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Properties of some self-dual monopoles

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The magnetic charge and the Brandt-Neri stability conditions are studied for a class of spherically symmetric self-dual monopoles previously exhibited by McGlinn. It is found that there is a connection between the requirement of Brandt-Neri stability and the charge: namely, a solution with the smallest charge is often, but not always, Brandt-Neri stable. Most of these solutions which have a larger magnetic charge are not Brandt-Neri stable.

I. INTRODUCTION

The study of self-dual monopoles has proved to be a fruitful field for the development of theoretical ideas concerning the nature and interactions of monopoles. Not many solutions to the self-dual equations, for a general simple gauge group, are known. Most known solutions are spherically symmetric, the spherical symmetry implemented by an SU(2) subgroup of the gauge group G. Ganoulis, Goddard, and Olive[1] (GGO) derived solutions in which the SU(2) was maximally embedded in G, and McGlinn[2] exhibited some generalizations of these solutions in which the spherical symmetry is implemented by an SU(2) maximal in a subgroup G' of G. This paper investigates the monopole strength of these solutions and their Brandt-Neri[3] stability.

The next section describes the GGO and McGlinn solutions and, for the latter case, more completely characterizes the asymptotic magnetic field then was done in Ref. 2. Section III translates the Brandt-Neri stability conditions to statements concerning the particular form of the asymptotic magnetic field of the GGO and McGlinn solutions. These statements, used in Sec. IV, characterize the solutions that are Brandt-Neri unstable. Concluding remarks are made in Sec. V.

II. SOME SPHERICAL SELF-DUAL MONOPOLES

We first describe the GGO solutions. The spherical symmetry of these solutions is implemented by an SU(2) subgroup which is maximal[1] in the full gauge group G. This maximal SU(2) subgroup of a simple group G is such that if the adjoint representation of G is reduced with respect to this SU(2), the number of multiplets obtained equals the rank of G. For this case,

$$T_3=\frac{\mathbf{R}\cdot\mathbf{H}}{2}\ , \tag{1}$$

where $\mathbf{R}$ is the level vector[4] of G. The physically interesting solutions are such that the unbroken-symmetry group is $U(1)\times K$ where K is semisimple. To obtain this, the asymptotic Higgs field

$$\mathbf{\Phi}=\mathbf{q}\cdot\mathbf{H} \tag{2}$$

has a weight space vector $\mathbf{q}$, which is proportional to a fundamental weight $\boldsymbol{\lambda}_\theta$. The resulting asymptotic radial magnetic field is given by[5]

$$eB_r=\frac{(\mathbf{m}-\mathbf{R})\cdot\mathbf{H}}{2r^2}\ . \tag{3}$$

The dual components of the weight vector $\mathbf{m}$ are those of the level vector of K for those components corresponding K and zero otherwise, i.e., for the component corresponding to the simple root $\boldsymbol{\alpha}_\theta$. The vector $\mathbf{m}-\mathbf{R}$ is proportional to $\mathbf{q}$; i.e., the asymptotic $\mathbf{B}$ field is in the same direction as the Higgs field.

McGlinn[2] considered a gauge group G, a subgroup G', with the maximal SU(2) subgroup of G' implementing the spherical symmetry. G' was restricted to being a subgroup such that there exists a Cartan-Weyl basis of G in which the simple roots of G' are a subset of the simple roots of G. Thus the Dynkin diagram for G' is obtained from that of G by removing dots. An ansatz is introduced which restricts the Higgs field (along the z axis) to be in the Cartan subalgebra. The dual components of the Higgs field corresponding to the removed dots are shown to be constant and can be chosen arbitrarily. However they are determined by the requirement that the asymptotic Higgs field is in the direction of a fundamental weight. The resulting asymptotic radial magnetic field is given by[2]

$$eB_r=\frac{(\mathbf{m}'-\mathbf{R}')\cdot\mathbf{H}}{2r^2}=\frac{\mathbf{Q}\cdot\mathbf{H}}{2r^2}\ . \tag{4}$$

Here the dual components of the vector $\mathbf{m}'$ are equal to those of the level vector of $K'=K\cap G'$ for the components of $\mathbf{m}'$ corresponding to K' and zero otherwise. $\mathbf{R}'$ has dual components equal to those of the level vector of G' for those components corresponding to G' and zero otherwise.

