Real-time finite-temperature evolution equation for the Higgs field in an expanding universe

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We display the *renormalized* one-loop evolution equation for the expectation value $\phi_c \equiv \langle \phi \rangle_T$ of the field operator ϕ in a $\lambda\phi^4$ theory in spatially flat Robertson-Walker space-times, subject to the initial condition that the system was in thermal equilibrium at a temperature T at some initial time.

In this note we discuss the problem of renormalization of the one-loop evolution equation for the Higgs field $\phi_c \equiv \langle \phi \rangle$ in a spatially flat Robertson-Walker (RW) universe. This equation determines the dynamics of the order parameter ϕ_c during a phase transition in the early Universe; in particular, it is needed in models of new inflation.[1-3] We extend the work of Semenoff and Weiss,[4] who represented the formal *unrenormalized* one-loop evolution equation derived under the assumption that at some early time t_0 [earlier than the grand-unified-theory (GUT) transition] the quantum part of the system is in thermal equilibrium at a temperature $T \equiv \beta^{-1} > T_{\rm crit}^{\rm GUT}$ and the further evolution is determined by the quantum equations of motion. In Ref. 4 the problem of renormalization was left open for the case of *nonadiabatically* varying background fields which will be treated in this paper.

For definiteness we consider a $\lambda\phi^4$ theory in a spatially flat RW space-time, given by the action

$$S = \int d^4x\, a^3(t) \left[\tfrac{1}{2}\dot{\phi}^2 - \frac{1}{2a^2}(\nabla\phi)^2 - V(\phi) \right] , \tag{1}$$

$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\xi}{2}R\phi^2 + \frac{\lambda}{4!}\phi^4 , \tag{2}$$

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2 , \tag{3}$$

$$R = 6(\dot{H} + 2H^2),\ H = \frac{\dot{a}}{a} , \tag{4}$$

where overdots denote the derivative with respect to t. Here ϕ, m, ξ, and λ are bare quantities. Using the operator field equations of motion of ϕ, making the background split

$$\phi = \phi_c + \varphi , \tag{5}$$

$$\phi_c \equiv \langle \phi \rangle \Longrightarrow \langle \varphi \rangle = 0 , \tag{6}$$

where $\langle \cdots \rangle$ denotes building an expectation value, and assuming that ϕ_c is a function of time only,[5] we get the one-loop-effective evolution equation[6]

$$\ddot{\phi}_c + 3H\dot{\phi}_c + (m_r^2 + \delta m^2)\phi_c + (\xi_r + \delta\xi)R\phi_c + \frac{\lambda_r + \delta\lambda}{3!}\phi_c^3 + \frac{\lambda_r}{2}\langle \varphi^2 \rangle \phi_c = 0 , \tag{7}$$

where the operator φ of the quantum fluctuation satisfies

$$\left[\frac{\partial^2}{\partial t^2} + 3H(t)\frac{\partial}{\partial t} - a^{-2}(t)\nabla_{\mathbf{x}}^2 + m_r^2 + \xi_r R(t) + \frac{\lambda_r}{2}\phi_c^2(t) \right] \varphi(\mathbf{x},t) = 0 . \tag{8}$$

In (7) we have introduced renormalized parameters and counterterms of one-loop order by the replacements $m^2 = m_r^2 + \delta m^2, \ldots$ in order to cancel the divergences in the last term in (7), the one-loop contribution. As usual the counterterms have to be fixed by renormalization conditions.

Introducing the conformal time τ

$$\tau \equiv \int^t \frac{dt'}{a(t')} \tag{9}$$

and the conformally transformed field

$$\overline{\varphi} \equiv a\varphi \tag{10}$$

we obtain

$$\langle \varphi^2 \rangle = \frac{1}{a^2}\langle \overline{\varphi}^2 \rangle \tag{11}$$

and

$$\left[\frac{\partial^2}{\partial \tau^2} - \nabla_{\mathbf{x}}^2 + M^2(\tau) \right] \overline{\varphi}(\mathbf{x},\tau) = 0 . \tag{12}$$

$M^2(\tau)$ denotes the time-dependent effective mass squared of $\overline{\varphi}$:

$$M^2(\tau) \equiv a^2(\tau) \left[m_r^2 + (\xi_r - \tfrac{1}{6})R(\tau) + \frac{\lambda_r}{2}\phi_c^2(\tau) \right] . \tag{13}$$

In this paper we will assume $M^2(\tau) > 0$. The Hamiltonian corresponding to (12) is given by

$$H_{\overline{\varphi}}(\tau) = \int d^3x \left[\tfrac{1}{2}\overline{\varphi}'^2 + \tfrac{1}{2}(\nabla\overline{\varphi})^2 + \frac{M^2(\tau)}{2}\overline{\varphi}^2 \right] \tag{14}$$

with a prime denoting differentiation with respect to τ.

