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BRST structure of general relativity in terms of new variables

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The structure of the Poisson-brackets algebra of constraints of general relativity is reexamined using the recently introduced spinorial variables. Three different combinations of constraints are analyzed and their relative merits are discussed. In each case we construct the corresponding expression of the Becchi-Rouet-Stora-Tyutin charge. These expressions provide a point of departure for a nonperturbative quantization scheme for general relativity.

I. INTRODUCTION

It is now generally recognized that the nonrenormalizability of quantum general relativity stems from the basic assumption of the perturbation theory that space-time geometry can be approximated by a classical background well below the Planck scale. The failure of perturbative quantum gravity does not, therefore, imply that it is impossible to construct a sensible quantum theory starting from general relativity; it is quite possible that the quantum theory exists nonperturbatively but predicts that the space-time geometry in the small class of solutions is drastically different from that described by classical general relativity. To see if this is in fact the case, one must approach the problem nonperturbatively. The canonical quantization scheme is, at least in principle, well suited to undertake this task since it does not require the introduction of a classical background geometry. In fact, it is the absence of a classical background metric that makes the Hamiltonian structure of general relativity novel and qualitatively different from that of other field theories of physical interest. This difference in turn implies that conceptual issues as well as technical problems which arise in canonical quantum gravity are of a very different nature from those normally faced in other theories (see, e.g., Refs. 1 and 2). Moreover, the difference prevails even when one brings in matter coupled to gravity; the Hamiltonian structure of the coupled theories resembles closely that of pure general relativity and is very different from that of theories of matter fields in Minkowski space-time.

A key feature of the nonperturbative Hamiltonian description is that constraints play a powerful role. In quantum theory, in particular, the appropriate incorporation of these constraints would be a major step. While in Yang-Mills theory in Minkowski space imposition of constraints is relatively easy and the key difficulties lie in quantum dynamics, it appears that the situation would be just the opposite in quantum gravity (possibly coupled with matter). Unfortunately, however, relatively little is known about exact solutions to the quantum constraints of general relativity. The main obstacle has been the complicated analytical form of these constraints in terms of the traditionally used canonically conjugate variables, the three-metric q_{ab} and its canonically conjugate momentum $\tilde{P}^{ab}$ (related to the extrinsic curvature k^{ab} via $\tilde{P}^{ab}=[(\det q)^{1/2}/G](k^{ab}-kq^{ab})$) (Ref. 3). As a result, in terms of these variables, not a single solution to the full scalar constraint of quantum general relativity is known. However, recently, the structure of these constraints was significantly simplified by the introduction of new canonically conjugate variables $(\tilde{\sigma}^{a}{}_{A}{}^{B}, A_{aA}{}^{B})$, where $\tilde{\sigma}^{a}$ is an SU(2) (densitized) soldering form, a "square root of the three-metric q^{ab}," and A_a is a (complexified) SU(2) connection.[4] When recast in terms of these variables, constraints are at worst quadratic in each of $\tilde{\sigma}^{a}$ and A_a. Furthermore, the new scalar constraint no longer has the "potential term"; in its form it resembles the well-understood strong-coupling limit[5] of the familiar $(q,\tilde{P})$-scalar constraint. These simplifications have been exploited to obtain two interesting results. First, a small class of solutions to all constraints has been obtained in the asymptotically flat context.[6] Thus, now we at least know that the exact theory is not empty. That it may in fact be interesting is indicated by the existence of a large class of solutions to the scalar constraint found by Jacobson and Smolin.[7] These solutions are analogous to the Wilson loops that feature in QCD; they represent gravitational excitations with support on one-dimensional loops. The picture of the microstructure of space-time geometry projected by these states is very different indeed from the one used in perturbative analyses. However, this program is still incomplete: it is not yet clear how one can incorporate the vector constraint in terms of these states.

In all this work, one has followed Dirac's prescription of imposing constraints as operator conditions on permissible physical states. One knows through examples, however, that this may be too strong a requirement to impose.[8] To recover the constraints in the classical limit, it is sufficient to require, for example, that only the expectation values of constraints in physical states should vanish. A convenient way to impose a weaker condition is provided by the Becchi-Rouet-Stora-Tyutin

(BRST) technique.[9,10] Here, one first constructs, starting from classical constraints, the BRST charge and represents physical states by (equivalence of classes of) wave functions that are annihilated by the quantum charge operator (two functions being regarded as equivalent if they differ by an element of the image of the charge operator). This technique may, in particular, provide a way to incorporate the vector constraint in the Jacobson-Smolin program.

The purpose of this paper is to construct expressions of the BRST charge in terms of the new variables $(\tilde{\sigma}^a, A_a)$. On the way to our goal, we shall reexamine the constraint algebra given in Ref. 4 and show that an alternate algebra, obtained by taking functional linear combination of constraints used in Ref. 4, is easier to work with both technically and conceptually. In particular, the second-order BRST structure functions of the new algebra vanish rather trivially.

The plan of the paper is as follows. The next section recalls briefly the steps leading to the BRST charge and the Hamiltonian structure of general relativity in terms of new variables, thereby fixing the notation used in the main discussion. Section III contains the main results: two alternative forms of constraints are discussed, their relative merits are pointed out and the expression of the BRST charge is given for each. In Appendix A, for completeness, we return to the constraint algebra given in Ref. 4. The second-order structure functions do not vanish for this algebra. We exhibit an explicit canonical transformation of the extended phase space (which includes ghosts) that transforms the corresponding BRST charge into the two simpler versions described in Sec. III. Appendix B recasts the key equations in a notation that is generally used in the particle-physics–gauge-theory literature.

Unless otherwise stated, our conventions are the same as those of Ref. 4.

II. PRELIMINARIES

This section is divided into two parts. In the first we summarize the BRST description of constrained systems and in the second we briefly recall the structure of Einstein constraints in terms of new variables.

A. The BRST formalism (Ref. 10)

Consider a (bosonic) physical system with phase space Γ. Let there be first-class constraints

$$C_\alpha \approx 0 \ , \tag{1}$$

with Poisson-brackets relations

$$[C_\alpha, C_\beta] = -2\,{}^{(1)}U_{\alpha\beta}{}^{\gamma} C_\gamma \ , \tag{2}$$

where ${}^{(1)}U_{\alpha\beta}{}^{\gamma}$ may be functions on Γ. We shall outline the steps leading to the BRST charge Q for this system.

