Algorithm to search for gravitational radiation from coalescing binaries

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This paper describes an algorithm to search for gravitational radiation from coalescing compact binary stars. As two stars rotate about each other they emit gravitational radiation, causing the stars to spiral together. One's best chance of detection is in the binary's last few moments, when the signal is strongest and the frequency is changing rapidly. This algorithm will track the change in frequency, no matter what the masses of the stars in the binary; thus, one can conduct a search for coalescing binaries without making assumptions about the stars' masses. This algorithm is capable of analyzing the data in real time.

One promising source of gravity waves is coalescing compact binary stars.[1–4] The event rate has been predicted to be approximately 3×10^{-4} per year per galaxy.[5] If a binary star coalescence were detected then the Hubble constant could be directly measured.[6] As two stars rotate about each other they emit energy in the form of gravitational radiation, causing the stars to spiral together. As the stars get closer together their angular velocity ω increases; hence, the radiation they emit changes frequency. The frequency of the radiation changes as

$$f_{\text{rad}}=f_0\left[1-\frac{t}{\tau}\right]^{-3/8},$$

where f_0=the radiation frequency at $t=0$, and τ=the time at which the two stars would collide, if no tidal disruption occurs. This calculation assumes nonrelativistic velocities and ignores the eccentricity of the orbit. Since emission of gravity waves tends to circularize the stars orbit it is reasonable to assume that by the time the stars are about to coalesce their orbits are nearly circular. It also treats the stars as point particles, a good approximation for compact stars such as black holes and neutron stars. The frequency of the stars orbit is a function of their masses and the time until coalescence:

$$\omega=\left[(\eta\tau)\left[1-\frac{t}{\tau}\right]\right]^{-3/8},$$

where

$$\eta=\frac{256}{5}\frac{m_1m_2}{(m_1+m_2)^{1/3}}$$

(in geometrized units $G=c=1$, $M_\odot=4.9255$ μsec). The gravitational radiation will have twice this frequency:

$$f_{\text{rad}}=\frac{2\omega}{2\pi}.$$

The amplitude of the radiation is also a function of the star masses and τ. The strain has maximum amplitude:[1,2]

$$h=\frac{4}{r}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\omega^{2/3}$$
$$=\frac{5}{64r}\eta^{3/4}(\tau-t)^{-1/4}.$$

(See Figs. 1 and 2.) The exact amplitude of the strain depends on the observers position relative to the binary. The expected signal at the detector would be

$$S(t)=\alpha h(t)\cos\left[\left[2\pi\int(f_{\text{rad}})dt\right]+\phi\right],$$

where α depends on the orientation of the detector relative to the binary and ϕ is the signal's phase. The larger the stars masses the greater h and the slower the frequency. The radiation will be elliptically polarized; since detectors tend to be sensitive to only one linear polarization this implies the strain measured at the detector will be at most $h/2$. The signal strength will almost certainly be less than this, since it is unlikely the detector will be optimally aligned with the source.

Although the strain is largest in the binary's last few moments, when the frequency is greatest, the Fourier transform of the strain does not increase with frequency. Because the frequency is changing more rapidly as the stars get closer to coalescence, the Fourier transform of the strain is larger at lower frequencies. To optimize the signal one should search through low frequencies and integrate the signal for as long as possible. If one searches for binaries long before coalescence then the frequency is nearly constant. Clearly the easiest way to search through many frequencies is to do a Fourier transform of the detector output. This will separate the signal from background noise at other frequencies. Unfortunately when the frequency of the gravitational radiation is nearly constant it is below the frequency at which Earth-bound detectors would probably be limited by gravity gradient noise.

One's best chance of detection is in the binary's last few moments. At this time the frequency is high enough to be separated from seismic noise. For two neutron stars, each with mass $1.4M_\odot$, the frequency is above 500 Hz for approximately 0.03 sec. The frequency remains approximately constant for only n cycles, where

$$n=[\tfrac{4}{3}f_{\min}(\tau-t_{\min})]^{1/2}$$

$$=\left[\frac{4}{3\eta}\right]^{1/2}\pi^{-4/3}f_{\min}{}^{-5/6}\ .$$

[$f_{\min}$ equals the minimum frequency of the detector and $(\tau-t_{\min})$ is the length of time from $f=f_{\min}$ until coalescence.] After n cycles the phase shift is greater than π. No longer can the signal be efficiently separated from background noise by doing a simple Fourier transform. Unless the change in frequency is tracked only a few cycles can be integrated over. For $m_1=m_2=1.4M_\odot$ and a detector which is sensitive down to 500 Hz less than five cycles can be integrated over. If one tracks the frequency then one is only limited by the time the signal is in the detector's frequency range. This means that with the same detector the signal could be integrated over 24 cycles. If the detector is sensitive down to $f=f_{\min}$, and the noise above this frequency is white, as expected in a laser interferometer detector, then the best signal-to-noise ratio which can be obtained through filtering is[7]