Let us suppose that the system is prepared at some initial time $t_0(\tau_0)$ in thermal equilibrium at a temperature $T \equiv \beta^{-1}$ with respect to $H_{\overline{\varphi}}(\tau_0)$. The density matrix at τ_0 is thus chosen to be

$$\rho(\tau_0) = \frac{\exp[-\beta H_{\overline{\varphi}}(\tau_0)]}{\mathrm{Tr}\{\exp[-\beta H_{\overline{\varphi}}(\tau_0)]\}} . \tag{15}$$

Since β and τ_0 are both left as parameters, the choice of initial conditions in our work is quite general (for further remarks regarding the choice of the density matrix see Ref. 4). The states of the system will evolve via the Hamiltonian $H_{\bar{\varphi}}(\tau)$, such that if we measure the expectation value of any operator Q in the evolved system we find

$$\langle Q\rangle_T(\tau)=\mathrm{Tr}[\rho(\tau_0)U(\tau,\tau_0)QU^\dagger(\tau,\tau_0)] \tag{16}$$

with

$$U(\tau,\tau_0)=P\left[\exp\left[i\int_{\tau_0}^{\tau}d\tau' H_{\bar{\varphi}}(\tau')\right]\right] . \tag{17}$$

Using path-integral methods Semenoff and Weiss[4] showed how to calculate $\langle\bar{\varphi}^2\rangle_T(\tau)$: it is given by the coincidence limit of

$$\bar{G}(\tau,\tau')=\int\frac{d^3k}{(2\pi)^3}\bar{G}_k(\tau,\tau') \tag{18}$$

with

$$\bar{G}_k(\tau,\tau')=\sum_{i,j=1}^{2}\gamma_{ij}^{(k)}\psi_i^{(k)}(\tau)\psi_j^{(k)}(\tau') . \tag{19}$$

$\psi_i^{(k)}$ are *any* two functions which are linearly independent solutions to the homogeneous equation

$$\left[\frac{d^2}{d\tau^2}+\Omega_k{}^2(\tau)\right]\psi_i^{(k)}(\tau)=0,\quad i=1,2 , \tag{20}$$

with

$$\Omega_k{}^2(\tau)\equiv k^2+M^2(\tau) , \tag{21}$$

and which satisfy the Wronskian condition

$$\psi_1^{(k)}\psi_2^{(k)\prime}-\psi_1^{(k)\prime}\psi_2^{(k)}=i . \tag{22}$$

The matrix $\gamma^{(k)}$ is given by

$$\gamma^{(k)}=\frac{1}{2\Omega_k(\tau_0)}\begin{bmatrix} -Q_2^{(k)}(\tau_0)\coth\left[\beta\frac{\Omega_k(\tau_0)}{2}\right] & Q^{(k)}(\tau_0)\coth\left[\beta\frac{\Omega_k(\tau_0)}{2}\right]-\Omega_k(\tau_0) \\ Q^k(\tau_0)\coth\left[\beta\frac{\Omega_k(\tau_0)}{2}\right]+\Omega_k(\tau_0) & -Q_1^{(k)}(\tau_0)\coth\left[\beta\frac{\Omega_k(\tau_0)}{2}\right]\end{bmatrix} \tag{23}$$

with

$$Q_i^{(k)}=\psi_i^{(k)\prime 2}+\Omega_k{}^2\psi_i^{(k)2} , \tag{24a}$$

$$Q^{(k)}=\psi_1^{(k)\prime}\psi_2^{(k)\prime}+\Omega_k{}^2\psi_1^{(k)}\psi_2^{(k)} . \tag{24b}$$

For the problem of renormalization it turns out to be convenient to convert (20) into a set of two linear first-order differential equations. We introduce two complex functions $\alpha_k(\tau)$ and $\beta_k(\tau)$ (not to be confused with $\beta\equiv T^{-1}$) via the ansatz[6,7]

$$\psi_1^{(k)}(\tau)=\frac{1}{[2\Omega_k(\tau)]^{1/2}}[\alpha_k(\tau)e_-(\tau)+\beta_k(\tau)e_+(\tau)] , \tag{25}$$

$$\psi_2^{(k)}(\tau)=[\psi_1^{(k)}(\tau)]^* , \tag{26}$$

with the additional condition

$$\psi_1^{(k)\prime}=-i\left[\frac{\Omega_k}{2}\right]^{1/2}(\alpha_k e_- -\beta_k e_+) . \tag{27}$$