It is convenient to change notation and label the constraint functions, C_α, by ${}^{(0)}U_\alpha$ and call them the zeroth-order structure functions. The ${}^{(1)}U_{\alpha\beta}{}^{\gamma}$ of (2) are the first-order structure functions. Note that if ${}^{(1)}U_{\alpha\beta}{}^{\gamma}$ fail to be constants, the constraints are not (strongly) closed under the Poisson brackets; Eq. (2) does not define a (closed) Lie algebra. In this case the commutator of two canonical transformations generated by constraints, each of which may be regarded as a gauge transformation following Dirac, does not yield another such canonical transformation except on the constraint surface. However, from the Jacobi identity satisfied by the Poisson brackets, it follows that

$$[{}^{(1)}U_{[\alpha\beta}{}^{\delta}, {}^{(0)}U_{\gamma]}] + 2\,{}^{(1)}U_{[\alpha\beta}{}^{\lambda}\,{}^{(1)}U_{\gamma]\lambda}{}^{\delta} = 2\,{}^{(2)}U_{\alpha\beta\gamma}{}^{\delta\lambda}\,{}^{(0)}U_\lambda \ , \tag{3}$$

everywhere on Γ for some functions ${}^{(2)}U_{\alpha\beta\gamma}{}^{\delta\lambda} = {}^{(2)}U_{[\alpha\beta\gamma]}{}^{[\delta\lambda]}$. The ${}^{(2)}U$ are referred to as second-order structure functions. If ${}^{(1)}U$ happen to be constants on Γ, then the constraints single out a subalgebra of the Poisson-brackets Lie algebra, the left side of (3) vanishes identically and ${}^{(2)}U$ can be set to zero. Even when ${}^{(1)}U$ are not constants, it can happen that ${}^{(2)}U$ vanish. This occurs, for example, in general relativity with $(q, \tilde{P})$ variables and, as we shall see, with new variables.

It turns out that[10] the second-order constraint functions also satisfy a "Jacobi-type" identity whence one is led to third-order structure functions. Proceeding along this way, one gets a "ladder" of structure functions ${}^{(n)}U_{\beta_1 \cdots \beta_{n+1}}{}^{\alpha_1 \cdots \alpha_n}$. A succinct way of defining these is as follows. Consider "potentials" ${}^{(n)}D_{\beta_1 \cdots \beta_{n+1}}{}^{\alpha_1 \cdots \alpha_n}$, antisymmetric in all contravariant and covariant indices, given by

$$\begin{aligned}{}^{(n)}D_{\beta_1 \cdots \beta_{n+2}}{}^{\alpha_1 \cdots \alpha_n} \equiv & \frac{1}{2} \sum_{p=0}^{n} ({}^{(p)}U_{\beta_1 \cdots \beta_{p+1}}{}^{\alpha_1 \cdots \alpha_p}, {}^{(n-p)}U_{\beta_{p+2} \cdots \beta_{n+2}}{}^{\alpha_{p+1} \cdots \alpha_n})(-1)^{(n-p+1)(p+1)+n-p} \\ & - \sum_{p=0}^{n-1} (p+1)(n-p+1)\,{}^{(p+1)}U_{\beta_1 \cdots \beta_{p+2}}{}^{\alpha_1 \cdots \alpha_p \gamma}\,{}^{(p+1)}U_{\beta_{p+3} \cdots \beta_{n+2}\gamma}{}^{\alpha_{p+1} \cdots \alpha_n}(-1)^{(n-p)(p+1)} \ .\end{aligned} \tag{4}$$

Then the structure functions ${}^{(n+1)}U_{\beta_1\cdots\beta_{n+2}}{}^{\alpha_1\cdots\alpha_{n+1}}$ of order $n+1$ are given by

$$(n+1)\,{}^{(n+1)}U_{\beta_1\cdots\beta_{n+2}}{}^{\alpha_1\cdots\alpha_{n+1}}\,{}^{(0)}U_{\alpha_{n+1}} = {}^{(n)}D_{[\beta_1\cdots\beta_{n+2}]}{}^{[\alpha_1\cdots\alpha_n]} . \quad (5)$$

If all structure functions of order n vanish for $s<n<2s+1$, then one can show[10] that the highest possible nonvanishing structure function is of order s. If all structure functions of order strictly greater than r vanish, we say that the constraint algebra (1),(2) is of rank r.

To construct the BRST charge, we need all nonvanishing structure functions. The charge itself is a (Grassmann-valued) function on an extended phase space. To obtain this extension, one introduces new (fermionic) degrees of freedom η^α, one for each constraint. These will be referred to as ghosts. Their conjugate momenta will be denoted by P_β. The nonvanishing (symmetric) Poisson brackets between these new fields are given by

$$[\eta^\alpha, P_\beta] = [P_\beta, \eta^\alpha] = -\delta^\alpha{}_\beta . \quad (6)$$

The ghosts and their momenta will be assumed to have vanishing Poisson brackets with the bosonic variables on Γ. Thus, we have an extension of the phase space Γ. Denote it by $\hat{\Gamma}$. The BRST charge Q is a (Grassmann-) odd function on $\hat{\Gamma}$, defined by

$$Q = \sum_{n=0}^{r} {}^{(n)}U_{\alpha_0\cdots\alpha_n}{}^{\beta_1\cdots\beta_n}\eta^{\alpha_n}\cdots\eta^{\alpha_0}P_{\beta_n}\cdots P_{\beta_1} , \quad (7)$$

where we have assumed that the constraints are of rank r. The characteristic properties of Q are

$$Q\mid_{P_\alpha=0} = \eta^{\alpha}\,{}^{(0)}U_\alpha \quad (8a)$$

and

$$[Q,Q]=0 . \quad (8b)$$

These properties contain, in a succinct way, all the information about constraints which is relevant to the issue of weeding out gauge freedom from true degrees of freedom.

A key difference between the Dirac and the BRST treatment of constrained systems is that, whereas one works essentially on the constraint surface in the Dirac approach, the BRST method requires one to work with the entire phase space. Put differently, while one is basically interested only in weak equalities in the Dirac treatment, one works with strong equalities in the BRST approach. The price paid in the BRST approach is that the structure functions, and hence Q, are not unique. As defined in (5), one has the freedom to add to a structure function of order n a totally antisymmetric object of covariant rank $(n+1)$ and contravariant rank $(n+1)$, contracted with a constraint function. However, these redefinitions can be compensated by a canonical transformation on the extended phase space[10] (also, see Appendix A).

B. Gravitational phase space using new variables (Ref. 4)

Fix a three-manifold Σ on which topological complications, if any, are restricted to a compact region. Thus, either Σ is compact or asymptotically flat, with only one asymptotic region. We shall use lower-case latin letters for tensor indices on Σ and upper-case latin ones for SU(2) internal [or SU(2) spinor] indices. The phase space of general relativity Γ, based on Σ, consists of pairs $(\tilde{\sigma}^a{}_A{}^B, A_{aA}{}^B)$, where $\tilde{\sigma}^a$ is a (densitized) SU(2) Infeld–Van der Waerden symbol[3] soldering the SU(2) internal indices to the tangent space indices of Σ, and A_a is a (complex) SU(2) connection one-form. More precisely, $\tilde{\sigma}^a$ is an isomorphism between Hermitian, trace-free second-rank fields $\lambda_A{}^B$ and real vector densities $\tilde{\lambda}^a$ of weight one on Σ, $\tilde{\lambda}^a = -\mathrm{Tr}\tilde{\sigma}^a\lambda$ and A_a is a complex one-form whose real and imaginary parts take values in the SU(2) Lie algebra. Given a pair $(\tilde{\sigma}^a, A_a)$ one can recover the traditional canonically conjugate pair $(q_{ab}, \tilde{P}^{ab})$ as follows. Let us first define q_{ab} in terms of $\tilde{\sigma}^a$ via