$$\frac{S}{N}=\frac{\left[\int[S(t)]^2dt\right]^{1/2}}{N}$$

$$=\frac{5\alpha}{64Nr}\eta^{3/4}[(\tau-t_{\min})^{1/2}-(\tau-t_{\text{final}})^{1/2}]^{1/2}\ ,$$

where $N=$the spectral noise density of the detector in strain/$\sqrt{\text{Hz}}$, and t_{final} is when the waveform ends. (t_{final} might be less than τ due to tidal disruption.[8]) A filter which gives this signal-to-noise ratio is called an optimal filter. This assumes that the signal's phase ϕ is known. If no assumption is made about the phase then the best signal-to-noise ratio possible would be

$$\frac{S}{N}=\frac{\left[\int[S(t)]^2dt\right]^{1/2}}{\sqrt{2}N}\ .$$

Laser interferometer gravity wave detectors have a wide bandwidth, and thus are well suited to searches for coalescing binaries. For the past decade detectors employing laser interferometry have been under development at Caltech, MIT, Glasgow University, and the

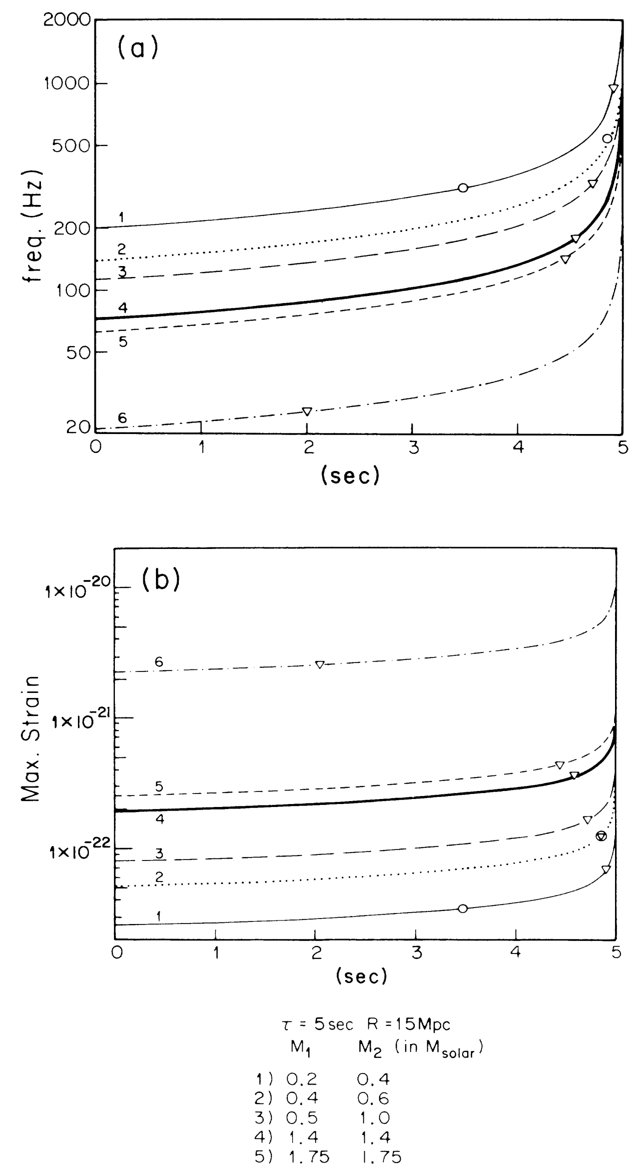

FIG. 1. The frequency and amplitude of the gravitational radiation emitted from coalescing binaries. The circles represent when tidal disruption is expected for a binary made of neutron stars (Ref. 8). The triangles represent when the velocity of the stars is 0.1c. If no tidal disruption occurs then the stars would coalesce at $t=\tau=5$ sec.