$e_\pm(\tau)$ are given by

$$e_\pm(\tau)\equiv\exp\left[\pm i\int_{\tau_0}^{\tau}d\tau'\Omega_k(\tau')\right] . \tag{28}$$

We choose $\alpha_k(\tau_0)=1$ and $\beta_k(\tau_0)=0$. Putting all together we obtain

$$\langle\varphi^2\rangle_T(\tau)=\frac{1}{a^2(\tau)}\langle\bar{\varphi}^2\rangle_T(\tau)=\frac{1}{a^2(\tau)}\bar{G}(\tau,\tau)=\frac{1}{4\pi^2}\frac{1}{a^2(\tau)}\int_0^\infty dk\, k^2\Omega_k{}^{-1}(\tau)\coth\left[\beta\frac{\Omega_k(\tau_0)}{2}\right][1+2s_k(\tau)+2\,\mathrm{Re}z_k(\tau)] , \tag{29}$$

with

$$s_k\equiv|\beta_k|^2 , \tag{30a}$$

$$z_k\equiv\alpha_k\beta_k^*e_-{}^2 . \tag{30b}$$

s_k and z_k satisfy, by virtue of (20), (22), (25), and (27),

$$s_k'=\frac{\Omega_k'}{\Omega_k}\mathrm{Re}z_k , \tag{31a}$$

$$z_k'=\frac{\Omega_k'}{\Omega_k}(s_k+\tfrac{1}{2})-2i\Omega_k z_k , \tag{31b}$$

with the initial conditions $s_k(\tau_0)=z_k(\tau_0)=0$. Equation (29) can be written as

$$\langle\varphi^2\rangle_T(\tau)=\langle\varphi^2\rangle_0(\tau)+\Delta\langle\varphi^2\rangle_T(\tau) \tag{32}$$

with

$$\langle\varphi^2\rangle_0(\tau)=\frac{1}{4\pi^2}\frac{1}{a^2(\tau)}\int_0^\infty dk\, k^2\Omega_k{}^{-1}(\tau)[1+2s_k(\tau)+2\,\mathrm{Re}z_k(\tau)] , \tag{33}$$

$$\Delta\langle\varphi^2\rangle_T(\tau)=\frac{1}{2\pi^2}\frac{1}{a^2(\tau)}\int_0^\infty dk\,k^2\Omega_k^{-1}(\tau)(e^{\beta\Omega_k(\tau_0)}-1)^{-1}[1+2s_k(\tau)+2\,\mathrm{Re}z_k(\tau)]\ . \tag{34}$$

Note that for $T\to 0$ ($\beta\to\infty$) (34) vanishes and only (33) survives, which coincides with the expression derived in Ref. 6 for the vacuum case.

The asymptotic behavior for large k of the integrands in (33) and (34), in particular of s_k and $\mathrm{Re}z_k$, can be obtained by an adiabatic expansion of the solutions of (31a) and (31b). This is valid for

$$\left|\frac{\Omega_k'}{\Omega_k^2}\right|\ll 1\ , \tag{35}$$

i.e., for large masses, large momenta, or slowly varying background fields [see the definition of Ω_k (21)]. In this region we have s_k, $|z_k|\ll 1$, and one finds[6,7]

$$s_k=s_k^{(2)}+s_k^{(4)}+\cdots\ , \tag{36a}$$

$$\mathrm{Re}z_k=\mathrm{Re}z_k^{(2)}+\mathrm{Re}z_k^{(4)}+\cdots\ , \tag{36b}$$

where superscripts inside the parentheses indicate the adiabatic order[8]

$$s_k^{(2)}=\frac{1}{16}\frac{\Omega_k'^{\,2}}{\Omega_k^4}\ , \tag{37a}$$

$$\mathrm{Re}z_k^{(2)}=\frac{1}{8}\frac{\Omega_k''}{\Omega_k^3}-\frac{1}{4}\frac{\Omega_k'^{\,2}}{\Omega_k^4}\ . \tag{37b}$$