$$\mathrm{Tr}\tilde{\sigma}^a\tilde{\sigma}^b = -(\det q)q^{ab} , \qquad q^{ab}q_{ac} = \delta_c{}^b , \quad (9)$$

where $(\det q)$ is the determinant of q_{ab}. Next, denote by $\Gamma_{aA}{}^B$ the SU(2) Lie-algebra-valued connection one-form of the unique torsion-free derivative operator D on Σ satisfying $D_a\tilde{\sigma}^b{}_A{}^B=0$. Then, given a point $(\tilde{\sigma}^a, A_a)$ of the gravitational phase space, we acquire a field $\Pi_{aA}{}^B$ via[11]

$$\Pi_a = -\sqrt{2}i(GA_a - \Gamma_a) , \quad (10)$$

where G is Newton's constant. Now the momentum $\tilde{P}^{ab}$ canonically conjugate to q_{ab} is defined by

$$\tilde{P}^{ab} \equiv \frac{1}{G}[\tilde{\Pi}^{(ab)} - \tilde{\Pi}q^{ab}] , \quad (11)$$

where

$$\tilde{\Pi}^{ab} = -q^{am}\mathrm{Tr}\tilde{\sigma}^b\Pi_m .$$

The basic Poisson-brackets relations between the new variables are given by

$$[\tilde{\sigma}^a{}_{AB}(x), \tilde{\sigma}^m{}_{MN}(y)]=0, \quad [A_a{}^{AB}(x), A_m{}^{MN}(y)]=0 , \quad (12)$$
$$[A_a{}^{AB}(x), \tilde{\sigma}^m{}_{MN}(y)] = \frac{i}{\sqrt{2}}\delta^m{}_a\delta_{(M}{}^A\delta_{N)}{}^B\delta(x,y) ,$$

where the internal indices are raised and lowered by the SU(2)-invariant two-form ϵ_{AB} and its inverse ϵ^{AB}. The constraint functions of general relativity can be recast in terms of these variables as[12]

$$U(\underline{N}) \equiv \frac{\sqrt{2}}{Gi}\int_\Sigma \mathrm{Tr}\underline{N}\mathcal{D}_a\tilde{\sigma}^a \equiv \frac{\sqrt{2}}{Gi}\int_\Sigma \mathrm{Tr}\underline{N}(\partial_a\tilde{\sigma}^a + G[A_a, \tilde{\sigma}^a]) , \quad (13)$$

$$U(\vec{N}) \equiv \frac{\sqrt{2}}{i} \int_\Sigma \mathrm{Tr} N^a \tilde{\sigma}^b F_{ab} \qquad (14)$$

and

$$U(\underset{\sim}{N}) \equiv \frac{\sqrt{2}}{i} \int_\Sigma \mathrm{Tr} \underset{\sim}{N} \tilde{\sigma}^a \tilde{\sigma}^b F_{ab} , \qquad (15)$$

where $\mathcal{D}$ is the SU(2) gauge-covariant derivative operator defined by $A_{aA}{}^B$ and $F_{abA}{}^B$ is the curvature of $\mathcal{D}$. [We shall need the action of $\mathcal{D}$ only on internal indices. In (13) for example, we can use any (torsion-free) extension of $\mathcal{D}$ to tensor indices since the divergence of a vector density of weight one is independent of the choice of the derivative operator used in its evaluation.] The smearing fields $\underline{N}$, $\vec{N}$, and $\underset{\sim}{N}$ are, respectively, a Lie-algebra-valued function on Σ, a vector field on Σ, and a scalar density of weight minus one on Σ (Ref. 3). [These fields now play the role of the index α of Eq. (1).] For convenience in what follows, we will take the constraint function (13) to be defined with its spinor indices both down, i.e., the constraint function in unsmeared form is

$$U_{AB} = -\frac{\sqrt{2}}{Gi} \mathcal{D}_a \tilde{\sigma}^a{}_{AB} .$$

Finally, note that the constraints are all polynomial in the new variables. In particular, the inverse $\underset{\sim}{\sigma}_a$ of the soldering form $\tilde{\sigma}^a$ never features in the expressions of the constraints.[13]

The Poisson brackets between these constraints determine the first-order structure functions[12] $U(\, , \, | \,)$, where the two entries in the parentheses on the left of the vertical line are to be thought of as the covariant greek indices of Eq. (2), and the entry on the right of the line as the contravariant greek index. The only nonvanishing first-order structure functions, in smeared form, are

$$U(\underline{N},\underline{M} \mid \underline{\tilde{L}}) = -\int_\Sigma \mathrm{Tr} \underline{N M \tilde{L}} , \qquad (16)$$

$$U(\vec{N},\vec{M} \mid \underline{\tilde{L}}) = \frac{G}{2} \int_\Sigma \mathrm{Tr} N^a M^b F_{ab} \underline{\tilde{L}} , \qquad (17)$$

$$U(\vec{N},\vec{M} \mid \tilde{\mathbf{L}}) = \tfrac{1}{2} \int_\Sigma (\mathcal{L}_{\vec{N}} M^a) \tilde{L}_a , \qquad (18)$$

$$U(\vec{N},\underset{\sim}{M} \mid \underline{\tilde{L}}) = G \int_\Sigma \mathrm{Tr} \underset{\sim}{M} N^b \tilde{\sigma}^a F_{ab} \underline{\tilde{L}} , \qquad (19)$$

$$U(\vec{N},\underset{\sim}{M} \mid \tilde{\tilde{L}}) = \tfrac{1}{2} \int_\Sigma (\mathcal{L}_{\vec{N}} \underset{\sim}{M}) \tilde{\tilde{L}} , \qquad (20)$$

$$U(\underset{\sim}{N},\underset{\sim}{M} \mid \tilde{\mathbf{L}}) = \int_\Sigma (\underset{\sim}{N} \partial_a \underset{\sim}{M} - \underset{\sim}{M} \partial_a \underset{\sim}{N})(\mathrm{Tr} \tilde{\sigma}^a \tilde{\sigma}^b) \tilde{L}_b . \qquad (21)$$

Here $\underline{\tilde{L}}$ is a density of weight one with values in the SU(2) Lie algebra (representing a "greek index" dual to $\underline{N}$), $\tilde{\mathbf{L}}$ is a covector field of weight one (representing a greek index dual to $\vec{N}$), and $\tilde{\tilde{L}}$ is a density of weight two (representing a greek index dual to $\underset{\sim}{N}$).