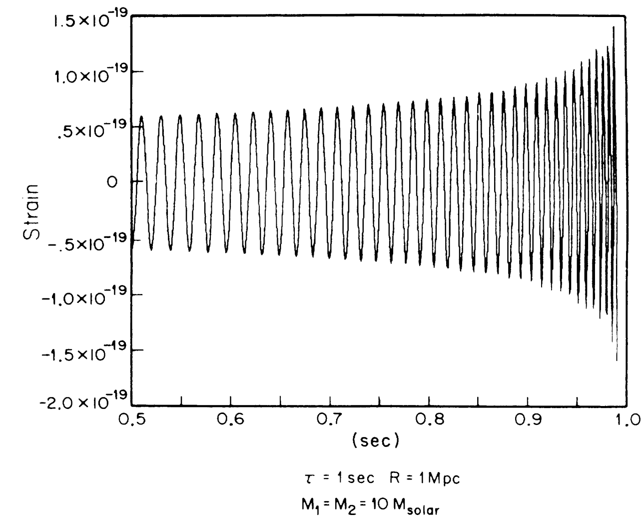

FIG. 2. The expected waveform from a binary made of two $10M_\odot$ black holes.

Max-Planck-Institut, Garching (near Munich). These detectors currently have a frequency range from approximately 500 Hz to 5 kHz, and ultimately may be sensitive to frequencies as low as 50 Hz. They have strain sensitivities down to $10^{-19}/\sqrt{\text{Hz}}$. Larger detectors are being proposed which could have a strain sensitivity better than $10^{-23}/\sqrt{\text{Hz}}$ (Ref. 9).

A major difficulty in searching for coalescing binaries is that η and τ are unknown (recall that η contains information about the star masses and that τ is when the coalescence occurs). In order to track the changing frequency both η and τ must be fixed, so one must search through all the possible values of η and τ. If one assumes a value for τ, then ignorance of η is the same as ignorance of the initial frequency f_0 $[f_0=(\eta\tau)^{-3/8}/\pi]$, so one must search through many frequencies. If the frequency were constant this would be simple, one would perform a fast Fourier transform (FFT):

$$F(S,f)=\int S(t)e^{-2\pi i f t}dt \ .$$

However, the frequency is not constant, so a normal Fourier transform will not work. The way to get around this is by changing variables. Instead of working with real frequency and real time, characterize the signal with initial frequency f_0 and a timelike variable $\mathcal{X}=-\frac{8}{5}\tau(1-t/\tau)^{5/8}$ then the signal will appear periodic when mapped versus $\mathcal{X}$:

$$S(\mathcal{X})=\alpha h(\mathcal{X})\cos(2\pi f_0\mathcal{X}+\phi) \ .$$

This timelike variable $\mathcal{X}$ works for all binaries, regardless of the stars mass. Resampling the data in even steps of $\mathcal{X}$ allows one to perform an FFT:

$$F(S,f_0)=\int S(\mathcal{X})e^{-2\pi i f_0\mathcal{X}}d\mathcal{X} \ .$$

This transform will track the gravity-wave frequency, and separate it from background noise at other frequencies. (I have neglected the Doppler shift due to Earth's motion, but over an integration period of 100 sec it causes a change in frequency of approximately 0.000 001%, which is negligible.) This filter assumes that one knows the time of coalescence, so one must search through τ in discrete steps. To use this filter fix τ, then resample the data in even steps of $\mathcal{X}$. One can then perform an FFT, which will search through all values of f_0, and hence η, simultaneously. Using an FFT cuts down on computation time, but at the cost of integrating all the signals over the same time period. Since different binaries will emit signals in the detector's frequency range for different time periods it is impossible to optimize the integration period for all binaries simultaneously (see Table I). How critical this problem depends on the frequency range of the detector used.

Just as the optimal filter for a sine wave of unknown frequency is a Fourier transform, an optimal filter for a coalescing binary is a Fourier transform using the variables f_0 and $\mathcal{X}$. Figure 3 shows the output of this sort of filter when applied to a data set consisting of computer-generated white noise with Gaussian distribution (to simulate shot noise in a detector) plus a signal. The signal strength is approximately equal to 1 standard deviation of the noise in the data set (it varies from 1.2σ to 3σ), the filter integrated from $t=0.5$ sec (when the frequency is 173 Hz) to $t=0.99$ sec, with $\tau=1.0$ sec. The sampling rate was 10 kHz. The peak can clearly be seen above the background noise, its relative height is approximately

$$\frac{\left[\int_{0.5\text{ sec}}^{0.99\text{ sec}}[S(t)]^2dt\right]^{1/2}}{\sqrt{2N}}=30$$

$(N=1/\sqrt{5000\text{ Hz}}, S(t)=(\tau-t)^{-1/4}\cos[2\pi\int(f_{\text{rad}})dt])$. This filter's output gives astrophysically relevant information immediately. When a signal is detected f_0 and τ

TABLE I. The length of time a signal will have $f>f_{\min}$ if no tidal disruption occurs.