The situation that the monopoles of (4) are topologically trivial with respect to G [but of course not with respect to $H=U(1)\times K$] is reflected by the fact that the dual components of $\mathbf{m}'-\mathbf{R}'$ are integers. The monopole magnetic charge (and topological charge) is given by $-R'_\theta$. Thus, for such a solution to have unit-strength magnetic charge, $G'=SU(2)$ since the dual components of the level vector of any other simple group are greater than one.[4]

III. THE BRANDT-NERI STABILITY CONDITION

Brandt and Neri[3] established that the magnetic monopole was stable against perturbations only for certain asymptotic **B** fields. Their arguments depend upon a rapid approach to the inverse-square law form which does not hold for self-dual monopoles.[6] Nevertheless one might expect that self-dual solutions may satisfy this condition. The Brandt-Neri stability condition, for the form (4), can be written as[7]

$$\mathbf{Q}\cdot\boldsymbol{\alpha}=0 \text{ or } \pm 1 \tag{5}$$

for all $\boldsymbol{\alpha}$ which is a root of K. Goddard and Olive[7] (GO) have reduced condition (5) to conditions which involve only simple roots of K. Since the roots of K are all perpendicular to $\boldsymbol{\lambda}_\theta$, (5) also holds for the projection $\mathbf{P}$ of $\mathbf{Q}$ perpendicular to $\boldsymbol{\lambda}_\theta$:

$$\mathbf{P}=\mathbf{Q}-\frac{\mathbf{Q}\cdot\boldsymbol{\lambda}_\theta}{\lambda_\theta^{\,2}}\boldsymbol{\lambda}_\theta\ ; \tag{6}$$

that is,

$$\mathbf{P}\cdot\boldsymbol{\alpha}=0 \text{ or } \pm 1\ . \tag{7}$$

GO argue that (7) can be replaced by

$$\mathbf{P}\cdot\boldsymbol{\alpha}_m=0 \text{ or } \pm 1 \quad\text{and}\quad \mathbf{P}\cdot\boldsymbol{\alpha}_j=0 \text{ for } j\neq m\ . \tag{8}$$

Here $\boldsymbol{\alpha}_m$ is a simple root of a simple factor of K restricted to being one which occurs with an expansion coefficient 1 in the expansion of the highest root of the simple factor. Such roots are conjugate to so-called minimal weights,[7,8] indicated in Fig. 1. $\boldsymbol{\alpha}_j$ is any other simple root of this factor.

We want to investigate (8) for solutions (4). If one denotes the dual components (in G) of $\mathbf{Q}$ as q_j, the dual components of $\mathbf{P}$, p_i, are

$$p_i=q_i-q_\theta\frac{G_{\theta i}}{G_{\theta\theta}}\ . \tag{9}$$

Here $G_{ij}=(K^{-1})_{ij}\alpha_j^{\,2}/2$ (no sum over j). K_{ij} is the Cartan matrix for G and $G_{\theta\theta}=\lambda_\theta^{\,2}$. Notice that the dual component p_θ is zero. The Dynkin components $\underline{p}_i$ (in K) of $\mathbf{P}$ are

$$\underline{p}_i=\sum_j \frac{2}{\alpha_i^{\,2}}K_{ij}\left[q_j-q_\theta\frac{G_{j\theta}}{G_{\theta\theta}}\right]\ . \tag{10}$$

But

$$\sum_i K_{ij}G_{i\theta}=\frac{\alpha_\theta^{\,2}}{2}\delta_{\theta i}$$

and thus

$$\underline{p}_i=\frac{2}{\alpha_i^{\,2}}\sum_j K_{ij}q_j,\quad i\neq\theta\ . \tag{11}$$

Equation (11) implies that Eq. (8) can be written as

$$\sum_i K_{mi}q_i=0 \text{ or } \pm 1 \quad\text{and}\quad \sum_i K_{ji}q_i=0,\quad j\neq m\ . \tag{12}$$

The sum is restricted to i such that $\boldsymbol{\alpha}_i$ is a simple root of G', since $q_i\neq 0$ only for such i. Also, $K_{ji}q_i=0$ for j, such that $\boldsymbol{\alpha}_j$ is a simple root of G' and thus (12) is satisfied for such j. However, as we shall see in the next section, (12) is generally not satisfied for $m\,(j)$, such that $\boldsymbol{\alpha}_m(\boldsymbol{\alpha}_j)$ is not a simple root of G'.

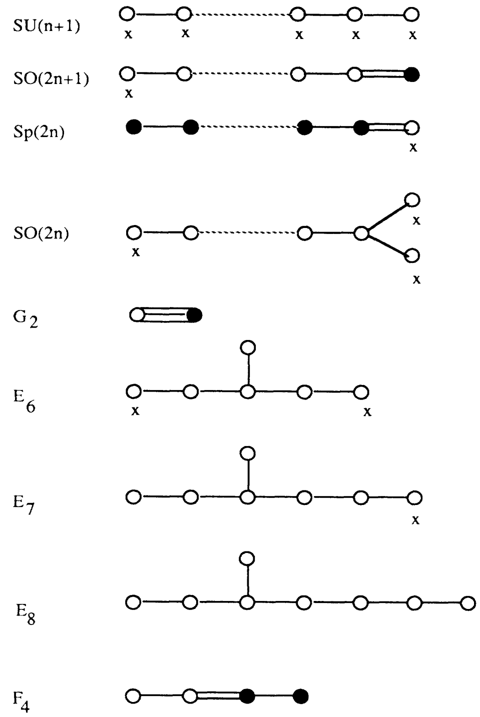

FIG. 1. Minimal weights. The minimal weights are those fundamental weights corresponding to the vertices of the Dynkin diagram marked by a cross. There are none for E_8, F_4, and G_2.