Constraint (13) is reminiscent of the "Gauss law" constraint of Yang-Mills theory and plays an analogous role: the canonical transformation it generates corresponds to local SU(2) gauge rotations. This constraint is absent in the familiar $(q,\tilde{P})$ framework; it arose here because we have introduced "internal" or "spinorial" indices. Modulo (13), (14) is equivalent to the standard vector constraint in the $(q,\tilde{P})$ variables and (15), to the standard scalar constraint. One can therefore think of the canonical transformations generated by (13) and (14) as "kinematical" and those generated by (15) as giving us "time evolution." Note, however, that the structure functions are not constant on the phase space Γ. In particular, (17) is not constant; hence, the infinitesimal kinematical canonical transformations do not form a (closed) Lie algebra. Moreover, the structure functions give rise, via (3), to second-order structure functions (see Appendix A). This is to be contrasted with the situation with $(q,\tilde{P})$ variables. There the kinematical canonical transformations do form a group which is in fact the diffeomorphism group of Σ (or rather, the connected component of identity of the group of asymptotically identity diffeomorphisms) and the constraints are of rank 1. We shall see, however, that, in the $(\tilde{\sigma}^a, A_a)$ picture, the group structure of the kinematical canonical transformations can in fact be restored by a redefinition of constraint functions.

III. BRST CHARGES

Let us begin by examining the structure of the group of geometric transformations acting on fields $T^{a \cdots b}{}_{c \cdots d}{}^{A \cdots B}{}_{M \cdots N}$ on Σ with both tensor and internal indices. Denote by B the vector bundle over Σ where the fiber F is a two-complex-dimensional vector space equipped with a Hermitian conjugation operation and a symplectic two-form. Thus, elements of F carry a single internal index; e.g., λ^A belongs to F. Let us denote, as before, the Hermitian conjugation operation by $\dagger$ and the symplectic two-form by ϵ_{AB}. The structure group of $(F,\dagger,\epsilon_{AB})$ is SU(2). Consider the group G of all diffeomorphisms of B which preserve its bundle structure, the $\dagger$ operation and ϵ_{AB}. This G has a natural action on fields $T^{a \cdots b}{}_{c \cdots d}{}^{A \cdots B}{}_{M \cdots N}$ on Σ. To unravel the structure of G, note first that because G maps entire fibers to entire fibers, each element of G projects down unambiguously to give us a diffeomorphism on the base space Σ. The subgroup of G that projects to the identity diffeomorphism on Σ is the group $\mathrm{SU(2)}_{\mathrm{loc}}$ of local SU(2) transformations which leave each fiber individually invariant. It is easy to check that $\mathrm{SU(2)}_{\mathrm{loc}}$ is a normal subgroup. The quotient $G/\mathrm{SU(2)}_{\mathrm{loc}}$ is naturally isomorphic with the diffeomorphism group $\mathrm{Diff}(\Sigma)$. Thus, G is a semidirect product of $\mathrm{SU(2)}_{\mathrm{loc}}$ and $\mathrm{Diff}(\Sigma)$. It is natural to regard G as the kinematical gauge group of general relativity in the setting of new variables.

The canonical transformations generated by the Gauss-law constraint $U(\underline{N})$ of Eq. (13) provide us with the natural realization of $\mathrm{SU(2)}_{\mathrm{loc}}$. This suggests that it would be fruitful to arrange matters so that the vector constraint provides the natural realization of the diffeomorphism group. To this end, let us redefine the vector constraint by taking the following combination of (13) and (14) (Ref. 14):

$$\mathbf{U}(\vec{N}) \equiv \frac{\sqrt{2}}{i} \int_\Sigma \mathrm{Tr} N^a (\tilde{\sigma}^b F_{ab} - A_a \mathcal{D}_b \tilde{\sigma}^b)$$

$$= \frac{\sqrt{2}}{i} \int_\Sigma \mathrm{Tr} N^a (2\tilde{\sigma}^b \partial_{[a} A_{b]} - A_a \partial_b \tilde{\sigma}^b) . \qquad (14')$$

Then, we have

$$[\mathbf{U}(\vec{N}),\tilde{\sigma}^{a}]=N^{m}\partial_{m}\tilde{\sigma}^{a}+(\partial_{m}N^{m})\tilde{\sigma}^{a}-\tilde{\sigma}^{m}\partial_{m}N^{a}=\mathcal{L}_{\vec{N}}\tilde{\sigma}^{a} \tag{22a}$$

and

$$[\mathbf{U}(\vec{N}),A_{a}]=N^{m}\partial_{m}A_{a}+A_{m}\partial_{a}N^{m}=\mathcal{L}_{\vec{N}}A_{a}\ , \tag{22b}$$

where the Lie derivative treats internal indices as scalars [the term $(\partial_m N^m)\tilde{\sigma}^a$ in (22a) arises because $\tilde{\sigma}^a$ is a vector *density* of weight one]. Thus the infinitesimal canonical transformation generated by $\mathbf{U}(\vec{N})$ does provide the action on $\underset{\sim}{\Gamma}$ of the diffeomorphism group generated by $\vec{N}$. Using (22) it is straightforward to compute the Poisson brackets between (13) and (14′). One has

$$[U(\underline{N}),U(\underline{M})]=-U([\underline{N},\underline{M}])\ , \tag{16a}$$

$$[\mathbf{U}(\vec{N}),U(\underline{M})]=-U(\mathcal{L}_{\vec{N}}\underline{M})\ , \tag{23}$$

and

$$[\mathbf{U}(\vec{N}),\mathbf{U}(\vec{M})]=-\mathbf{U}([\vec{N},\vec{M}])\ . \tag{24}$$

Since the structure functions are all constants, (13) and (14′) do generate a Lie algebra on all of Γ. Equation (13) generates a Lie ideal of this algebra. Thus, (13), (14′), (16a), (23), and (24) provide us with the natural realization of the Lie algebra of G.

Let us now use (13), (14′), and (15) as our constraints. Then, the commutation relations, in addition to (16a), (23), and (24) are

$$[U(\underset{\sim}{N}),U(\underset{\sim}{M})]=0\ , \tag{25}$$

$$[\mathbf{U}(\vec{N}),U(\underset{\sim}{M})]=-U(\mathcal{L}_{\vec{N}}\underset{\sim}{M})\ , \tag{26}$$

$$[U(\underset{\sim}{N}),U(\underset{\sim}{M})]=-U(\vec{K})=-\mathbf{U}(\vec{K})-U(GK^{m}A_{m})\ , \tag{27}$$

where

$$K^{a}=2(\underset{\sim}{N}\partial_{m}\underset{\sim}{M}-\underset{\sim}{M}\partial_{m}\underset{\sim}{N})\,\mathrm{Tr}\tilde{\sigma}^{a}\tilde{\sigma}^{m}\ ,$$

so that the new nonvanishing first-order structure functions are given by

$$U(\underline{N},\underline{M}\mid\underline{\tilde{L}})=\tfrac{1}{2}\int_{\Sigma}\mathrm{Tr}(-2)\underline{NM\tilde{L}}\ , \tag{16b}$$

$$\mathbf{U}(\vec{N},\underline{M}\mid\underline{\tilde{L}})=\tfrac{1}{2}\int_{\Sigma}-\mathrm{Tr}(\mathcal{L}_{\vec{N}}\underline{M})\underline{\tilde{L}}\ , \tag{23a}$$

$$\mathbf{U}(\vec{N},\vec{M}\mid\tilde{\mathbf{L}})=\tfrac{1}{2}\int_{\Sigma}(\mathcal{L}_{\vec{N}}M^{a})\tilde{L}_{a}\ , \tag{24a}$$