M_1	M_2	Seconds from $f=f_{\min}$ until coalescence		
(in $M_\odot$)		$f_{\min}=500$ Hz	$f_{\min}=200$ Hz	$f_{\min}=50$ Hz
0.2	0.4	0.43	5.0	200
0.4	0.6	0.17	2.0	79
1	1	0.05	0.60	24
1.4	1.4	0.03	0.34	14
1.75	1.75	0.02	0.23	9.4
10	10	0.001	0.013	0.52

$f_{\min}$ = the minimum frequency of the detector

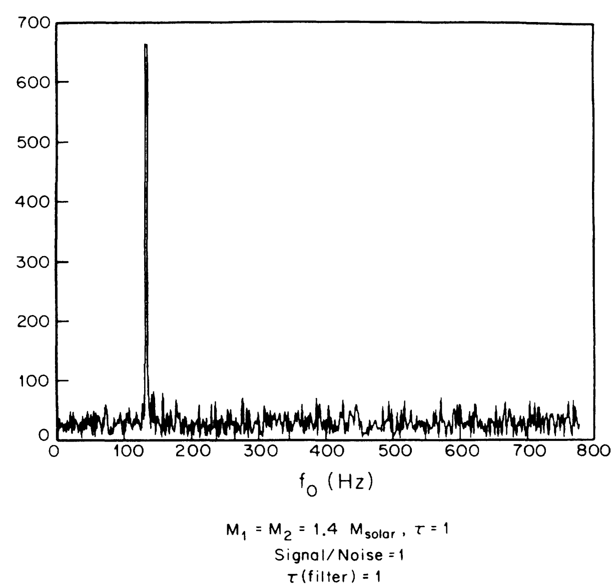

FIG. 3. Output of the filter when there is no error in τ. The input was computer-generated white noise with a Gaussian distribution plus a signal equal to that predicted for a binary made of two $1.4M_\odot$ neutron stars. The signal amplitude varied from 1.2σ to 3σ (σ is the standard deviation of the noise). The filter analyzed from $(\tau-t)=0.5$ sec until $(\tau-t)=0.01$ sec. The peak can be seen at $f_0=133$ Hz which implies that $\eta=1.0^{-7}$ sec$^{5/3}$, as expected for a binary made of two $1.4M_\odot$ stars.

are known. From these η can be immediately calculated, giving information about the mass of the stars in the binary. The peak in Fig. 3 is at $f_0=133$ Hz which implies that $\eta=1.0\times10^{-7}$ sec$^{5/3}$ as expected for a binary made of $1.4M_\odot$ stars. (It is important to note that there is nothing special about $m_1=m_2=1.4M_\odot$, this filter will work for any mass combination; for example, if $m_1=m_2=10M_\odot$, a peak would appear at 39 Hz.) The filter's performance will deteriorate if there is an error in τ. If there is a 0.1-sec error in τ then the peak will only be 40% as high.

The speed of this algorithm was checked on a Masscomp 500 computer. A detector frequency range of 200 to 2000 Hz was assumed, and the filter was optimized for two $1.4M_\odot$ stars. The FFT was performed from $(\tau-t)=0.34$ sec (at which time a binary with two $1.4M_\odot$ would have $f=200$ Hz) until $(\tau-t)=0.01$ sec. This was repeated every 0.1 sec. The data were sampled at 10 kHz. The algorithm analyzed 10 sec of data in 8.2 sec (5.3 sec were spent reading in the data and 2.9 sec were spent performing the calculation). The Masscomp 500 computer has an array processor which is capable of performing a 1024 element FFT in 4.5 msec.

Consider the signal from two 1.4 solar mass neutron stars. Assume one has a detector with spectral noise density $N=10^{-20}/\sqrt{\mathrm{Hz}}$, which is sensitive down to 200 Hz. Also assume the strain at the detector is $0.25\times$the maximum strain (due to polarization and orientation considerations this is a reasonable estimate of the strain which could produce a signal at the detector). The range of the detector is 57 kpc. If the spectral noise density were $N=10^{-23}/\sqrt{\mathrm{Hz}}$ then the range would be 57 Mpc, given an expected event rate of approximately 3 coalescences per year.[5]

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