IV. BRANDT-NERI INSTABILITIES

To simplify the discussion we restrict G' to be simple and K to be semisimple, but not simple. Consider, as an example, $G=\mathrm{SU}(n+1)$. The generic diagram for such a case is indicated by Fig. 2.

For this case $K=\mathrm{SU}(o+p+1)\times\mathrm{SU}(q+r+1)$, $G'=\mathrm{SU}(p+q+2)$, and $K'=\mathrm{SU}(p+1)\times\mathrm{SU}(q+1)$. As noted in Sec. II, (12) is satisfied for an index corresponding to a noneliminated root. The question to be answered is the following: What is the value of $K_{ji}q_i$ for j corresponding to an eliminated root? For $\mathrm{SU}(n+1)$, $K_{ij}=0$ except for $i=j$ ($K_{ii}=2$) or $i=j\pm 1$ ($K_{ii\pm 1}=-1$). Thus, the only nonvanishing $K_{ji}q_i$ is for j corresponding to an eliminated root at the end of a string of eliminated roots and then $K_{ji}q_i=-q_{j\pm 1}$. Recall that $\mathbf{q}=\mathbf{m}'-\mathbf{R}'$, where $\mathbf{m}'$ have the dual components of the level vector of K' and $\mathbf{R}'$ those of the level vector of G'. Since the

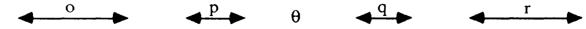

FIG. 2. Dynkin diagram for $G=\mathrm{SU}(n+1)$. ⊗ indicates a root eliminated in defining G' and θ indicates the root conjugate to $\boldsymbol{\lambda}_\theta$.

components of the level vector for $SU(m+1)$ are $[m, 2(m-1), 3(m-2), \ldots, (m-1)2, m]$, the values for $q_{j\pm 1}$ are $-q-1$ and $-p-1$. Thus, the Brandt-Neri conditions are satisfied only for $q=p=0$, in which case, as we have noted, $G'=SU(2)$ and the monopole has unit magnetic strength. We call an embedding when $G'=SU(2)$ a minimal embedding. Of course, the conditions are also satisfied if $G'=G$ when there are no eliminated roots; this is maximal embedding.[1]

The generalization of these arguments to other simple groups is straightforward. Slight complications occur because the group (i) has a Dynkin diagram that branches [SO(2n), E_6, E_7, E_8] or (ii) has simple roots of different lengths [SO(2n+1), Sp(2n), G_2, and F_4].

First consider (i). It is still true that $K_{ij}=0$ or -1 for $i\neq j$ and it is easy to see that the result is the same as for $SU(n+1)$; the Brandt-Neri conditions hold only for $G'=G$ and $G'=SU(2)$.

For case (ii) one can again easily argue that the Brandt-Neri conditions are not satisfied unless $G'=G$ or $G'=SU(2)$, but the latter is not a sufficient condition. Suppose the breaking is in the direction of a fundamental weight $\boldsymbol{\lambda}_\theta$ whose conjugate simple root $\boldsymbol{\alpha}_\theta$ is a short root of G. (Such a root corresponds to a black dot in the Dynkin diagrams listed in Fig. 1.) We intend to show that all minimally embedded solutions that correspond to these short simple roots are not Brandt-Neri stable.

First, consider the case where $\boldsymbol{\alpha}_\theta$ corresponds to the unique black dot in the Dynkin diagram connected to a white dot; denote the simple root corresponding to the white dot as $\boldsymbol{\alpha}_j$. $-K_{j\theta}$ is 2 for SO(2n+1), Sp(2n), and F_4, and 3 for G_2. Thus, the minimal embedding corresponding to $\boldsymbol{\alpha}_\theta$ is not Brandt-Neri stable. (The magnitude of each of the other nonzero off-diagonal elements of the Cartan matrix is one.)