$$\mathbf{U}(\vec{N},\underset{\sim}{M}\mid\tilde{\tilde{L}})=\tfrac{1}{2}\int_{\Sigma}(\mathcal{L}_{\vec{N}}\underset{\sim}{M})\tilde{\tilde{L}}\ , \tag{26a}$$

$$\mathbf{U}(\underset{\sim}{N},\underset{\sim}{M}\mid\tilde{\mathbf{L}})=\tfrac{1}{2}\int_{\Sigma}K^{a}\tilde{L}_{a}\ , \tag{27a}$$

and

$$\mathbf{U}(\underset{\sim}{N},\underset{\sim}{M}\mid\underline{\tilde{L}})=\tfrac{1}{2}\int_{\Sigma}(-G)\,\mathrm{Tr}K^{a}A_{a}\underline{\tilde{L}}\ , \tag{27b}$$

where, as before, $K^{a}=2(\underset{\sim}{N}\partial_{m}\underset{\sim}{M}-\underset{\sim}{M}\partial_{m}\underset{\sim}{N})\,\mathrm{Tr}\tilde{\sigma}^{a}\tilde{\sigma}^{m}$.

Although we again have six nonvanishing first-order structure functions, the detailed form of the new structure functions makes them easier to handle. The first four structure functions, (16), (23a), (24a), and (26a) are now constants on Γ and the only nonconstant ones, (27a) and (27b), both involve two lapse fields. This makes the computation of the second-order structure functions very simple. The only nontrivial triple Poisson brackets are $[[U(\underset{\sim}{N}),U(\underset{\sim}{M})],\mathbf{U}(\vec{L})]$, $[[U(\underset{\sim}{N}),U(\underset{\sim}{M})],U(\underset{\sim}{L})]$, and $[[U(\underset{\sim}{N}),U(\underset{\sim}{M})],U(\underline{L})]$. By explicitly computing these (and their cyclic permutations), it is straightforward to show that all the second-order structure functions can be made to vanish identically. Thus, the expression of the BRST charge Q is completely dictated by the (new) zeroth- and first-order structure functions.

To obtain Q, let us first extend the phase-space Γ to include ghosts $\eta_{A}{}^{B}$, η^{a}, and $\underset{\sim}{\eta}$ and their conjugate momenta $\tilde{P}_{A}{}^{B}$, $\tilde{P}_{a}$, and $\tilde{\tilde{P}}$, where[3] $\underset{\sim}{\eta}$, $\tilde{P}_{A}{}^{B}$, $\tilde{P}_{a}$, and $\tilde{\tilde{P}}$ are, respectively, densities of weight -1, $+1$, and $+2$. The ghosts and their momenta have vanishing Poisson brackets with $\tilde{\sigma}^{a}{}_{A}{}^{B}$ and $A_{aA}{}^{B}$; their mutual nonvanishing Poisson brackets are given by

$$[\eta^{AB}(x),\tilde{P}_{MN}(y)]=-\delta_{(M}{}^{A}\delta_{N)}{}^{B}\delta(x,y)\ , \tag{28.a}$$

$$[\eta^{a}(x),\tilde{P}_{b}(y)]=-\delta^{a}{}_{b}\delta(x,y)\ , \tag{28b}$$

$$[\underset{\sim}{\eta}(x),\tilde{\tilde{P}}(y)]=-\delta(x,y)\ . \tag{28c}$$

Using the definition of the BRST charge given in Sec. II A, we can now set

$$Q=\int_{\Sigma}\mathrm{Tr}\left[\frac{\sqrt{2}}{i}\left[\frac{1}{G}\underline{\eta}(\mathcal{D}_{a}\tilde{\sigma}^{a})+\eta^{a}(\tilde{\sigma}^{b}F_{ab}-A_{a}\mathcal{D}_{b}\tilde{\sigma}^{b})+\underset{\sim}{\eta}\tilde{\sigma}^{a}\tilde{\sigma}^{b}F_{ab}\right]+\underline{\eta\eta}\underline{\tilde{P}}+(\eta^{a}\partial_{a}\underline{\eta})\underline{\tilde{P}}\right.$$
$$\left.-(\eta^{b}\partial_{b}\eta^{a})\tilde{P}_{a}-(\eta^{a}\partial_{a}\underset{\sim}{\eta}+\underset{\sim}{\eta}\partial_{a}\eta^{a})\tilde{\tilde{P}}-2\underset{\sim}{\eta}(\partial_{a}\underset{\sim}{\eta})(\mathrm{Tr}\tilde{\sigma}^{a}\tilde{\sigma}^{b})(\tilde{P}_{b}-\mathrm{Tr}GA_{b}\underline{\tilde{P}})\right]\ . \tag{29}$$

How does this expression compare with that of the BRST charge in terms of $(q_{ab},\tilde{P}^{ab})$? In both cases the BRST charge contains only up to cubic ghost terms because the second-, and, hence, also higher-order structure functions vanish. The cubic ghost terms in (29) contain coupling to both $\tilde{\sigma}^{a}$ and A_{a} through the last term.[15] In the $(q_{ab},\ \tilde{P}^{ab})$ framework, the three ghost terms do not involve a coupling to $\tilde{P}^{ab}$. On the other hand, whereas in that case the coupling involves nonpolynomial functionals of q_{ab}, in (29) Q has a polynomial dependence on $\tilde{\sigma}^{a}$ and A_{a}. Finally, in both cases, the canonical transformations generated by the "kinematic constraints" provide us with a natural realization of the corresponding "kinematic gauge group" associated with Σ.

The Poisson-brackets relations between constraints are, as usual, succinctly expressed as $[Q,Q]=0$. To go over to quantum theory one has to replace the (bosonic) physical as well as the (fermionic) ghost fields by operators satisfying the appropriate canonical commutation and anticommutation relations, regularize the expression of the BRST charge operator, and find states which are annihilated by this operator.[13]

Remark. To simplify the constraint algebra, we added to the vector constraint $U(\vec{N})$ of Eq. (14), a multiple of the Gauss-law constraint (13). Now the only complicated Poisson brackets is the one between two scalar constraints [Eq. (27)]; since we have left the scalar constraint untouched, this bracket continues to equal the *old* vector constraint [weighted by $(2\underset{\sim}{M}\overleftrightarrow{\partial}_m\underset{\sim}{N})(\mathrm{Tr}\tilde{\sigma}^{m}\tilde{\sigma}^{a})$] rather than the new one. Can we not simplify matters further by redefining the scalar constraint as well? This is indeed possible to some extent. Set

$$\mathbf{U}(\underset{\sim}{N})\equiv\frac{\sqrt{2}}{i}\int_\Sigma \mathrm{Tr}\underset{\sim}{N}[\tilde{\sigma}^{a}\tilde{\sigma}^{b}F_{ab}+2\tilde{\sigma}^{a}A_a(\mathcal{D}_b\tilde{\sigma}^{b})] \ . \tag{15'}$$

