X'')[G_u^{(1)}(X',X'')-G_{\rm DS}^{(1)}(X',X'')]\ , \tag{A1}$$

with $\mathcal{D}_{\mu\nu}(X',X'')$ being a derivative operator.[21]

If we compute $G_{\rm ad}^{(1)}(X',X'')={}_A\langle 0|\phi(X')\phi(X'')|0\rangle_A$, act on it with $\mathcal{D}_{\mu\nu}(X',X'')$ and set $X'=X''=X$, we must end up with ${}_A\langle 0|T_{\mu\nu}|0\rangle_A$. Then the regularized stress tensor from adiabatic regularization must be given by

$$\langle T_{\mu\nu}\rangle_{\rm ad}=\langle T_{\mu\nu}\rangle_u-{}_A\langle 0|T_{\mu\nu}|0\rangle_A=\lim_{\substack{X'\to X\\ X''\to X}}\mathcal{D}_{\mu\nu}(X',X'')[G_u^{(1)}(X',X'')-G_{\rm ad}^{(1)}(X',X'')]\ . \tag{A2}$$

Thus to show that adiabatic regularization is equivalent to point splitting, i.e., $\langle T_{\mu\nu}\rangle_{\rm ad}=\langle T_{\mu\nu}\rangle_{\rm PS}$, it suffices to show that $G_{\rm ad}^{(1)}(X',X'')=G_{\rm DS}^{(1)}(X',X'')$ to whatever order in the separation between the points it is necessary to retain so that the contribution to $\langle T_{\mu\nu}\rangle$ does not vanish when the points come together. This turns out to be to $O(\epsilon^4)$, where ϵ is one-half the geodesic distance between X' and X''. Since $\langle T_{\mu\nu}\rangle_{\rm PS}$ does not depend on the choice of the points X' and X'' (Refs. 21 and 28) it is sufficient to prove that $G_{\rm ad}^{(1)}(X',X'')=G_{\rm DS}^{(1)}(X',X'')+O(\epsilon^4)$ for a particular choice of X' and X''.

The proof that $G_{\rm ad}^{(1)}(X',X'')=G_{\rm DS}^{(1)}(X',X'')+O(\epsilon^4)$ proceeds as follows: Compute $G_{\rm ad}^{(1)}(X',X'')$ using the Minkowski measure for the mode sum and the adiabatic expansion for the modes truncated at fourth adiabatic order. One finds that

$$G_{\rm ad}^{(1)}(X',X'')=\int_0^\infty dk\left[[2a(\eta')a(\eta'')W^{1/2}(\eta')W^{1/2}(\eta'')]^{-1}\exp\left[-i\int_{\eta''}^{\eta'}W(\eta)d\eta\right]\sum_{l,m}\mathcal{Y}_{\mathbf{k}}(\mathbf{x}')\mathcal{Y}_{\mathbf{k}}^*(\mathbf{x}'')+\text{complex conjugate}\right]\ . \tag{A3}$$

Here $W^{-1/2}(\eta')W^{-1/2}(\eta'')$ are to be expanded to fourth adiabatic order.[30]

In the coordinate systems with

$$ds^2=a^2(\eta)[-d\eta^2+d\chi^2+\Sigma^2(\chi)d\Omega^2]\ ,$$
$$\Sigma^2(\chi)=\chi^2,\ \ K=0=\sinh^2\chi\ , \tag{A4}$$
$$K=-1=\sin^2\chi,\ \ K=1\ ,$$

and with X' and X'' connected by a geodesic with constant values of the angular coordinates θ and ϕ one finds[31]

$$\sum_{l,m}\mathcal{Y}_{\mathbf{k}}(\mathbf{x}')\mathcal{Y}_{\mathbf{k}}^*(\mathbf{x}'')=\sum_{l,m}\Pi_{kl}^{(\pm)}(x')\Pi_{kl}^{(\pm)*}(x'')\,|\,Y_{lm}(\theta,\phi)\,|^2=\frac{k}{2\pi^2}\frac{\sin k\Delta_\chi}{\Sigma(\Delta_\chi)},\ \ \Delta_\chi=\chi'-\chi''\ . \tag{A5}$$

Next one expands η',η'',Δ_χ in powers of ϵ with the result that[21,29]

$$\eta(\epsilon)=\eta+\epsilon t_1^0+\frac{\epsilon^2}{2!}t_2^0+\frac{\epsilon^3}{3!}t_3^0+\cdots\ ,$$
$$\chi(\epsilon)=\chi+\epsilon t_1^1+\frac{\epsilon^2}{2!}t_2^1+\frac{\epsilon^3}{3!}t_3^1+\cdots\ . \tag{A6}$$

Here $\eta(\epsilon)=\eta'$, $\eta(-\epsilon)=\eta''$, $\chi(\epsilon)=\chi'$, $\chi(-\epsilon)=\chi''$, η and χ are the points approached in the limit $\epsilon\to 0$, and t^u is the tangent vector to the geodesic. t^μ is expanded in powers of ϵ such that

$$t^\mu=t_1^\mu+\epsilon t_2^\mu+\frac{\epsilon^2}{2!}t_3^\mu+\cdots\ . \tag{A7}$$

The geodesic equation can be solved as a power series in ϵ to obtain the t_i^μ. t_1^μ is the tangent vector at the point X.

To make the computation of the integrals in (A3) easier we choose X' and X'' so that they are separated by a geodesic with constant values of the angular coordinates θ and ϕ and with the tangent vector at χ given by $t_1^0=0$, $t_1^1=a^{-1}$. Then one finds that

$$\eta(\epsilon)=\eta+\frac{\epsilon^2}{2!}t_2^0+\frac{\epsilon^4}{4!}t_4^0+O(\epsilon^6)\ , \tag{A8a}$$

while

$$\chi(\epsilon)=\chi+\epsilon t_1^1+\frac{\epsilon^2}{2!}t_2^1+\cdots\ , \tag{A8b}$$

as before. Since $\eta(\epsilon)$ is symmetric in ϵ, i.e., $\eta(\epsilon)=\eta(-\epsilon)$ to at least $O(\epsilon^7)$, so that $\eta'-\eta''=O(\epsilon^7)$, (A3) can be written as

$$G_{\rm ad}^{(1)}(X',X'')=\frac{1}{2\pi^2a^2(\eta')}\int_0^\infty dk\frac{k\sin(k\Delta_\chi)}{W(\eta')\Sigma(\Delta_\chi)}\ . \tag{A9}$$

One next expands W to fourth adiabatic order, does the momentum integrals, and then expands η' and Δ_χ in powers of ϵ such that each term in $G_{\rm ad}^{(1)}$ is of order ϵ^2 or less. The result agrees exactly with that given by Christensen[28,29,21] with $t_1^0=0$ with $t_1^1=a^{-1}$. Thus for this separation of the points, $G_{\rm ad}^{(1)}(X',X'')=G_{\rm DS}^{(1)}(X',X'')+O(\epsilon^4)$ completing the proof of the equivalence of adiabatic regularization and point splitting.

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