In order to see why the rest of the short roots also correspond to unstable solutions, we must emphasize a point ignored in the analysis so far, namely, that to invoke the first of Eqs. (12), we must check that the simple root $\boldsymbol{\alpha}_\theta$, which identifies the minimal embedding, is connected in the Dynkin diagram to simple roots $\boldsymbol{\alpha}_m$, each of which are conjugate to minimal weights of K. For simple Lie algebras with one root length, this is easily verified for all choices of $\boldsymbol{\alpha}_\theta$. For simple Lie algebras having more than one root length, it is also true for all $\boldsymbol{\alpha}_\theta$ which are either long simple roots or connected to long simple roots in the Dynkin diagram. The remainder of the short simple roots violate the condition, that is, deleting one of these black dots in the Dynkin diagram creates new vertices, one of which is not conjugate to a fundamental weight of K. This confirms our statement that all minimally embedded solutions that correspond to short roots are not Brandt-Neri stable.

V. DISCUSSION

After submitting this report, it came to our attention that the SU(2) embeddings we have referred to as minimal embeddings have been extensively studied by Weinberg.[9,10] He calls these solutions "fundamental monopoles" based on the form of the energy and topological charge for an arbitrary magnetic charge consistent with the quantization condition, and shows that a counting of the zero-mode perturbations about a self-dual solution supports this interpretation. He asserts that the monopoles of higher topological charge will not correspond to particles in the spectrum of the quantized theory. In his investigation of the properties of these rank-(G) fundamental monopoles for a non-Abelian unbroken-symmetry group, Weinberg classifies the fundamental monopoles as "degenerate" or "nondegenerate." A fundamental monopole is said to be degenerate if there exists another monopole solution [with an SU(2) embedding implemented by a root of G which is not a simple root[11]] that has the same topological charge as the fundamental monopole and yet is not gauge equivalent to the fundamental-monopole solution under the unbroken-symmetry group. Weinberg lists the breakings which admit fundamental monopoles. Finally, he observes that precisely those fundamental monopoles which are degenerate violate the Brandt-Neri stability conditions. We find our list of minimally embedded solutions which are not Brandt-Neri stable identical to Weinberg's list of degenerate fundamental monopoles for the case when the unbroken-symmetry group is $U(1)\times K$.

It has been conjectured[12] that monopoles in a theory with an unbroken symmetry H might be described by field operators which transform according to the so-called dual group H^v. GO argue that self-dual monopoles that are stable with respect to both the Brandt-Neri conditions and the stability criterion of Bais,[11] the latter requiring that the monopole strength be fundamental, can indeed be interpreted as heavy gauge particles of H^v. The discussion in this paper confirms that there is a class of solutions of self-dual monopoles which do not satisfy Brandt-Neri conditions, some of which, for some groups, satisfy the stability criterion of Bais.

The main purpose of this paper was to investigate properties of a class of spherically symmetric self-dual monopoles exhibited in Ref. 2. We have shown that for a given direction of symmetry breaking $\boldsymbol{\lambda}_\theta$, all these solutions characterized by different SU(2) embeddings have a monopole strength greater than the fundamental strength, except for the minimal SU(2) embedding. In addition, we have found that in general the Brandt-Neri conditions are not satisfied. Under the restrictions stated in Sec. IV, a necessary condition for these conditions to be satisfied is that the SU(2) embedding be either maximal or minimal. However, for groups which have roots of different lengths, an SU(2) embedding may be minimal yet the Brandt-Neri conditions not satisfied. This is true if the simple root $\boldsymbol{\alpha}_\theta$ is a short root of G. If the restriction that K not be simple is relaxed, there will in general be solutions that are Brandt-Neri stable for which the SU(2) embedding is not maximal or minimal. Our results in the case of minimal embeddings are consistent with the earlier observations of Weinberg.

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[1]N. Ganoulis, P. Goddard, and D. Olive, Nucl. Phys. **B205**, 601 (1982).
[2]W. D. McGlinn, Phys. Rev. D **30**, 2249 (1984).
[3]R. Brandt and F. Neri, Nucl. Phys. **B161**, 253 (1979).
[4]R. Slansky, Phys. Rep. **79**, 1 (1981).
[5]S. K. Bose and W. D. McGlinn, Phys. Rev. D **29**, 1819 (1984).
[6]D. Olive, in *Monopoles in Quantum Field Theory,* edited by N. S. Craigie, P. Goddard, and W. Nahm (World Scientific, Singapore, 1982).
[7]P. Goddard and D. Olive, Nucl. Phys. **B191**, 528 (1981).
[8]J. Humphreys, *Introduction to Lie Algebras and Representation Theory* (Springer, Berlin, 1972).
[9]E. J. Weinberg, Nucl. Phys. **B167**, 500 (1980).
[10]E. J. Weinberg, Nucl. Phys. **B203**, 445 (1982).
[11]F. A. Bais, Phys. Rev. D **18**, 1206 (1978).
[12]P. Goddard, J. Nuyts, and D. Olive, Nucl. Phys. **B125**, 1 (1977).