Then it does turn out that

$$[\mathbf{U}(\underset{\sim}{N}),\mathbf{U}(\underset{\sim}{M})]=-\mathbf{U}(\vec{K}) \ ,$$

where, as before,

$$K^a=2(\underset{\sim}{N}\partial_m\underset{\sim}{M}-\underset{\sim}{M}\partial_m\underset{\sim}{N})(\mathrm{Tr}\tilde{\sigma}^{a}\tilde{\sigma}^{m}) \ . \tag{30}$$

However, since, unlike $U(\underset{\sim}{N})$, $\mathbf{U}(\underset{\sim}{N})$ fails to be gauge invariant, the Poisson brackets between (13) and (15′) are no longer zero. We have

$$[U(\underline{N}),\mathbf{U}(\underset{\sim}{M})]=-U(2\underset{\sim}{M}\tilde{\sigma}^{a}\partial_a\underline{N}) \ . \tag{31}$$

Consequently, the nonvanishing first-order structure functions are now given by (16), (23a), (24a), (26a), (27a), and

$$\mathbf{U}(\underline{N},\underset{\sim}{M}\mid\underline{\tilde{L}})=-\int_\Sigma \mathrm{Tr}(M\tilde{\sigma}^{a}\partial_a\underline{N})\underline{\tilde{L}} \tag{32}$$

and the new BRST charge is given by

$$Q'=\int_\Sigma \mathrm{Tr}\left[\frac{\sqrt{2}}{i}\left[\frac{1}{G}\underline{\eta}(\mathcal{D}_a\tilde{\sigma}^{a})+\eta^a(\tilde{\sigma}^{b}F_{ab}-A_a\mathcal{D}_b\tilde{\sigma}^{b})+\underset{\sim}{\eta}(\tilde{\sigma}^{a}\tilde{\sigma}^{b}F_{ab}+2\tilde{\sigma}^{a}A_a\mathcal{D}_b\tilde{\sigma}^{b})\right]+\underline{\eta\eta}\underline{\tilde{P}}+(\eta^a\partial_a\underline{\eta})\underline{\tilde{P}}\right.$$
$$\left.-(\eta^b\partial_b\eta^a)\tilde{P}_a-(\eta^a\partial_a\underset{\sim}{\eta}+\underset{\sim}{\eta}\partial_a\eta^a)\tilde{\tilde{P}}-2\underset{\sim}{\eta}\tilde{\sigma}^{a}(\partial_a\underline{\eta})\underline{\tilde{P}}-2\underset{\sim}{\eta}(\partial_a\underset{\sim}{\eta})(\mathrm{Tr}\tilde{\sigma}^{a}\tilde{\sigma}^{b})\tilde{P}_b\right] \ . \tag{33}$$

Q' has the advantage that the cubic ghost terms contain coupling *only* to $\tilde{\sigma}^{a}$. On the other hand, since the scalar constraint is no longer gauge invariant, we now have the ($\tilde{\sigma}^{a}$-dependent) cubic ghost term $\eta\tilde{\sigma}^{a}(\partial_a\eta)\underline{P}$ which complicates the relation between the kinematic gauge group and the dynamical canonical transformations generated by the scalar constraint. More precisely, if one works with the set $\{U(\underline{N}),\mathbf{U}(\vec{N}),U(\underset{\sim}{N})\}$, we have a (generalized) symmetric system: Infinitesimal kinematical gauge transformations, generated by (13) and (14′), form a Lie algebra g; the commutators of elements of g with the set h of infinitesimal canonical transformations generated by (15) *are again in* h and the commutator of any two elements of h is in g. This structure is lost if one works with the set $\{U(\underline{N}),\mathbf{U}(\vec{N}),\mathbf{U}(\underset{\sim}{N})\}$ instead. Preliminary investigations indicate that this drawback would complicate the calculation of quantum transition amplitudes substantially.[16]

After this work was completed, Ref. 17 was brought to our attention. This paper examines the constraint structure of Einstein gravity using the traditional tetrad variables. It is found that a redefinition of the vector constraint, analogous to (14′), also leads to simplifications in this case. We thank M. Henneaux for bringing this work to our attention.

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APPENDIX A

Here we complete the BRST analysis for the constraints (13), (14), and (15). We also exhibit canonical transformations which map the resulting (rank-2) BRST charge into the rank-1 charges (29) and (33).

It is a straightforward, albeit lengthy, affair to substitute (13)–(21) into the defining equation for the second-order structure functions (3). We find the following nonvanishing functions:

$$U(\underset{\sim}{L},\underset{\sim}{M},\vec{K}\mid\tilde{\mathbf{N}},\underline{\tilde{J}}) \tag{A1}$$

$$=\frac{\sqrt{2}i}{6}\mathrm{Tr}\int_\Sigma(\underset{\sim}{M}\partial_a\underset{\sim}{L}-\underset{\sim}{L}\partial_a\underset{\sim}{M})\tilde{N}_bK^{(a}\tilde{\sigma}^{b)}\underline{\tilde{J}} \ ,$$

$$U(\underset{\sim}{L},\vec{M},\vec{N}\mid\underline{\tilde{K}},\underline{\tilde{J}})=\frac{\sqrt{2}i}{6}\mathrm{Tr}\int_\Sigma \underset{\sim}{L}N^aM^bF_{ab}\underline{\tilde{K}}\underline{\tilde{J}} \ . \tag{A2}$$

Next we must examine the third-order structure functions. A combination of inspection and direct calculation reveals that they all vanish. Similarly, it is not hard to see that the fourth-order functions also vanish. As indicated in Sec. II, if the third- and fourth-order structure functions vanish, the theory is at most of rank 2. Given (A1) and (A2), we see that this set of constraints is of rank 2. The BRST charge takes the rather unwieldy form

$$Q''=\int_\Sigma \mathrm{Tr}\left[\frac{\sqrt{2}}{i}\left[\frac{1}{G}\underline{\eta}(\mathcal{D}_a\tilde{\sigma}^{\,a})+\eta^a\tilde{\sigma}^{\,b}F_{ab}+\underset{\sim}{\eta}\tilde{\sigma}^{\,a}\tilde{\sigma}^{\,b}F_{ab}\right]+\underline{\eta\eta}\,\underline{\tilde{P}}-(\eta^b\partial_b\eta^a)\tilde{P}_a\right.$$
$$-(\eta^a\partial_a\underset{\sim}{\eta}+\underset{\sim}{\eta}\partial_a\eta^a)\tilde{\tilde{P}}-2\underset{\sim}{\eta}(\partial_a\underset{\sim}{\eta})(\mathrm{Tr}\tilde{\sigma}^{\,a}\tilde{\sigma}^{\,b})\tilde{P}_b+2\eta^a\underset{\sim}{\eta}\underline{\tilde{P}}\tilde{\sigma}^{\,b}F_{ab}-\tfrac{1}{2}\eta^a\eta^b\underline{\tilde{P}}F_{ab}$$
$$\left.-2i\sqrt{2}\underset{\sim}{\eta}(\partial_a\underset{\sim}{\eta})\eta^{(a}\tilde{\sigma}^{\,b)}\tilde{P}_b\underline{\tilde{P}}-\frac{i\sqrt{2}}{2}\underset{\sim}{\eta}\eta^a\eta^bF_{ab}\underline{\tilde{P}\tilde{P}}\right]. \tag{A3}$$

This expression of the charge is significantly more complicated than the other two previously considered. The cubic ghost terms involve additional couplings to the gravitational variables and, in addition, we have five-ghost couplings which involve gravitational variables.