Based on the fact that the adiabatic solution is an asymptotic solution for large k we conclude that $\Delta\langle\varphi^2\rangle_T$ is finite and that the only UV-divergent term in $\langle\varphi^2\rangle_0$ is that part which remains if all background fields are set to be constant ($s_k\equiv z_k\equiv 0$), i.e., the part

$$\frac{1}{4\pi^2}\frac{1}{a^2}\int_0^\infty dk\,k^2\Omega_k^{-1}\ . \tag{38}$$

In coincidence with Refs. 9–12 we denote that part of the evolution equation which remains if all background fields are set to be constant the *derivative with respect to* ϕ_c *of the one-loop finite-temperature effective potential* $\partial V_{\mathrm{eff}}^T/\partial\phi_c$.

In order to regularize the divergent zero-temperature contribution (38) we introduce the physical wave number $k_p\equiv k/a$ and use a momentum cutoff in physical momenta Λ_p. In this manner we obtain for the zero-temperature contribution to the derivative of the effective potential (details of the following calculations can be found in Ref. 6)

$$\frac{\partial V_{\mathrm{eff}}^0}{\partial\phi_c}=(m_r^2+\delta m^2)\phi_c+(\xi_r+\delta\xi)R\phi_c+\frac{\lambda_r+\delta\lambda}{3!}\phi_c^3+\frac{\lambda_r}{16\pi^2}\Lambda_p^2\phi_c+\frac{\lambda_r}{32\pi^2}\left[m_r^2+(\xi_r-\tfrac{1}{6})R+\frac{\lambda_r}{2}\phi_c^2\right]\phi_c$$
$$+\frac{\lambda_r}{32\pi^2}\left[m_r^2+(\xi_r-\tfrac{1}{6})R+\frac{\lambda_c}{2}\phi_c^2\right]\phi_c\ln\left[\frac{m_r^2+(\xi_r-\frac{1}{6})R+\frac{\lambda_r}{2}\phi_c^2}{4\Lambda_p^2}\right]\ . \tag{39}$$

The counterterms δm^2, $\delta\xi$, and $\delta\lambda$ have to be fixed by renormalization conditions. We introduce the following renormalization conditions:[9,13]

$$m_r^2=\left.\frac{\partial^2V_{\mathrm{eff}}^0}{\partial\phi_c^2}\right|_{\phi_c=0,R=0}\ ,$$
$$\xi_r=\left.\frac{\partial^3V_{\mathrm{eff}}^0}{\partial R\,\partial\phi_c^2}\right|_{\phi_c=0,R=\mu_2^2}\ , \tag{40}$$
$$\lambda_r=\left.\frac{\partial^4V_{\mathrm{eff}}^0}{\partial\phi_c^4}\right|_{\phi_c=\mu_1,R=0}\ .$$

The above choice of renormalization conditions means that we have chosen to define the parameters of the theory in the limit of *constant* background fields and of zero temperature. The coupling constants λ_r and ξ_r are defined at the energy scales corresponding to the values $\phi_c=\mu_1$ and $R=\mu_2^2$, which may in general be different as the energy scales at which they are measured need not be the same. The renormalization points μ_1 and μ_2 are completely arbitrary; different choices will lead to different definitions of the coupling constants λ_r and ξ_r (Ref. 9). How the coupling constants behave under the change of scales is determined by their associated renormalization-group equations, which are discussed in the case under consideration in Ref. 13. Using (39) and the renormalization conditions (40) the counterterms are calculated as

$$\delta m^2=-\frac{\lambda_r}{16\pi^2}\left[\Lambda_p^2+\frac{m_r^2}{2}+\frac{m_r^2}{2}\ln\left[\frac{m_r^2}{4\Lambda_p^2}\right]\right]\ , \tag{41a}$$

$$\delta\xi=-\frac{\lambda_r}{16\pi^2}(\xi_r-\tfrac{1}{6})\left[1+\tfrac{1}{2}\ln\left[\frac{m_r^2+(\xi_r-\frac{1}{6})\mu_2^2}{4\Lambda_p^2}\right]\right]\ , \tag{41b}$$

$$\delta\lambda=-\frac{\lambda_r^2}{16\pi^2}\left[3+\tfrac{3}{2}\ln\left[\frac{m_r^2+\frac{\lambda_r}{2}\mu_1^2}{4\Lambda_p^2}\right]+3\frac{\lambda_r\mu_1^2}{m_r^2+\frac{\lambda_r}{2}\mu_1^2}-\frac{\frac{\lambda_r^2}{2}\mu_1^4}{\left[m_r^2+\frac{\lambda_r}{2}\mu_1^2\right]^2}\right]\ . \tag{41c}$$