In Sec. III we found two alternative sets of constraints which were of rank 1, and we constructed the corresponding BRST charges. There is an interesting "trick"[10] which often provides a short cut when examining the effect of a constraint redefinition on the BRST charge; it comes from the following considerations. We observe that (8) effectively *defines* the BRST charge; the solutions of (8) being given by (4), (5), and (7). Now, any canonical change of variables (of $\hat{\Gamma}$) will preserve (8b) but, in general, will modify (8a). If we want to examine the effect on Q of a redefinition of the constraints, we simply perform a canonical transformation which implements the desired redefinition in (8a). The new BRST charge, which results from the change of variables, will be the same charge obtained from (4), (5), and (7) using the new constraints.

Let us now apply this reasoning to the case at hand. It is easily verified that the following transformation is canonical (on $\hat{\Gamma}$) and implements the desired constraint redefinition (14′):

$$\underline{\eta}\to\underline{\eta}-\eta^aA_a,\quad \tilde{P}_a\to\tilde{P}_a-\mathrm{Tr}(A_a\underline{\tilde{P}}),$$
$$\tilde{\sigma}^{\,a}\to\tilde{\sigma}^{\,a}+\frac{i}{\sqrt{2}}\eta^a\underline{\tilde{P}}, \tag{A4}$$

all other variables unchanged. Substituting (A4) into (A3) we find the various five-ghost terms cancel, leaving a rank-1 charge which is exactly (29). Similarly, in order to implement the modification of the scalar constraint in (15′), we make the additional change of variables in (29):

$$\underset{\sim}{\eta}\to\underset{\sim}{\eta}+\underline{\eta}[\tilde{\sigma}^{\,a},A_a],$$
$$\tilde{\tilde{P}}\to\tilde{\tilde{P}}+\mathrm{Tr}(\underline{\tilde{P}}[\tilde{\sigma}^{\,a},A_a]),$$
$$\tilde{\sigma}^{\,a}\to\tilde{\sigma}^{\,a}+\frac{i}{\sqrt{2}}\underset{\sim}{\eta}[\tilde{\sigma}^{\,a},\underline{\tilde{P}}], \tag{A5}$$
$$A_a\to A_a+\frac{i}{\sqrt{2}}\underset{\sim}{\eta}[A_a,\underline{\tilde{P}}].$$

It is easily seen that no five-ghost couplings can result from this transformation. Hence the constraints remain of rank 1. As expected, the resulting charge is identical to Q' of (33). In this latter case it is actually easier to construct Q' directly, i.e., by evaluating the various structure functions, than to substitute (A5) into (29).

APPENDIX B

For the convenience of readers who are more familiar with the notation used in particle physics rather than that of general relativity, we shall recast the main results of the paper using a fixed basis, $T^i{}_A{}^B$, $i=1,2,3$, in the Lie algebra of SU(2). We assume that T^i satisfy

$$T^iT^j=-\tfrac{1}{2}1\delta^{ij}+\frac{1}{\sqrt{2}}\epsilon^{ijk}T^k, \tag{B1}$$

thus T^k is $(-i/\sqrt{2})$ times the Pauli matrix. Given an orthonormal basis, $\tilde{e}^{\,a}{}_i$ of vector densities of weight one, $\tilde{e}^{\,a}{}_i\tilde{e}^{\,b}{}_j\delta^{ij}=(\det q)q^{ab}$ and $\tilde{e}^{\,a}{}_i\tilde{e}^{\,b}{}_jq_{ab}=(\det q)\delta_{ij}$, we can expand out the densitized soldering forms $\tilde{\sigma}^{\,a}$ as

$$\tilde{\sigma}^{\,a}=\tilde{e}^{\,a}{}_iT^i. \tag{B2}$$

The connection one-form can also be expanded as

$$A_a=A_a{}^iT_i, \tag{B3}$$

where $A_a{}^i$ are three (complex-valued) one-forms. (The indices i,j,k, being numerical, can be raised and lowered freely with δ^{ij} and δ_{ij}.)

Now the basic variables of general relativity are pairs[3] $(\tilde{e}^{\,a}{}_i,\ A_a{}^i)$, consisting of a basis of vector densities $\tilde{e}^{\,a}{}_i$ of weight one, and a triplet of one-forms $A_a{}^i$, satisfying the Poisson-brackets relations

$$[\tilde{e}^{\,a}{}_i(x),\tilde{e}^{\,b}{}_j(y)]=0=[A_a{}^i(x),A_b{}^j(y)],$$
$$[A_a{}^i(x),\tilde{e}^{\,b}{}_j(y)]=\frac{i}{\sqrt{2}}\delta_a{}^b\delta_j{}^i\delta(x,y). \tag{B4}$$

The constraint functions of the theory (see Sec. III) are[13]

$$U(\underline{N})\equiv\frac{i\sqrt{2}}{G}\int_\Sigma N^i\mathcal{D}_a\tilde{e}^{\,a}{}_i\equiv\frac{i\sqrt{2}}{G}\int_\Sigma N^i(\partial_a\tilde{e}^{\,a}{}_i+\sqrt{2}G\epsilon_{ijk}A_a{}^j\tilde{e}^{\,ak}), \tag{B5}$$

$$\mathbf{U}(\vec{N})\equiv i\sqrt{2}\int_\Sigma N^a(\tilde{e}^{\,b}{}_iF_{ab}{}^i-A_a{}^i\mathcal{D}_b\tilde{e}^{\,b}{}_i) \tag{B6}$$

and

$$U(\underset{\sim}{N})\equiv i\sqrt{2}\int_\Sigma \frac{1}{\sqrt{2}}\underset{\sim}{N}\epsilon^{ijk}\tilde{e}^{a}{}_{i}\tilde{e}^{b}{}_{j}F_{abk}\ , \tag{B7}$$

where $F_{ab}{}^{i}=2\partial_{[a}A_{b]}{}^{i}+\sqrt{2}G\epsilon^{ijk}A_{aj}A_{bk}$ and $\underline{N}$, $\vec{N}$, and $\underset{\sim}{N}$ are, respectively, a triplet N^i of functions, a vector field N^a, and a scalar density of weight minus one on Σ. By computing Poisson brackets between these constraint, or, zeroth-order structure, functions, one obtains the following first-order structure functions:[3]

$$U(\underline{N},\underline{M}\mid\underline{\tilde{L}})=\tfrac{1}{2}\int_\Sigma \sqrt{2}\epsilon^{ijk}N_iM_j\tilde{L}_k\ , \tag{B8}$$

$$\mathbf{U}(\vec{N},\underline{M}\mid\underline{\tilde{L}})=\tfrac{1}{2}\int_\Sigma (\mathcal{L}_{\vec{N}}M^i)\tilde{L}_i\ , \tag{B9}$$

$$\mathbf{U}(\vec{N},\vec{M}\mid\tilde{\mathbf{L}})=\tfrac{1}{2}\int_\Sigma (\mathcal{L}_{\vec{N}}M^a)\tilde{L}_a\ , \tag{B10}$$

$$\mathbf{U}(\vec{N},\underset{\sim}{M}\mid\tilde{\tilde{L}})=\tfrac{1}{2}\int_\Sigma (\mathcal{L}_{\vec{N}}\underset{\sim}{M})\tilde{\tilde{L}}\ , \tag{B11}$$

$$\mathbf{U}(\underset{\sim}{N},\underset{\sim}{M}\mid\tilde{\mathbf{L}})=\tfrac{1}{2}\int_\Sigma K^a\tilde{L}_a\ , \tag{B12}$$

$$\mathbf{U}(\underset{\sim}{N},\underset{\sim}{M}\mid\underline{\tilde{L}})=\tfrac{1}{2}\int_\Sigma GK^aA_a{}^{i}\tilde{L}_i\ , \tag{B13}$$

where we have set $K^a=2(\underset{\sim}{N}\partial_b\underset{\sim}{M}-\underset{\sim}{M}\partial_b\underset{\sim}{N})(\mathrm{Tr}\tilde{\sigma}^{a}\tilde{\sigma}^{b})$. The second-order structure functions are "pure gauge" and can be set equal to zero.

To define the BRST charge, we first enlarge the phase space to include ghosts, η^i, η^a, and $\underset{\sim}{\eta}$, and their conjugate momenta $\tilde{P}_i$, $\tilde{P}_a$, and $\tilde{\tilde{P}}$. These fermionic fields have vanishing Poisson brackets with $\tilde{e}^{a}{}_{i}$ and $A_a{}^{i}$. Their mutual (nonvanishing) Poisson brackets are given by

$$[\eta^i(x),\tilde{P}_j(y)]=-\delta^i{}_j\delta(x,y)\ ,$$
$$[\eta^a(x),\tilde{P}_b(y)]=-\delta^a{}_b\delta(x,y)\ , \tag{B14}$$
$$[\underset{\sim}{\eta}(x),\tilde{\tilde{P}}(y)]=-\delta(x,y)\ .$$

The BRST charge Q is the (Grassmann-)odd function on the extended phase space:

$$Q=\int_\Sigma\Bigg[i\sqrt{2}\left[\frac{1}{G}\eta^i\mathcal{D}_a\tilde{e}^{a}{}_{i}+\eta^a[\tilde{e}^{b}{}_{i}F_{ab}{}^{i}-A_a{}^{i}(\mathcal{D}_b\tilde{e}^{b}{}_{i})]+\frac{i}{\sqrt{2}}\epsilon^{ijk}\underset{\sim}{\eta}\tilde{e}^{a}{}_{i}\tilde{e}^{b}{}_{j}F_{abk}\right]-\frac{1}{\sqrt{2}}\epsilon^{ijk}\eta_i\eta_j\tilde{P}_k-(\eta^a\partial_a\eta^i)\tilde{P}_i$$
$$-(\eta^b\partial_b\eta^a)\tilde{P}_a-(\eta^a\partial_a\underset{\sim}{\eta}+\underset{\sim}{\eta}\partial_a\eta^a)\tilde{\tilde{P}}+2\underset{\sim}{\eta}(\partial_a\underset{\sim}{\eta})(\tilde{e}^{a}{}_{i}\tilde{e}^{bi})(\tilde{P}_b+GA_b{}^{j}\tilde{P}_j)\Bigg]\ . \tag{B15}$$

Because the second-order structure functions vanish, the dependence on ghosts is at worst cubic. All cubic ghost terms, except the last one in (B15) involving the densitized metric $\tilde{e}^{a}{}_{i}\tilde{e}^{bi}=(\det q)q^{ab}$, are independent of the gravitational variables. This is a reflection of the fact that the kinematic constraints, (B5) and (B6), generate a genuine Lie algebra. Finally, note that the charge depends polynomially on all basic canonically conjugate fields on the extended phase space.

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[1]K. Kuchař, in *Quantum Gravity 2,* edited by C. J. Isham and R. Penrose (Oxford University Press, London, 1980).

[2]A. Ashtekar, in *Constraint's Theory and Relativistic Dynamics,* edited by G. Longhi and L. Lusanna (World Scientific, Singapore, 1987); in Proceedings of the Eighth Workshop on Grand Unification, edited by K. C. Wali (World Scientific, Singapore, to be published).

[3]Throughout, our notation is the following: A tilde over a letter (e.g., $\tilde{P}^{ab}$) denotes a density of weight + 1 and a tilde under a letter (e.g., $\underset{\sim}{N}$) denotes a density of weight − 1.

[4]For details, see A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986), and especially, Phys. Rev. D **36**, 1587 (1987).

[5]M. Pilati, Phys. Rev. D **26**, 2645 (1982); M. Henneaux, M. Pilati, and C. Teitelboim, Phys. Lett. **110B**, 123 (1982).

[6]A. Ashtekar, in *Quantum Concepts in Space and Time,* edited by C. J. Isham and R. Penrose (Oxford University Press, London, 1986).

[7]T. Jacobson and L. Smolin, report, 1987 (unpublished).

[8]For example, in the bosonic string theory, there are no wave functions satisfying the full (Virasoro) operator constraints. However, there do exist states annihilated by the normal-ordered BRST charge operator.

[9]C. Becchi, A. Rouet, and R. Stora, Ann. Phys. (N.Y.) **98**, 287 (1976); E. S. Fradkin and G. A. Vilkovisky, Phys. Lett. **55B**, 224 (1975); I. A. Batalin and G. A. Vilkovisky, *ibid.* **69B**, 309 (1977); M. Henneaux, Phys. Rev. Lett. **55**, 769 (1985).

[10]M. Henneaux, Phys. Rep. **126**, 1 (1985).

[11]In the notation of Ref. 4 we are using $A_a={}^{+}A_a$.

[12]For simplicity we shall drop the symbols (n) in the nth-order structure functions. The index structure, in any case, is sufficient to determine n. Thus, for example, $U(\vec{N})$ is a structure function of order zero, and $U(\vec{N},\vec{M}\mid\tilde{\mathbf{L}})$ of order one.

[13]In addition to the constraints, fields $\tilde{\sigma}^{a}$ and A_a are subject to Hermiticity conditions: $\tilde{\sigma}^{a}$ and Π_a are both required to be Hermitian. These conditions must be satisfied if, in the classical theory, $(\tilde{\sigma}^{a},A_a)$ is to evolve to yield a *real* solution to Einstein's equations. In quantum theory, these conditions are to be incorporated by making suitable operators Hermitian. Thus, they will restrict the inner product to be imposed on physical states.

[14]Note that this combination of constraints actually yields a Hermitian function on Γ.

[15]The A_a coupling comes from the structure function (27b). It would be useful to have a geometric interpretation of this particular structure function, e.g., in terms of hypersurface deformations.

[16]See, for example, C. G. Torre, University of Utah report, 1987 (unpublished).

[17]M. Henneaux, Phys. Rev. D **27**, 986 (1982).