For example, in the case $m_r=0$, which is usually called the Coleman-Weinberg (CW) case,[9] one obtains, for the derivative of the one-loop zero-temperature effective potential,

$$\frac{\partial V_{\rm eff}^0}{\partial \phi_c}=\xi_r R\phi_c+\frac{\lambda_r}{3!}\phi_c{}^3+\frac{\lambda_r{}^2}{64\pi^2}\phi_c{}^3\left[\ln\left[\frac{(\xi_r-\frac{1}{6})R+\frac{\lambda_r}{2}\phi_c{}^2}{\frac{\lambda_r}{2}\mu_1{}^2}\right]-\frac{11}{3}\right]+\frac{\lambda_r}{32\pi^2}(\xi_r-\tfrac{1}{6})R\phi_c\left[\ln\left[\frac{(\xi_r-\frac{1}{6})R+\frac{\lambda_r}{2}\phi_c{}^2}{(\xi_r-\frac{1}{6})\mu_2{}^2}\right]-1\right]. \tag{42}$$

The finite-temperature correction to the derivative of the effective potential reads [(34) has to be multiplied by $(\lambda_r/2)\phi_c$, and s_k and z_k have to be set equal to zero]

$$\frac{\partial \Delta V_{\rm eff}^T}{\partial \phi_c}[\phi_c(\tau)] =\frac{\lambda_r}{4\pi^2}\frac{\phi_c}{a^2(\tau)}\int_0^\infty dk\, k^2\Omega_k{}^{-1}(\tau)(e^{\beta\Omega_k(\tau_0)}-1)^{-1}. \tag{43}$$

The momentum integral in (43) can be approximately evaluated in the initially high- ($\beta M \ll 1$) or low- ($\beta M \gg 1$) temperature regimes using the results derived in Ref. 11. In the high-temperature case and for $|M^2(\tau)-M^2(\tau_0)| \ll M^2(\tau_0)$ we obtain, for example,

$$\frac{\partial \Delta V_{\rm eff}^T}{\partial \phi_c}[\phi_c(\tau)]\simeq\frac{\lambda_r}{24}\phi_c(\tau)\frac{T^2}{a^2(\tau)}, \tag{44}$$

which coincides in Minkowski space ($a\equiv 1$) with the well-known result derived in Ref. 10. In our RW space-time we explicitly see that this term will be red-shifted away.

Putting all things together we obtain the *renormalized* one-loop effective evolution equation

$$\ddot\phi_c+3H\dot\phi_c+\frac{\partial V_{\rm eff}^T}{\partial\phi_c}+\frac{\lambda_r}{4\pi^2}\frac{1}{a^2}\phi_c\int_0^\infty dk\, k^2\Omega_k{}^{-1}\times\coth\left[\beta\frac{\Omega_k(\tau_0)}{2}\right](s_k+{\rm Re}z_k)=0, \tag{45}$$

with $V_{\rm eff}^T\equiv V_{\rm eff}^0+\Delta V_{\rm eff}^T$. s_k and z_k are determined by (31a) and (31b) with the initial condition $s_k(\tau_0)=z_k(\tau_0)=0$. The asymptotic behavior of s_k and ${\rm Re}z_k$ for large k (37a) and (37b) ensures the convergence of the momentum integral in (45).

Since s_k and ${\rm Re}z_k$ are integrals of Eqs. (31a) and (31b), which involve the background fields ϕ_c and a, (45) is, essentially, an integro-differential equation, the last term on the left-hand side being a nonlocal functional depending on the precise history of the background fields. In the *adiabatic limit,* when (35) is satisfied for all momenta k, the nonlocal term in (45) can be neglected in comparison with the other terms since then s_k, $|z_k| \ll 1$ for all k. Equation (45) becomes in this limit similar (although not identical, because we have worked in curved space-time) to the evolution equation usually taken in the discussion of phase transitions in the early Universe,[14] in particular of new inflation based on CW potentials.[2,3] In the *nonadiabatic* case, when the background fields vary more rapidly, it is not possible to neglect the nonlocal contribution in (45) from the beginning. Equations (31a), (31b), and (45) can be used in this case to study the influence of quantum processes such as particle production and vacuum polarization on phase transitions in the very early Universe when space-time curvature and dynamical field effects are important.

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