Continuum limit of lattice gauge theory: A perturbative study

S. Sakai and H. Iso
Institute of Physics, University of Tsukuba, Sakura-mura, Ibaraki, 305 Japan
(Received 15 December 1986)

In a weak-coupling expansion, the Creutz ratio is calculated on relatively large lattices up to $O(g^4)$ for improved actions. A universality in size dependence of the Creutz ratio is found among lattice actions. By extrapolating the results on finite lattices to an infinite lattice, the artifacts are studied, and the minimum size of Creutz ratios for which the artifact becomes less than 10% is determined. The scale transformation and universality between lattice actions on finite lattices are studied.

I. INTRODUCTION

Lattice gauge theory[1] is a well-defined quantum system as long as the lattice size L, lattice spacing a, and coupling constant g_a are kept finite. The real world corresponds to the continuum limit, $L\rightarrow\infty$, $a\rightarrow 0$, and $g\rightarrow 0$. In this limit, the correlation length ξ diverges. The nonperturbative effects of the theory are taken into account under the condition $l \gg \xi$, where l is a characteristic length of some physical quantity under consideration.

The nonperturbative aspects of the SU(N) gauge theory are mainly studied by Monte Carlo (MC) simulations[2] on finite lattices, but from the ability of present-day computers, the size of lattices L could not be taken large enough and therefore, the coupling constant g_a is also limited. The size effects and lattice artifacts of MC calculations are not studied systematically and quantitatively.

The weak-coupling perturbative expansion[3] corresponds to the $g_a\rightarrow 0$ limit. Then for all the quantities with finite l, the nonperturbative effects could not be taken into account. However, perturbatively the physical quantities are calculable on continuous space from the beginning if a renormalization scheme is specified. Then by comparing the corresponding calculation on a finite lattice with those on continuous space, we definitely can know the size dependences and artifacts.

On a lattice there is an ambiguity in the choice of action, and the improved actions[4,5] have been proposed to reduce the lattice artifacts. However, we must carry out actual calculations on finite lattices with these actions to know how large L and l/a should be in order that the difference between lattice and continuum calculations becomes less than 10%, for example.

In this paper we study the continuum limit of perturbative Creutz ratios[2] for three kinds of actions, a simple plaquette action[1] ($S_{\rm pq}$), Symanzik and Weisz's improved action[5,6] ($S_{\rm SW}$), and Iwasaki's improved action[7] ($S_{\rm Iw}$), because MC simulations have been mainly done with these actions on relatively large lattices and the continuum limit and universality of string tension have been discussed.[8–10] Another reason we have limited the actions stated above is that the CPU time of our numerical calculations is rather large.

In Creutz ratios the renormalization of the coupling constant begins at $O(g^4)$. Therefore, from the calculation of the $O(g^4)$ coefficient, we can find how the renormalization-group (RG) equations for Creutz ratios with finite characteristic length l are affected on finite lattices L. This information is important in taking the continuum limit of nonperturbative quantities, because to take that limit the scaling relation obtained by the RG equation on continuous space is assumed. The correction to the RG equation on finite L and (l/a) might be another origin of systematic errors in the continuum limit of nonperturbative quantities.

In Sec. II we outline the weak-coupling expansion of SU(N) pure gauge theory on a lattice, and the calculation of vacuum expectation value of Wilson loops. We relate the calculation of physical quantities on a lattice and on continuum space and the meaning of universality is clarified.

In Sec. III the numerical results on Creutz ratios for three kinds of actions are presented and we study the size effects and lattice artifacts. It is found that the size dependence of the $O(g^2)$ and $O(g^4)$ coefficients have universality. We take the $L\rightarrow\infty$ limit of Creutz ratios by extrapolation method and find the minimum size of characteristic length l_m, for which the lattice artifacts become less than 10%. We find that improved actions have in fact, smaller l_m than the simple plaquette action. In the case of the $O(g^2)$ coefficients, $S_{\rm SW}$ is the most improved action while for the $O(g^4)$ coefficient, it is $S_{\rm Iw}$.

Section IV is devoted to discussions and comments. We discuss the scale transformation and universality of Creutz ratios between lattice actions, and make a few comments.

II. WEAK-COUPLING EXPANSION ON A LATTICE

A. Improved actions

In this paper we study SU(N) pure gauge theory (without dynamical quarks). The action is written as

$$S=\frac{1}{{g_a}^2}\sum_{i=0}^{I}\left[c_i\sum_{\rm site}{\rm Tr}\left[\prod_{k\in\Gamma_i}U_k-1\right]\right], \qquad (2.1)$$

where Γ_i represent closed loops on a lattice, U_k is a SU(N) matrix defined on the link k, and g_a is the coupling constant on a lattice. In the $L \rightarrow \infty$ and $a \rightarrow 0$ ($l/a \rightarrow \infty$) limit, it is expected that all choices of c_i give the same results for physical quantities (universality), as long as c_i's satisfy one restriction which ensures that in the $a \rightarrow 0$ limit the action given by (2.1) reduces to the classical action.

To get the continuum limit for small l/a, the improved actions have been proposed by two methods: (i) the (a/l) expansion method of Symanzik and Weisz[5] and (ii) the renormalization-group method of Wilson.[4]

In a weak-coupling expansion at $O(g^2)$, these two methods are shown to be equivalent as long as we are working on infinite parameter space of action ($I=\infty$) (Ref. 11). However, the number of parameters is usually limited to 4 or 2, which are not found to be large enough[11] and different results for improved actions have been obtained from the two methods.[6,7]

In this paper we limit ourselves to $I=2$, and the normalization condition becomes

$$c_0+8c_1=1 , \tag{2.2}$$

where Γ_0 is the simple plaquette loop and Γ_1 represent the planar $a\times 2a$ rectangular loops. The actions considered in this paper are parametrized as (i) standard plaquette action[1] $S_{\rm pq}$, $c_1=0$, (ii) Symanzik and Weisz's tree-improved action[6] $S_{\rm SW}$, $c_1=-\frac{1}{12}$, and (iii) Iwasaki's tree-improved action[7] $S_{\rm Iw}$, $c_1=-0.331$.

B. The weak-coupling expansion on a lattice

The Wilson loop W_Γ is defined as

$$W_\Gamma=\frac{1}{N}\mathrm{Tr}\left[\prod_{k\in\Gamma} U_k\right] , \tag{2.3}$$

where Γ is a closed loop on the lattice.

In the weak-coupling expansion,[3] the link variable U_k is parametrized as

$$U_k=\exp(iag_a A_k) . \tag{2.4}$$

A_k is regarded as the dynamical variable and the action in (2.1) is expanded in powers of g_a. As the details of the weak-coupling expansion of the action and Wilson loops are reported in a previous publication,[12] we do not repeat them here.

We should like to notice that all the algebraic treatment of the weak-coupling expansion and the calculation of vacuum expectation values (VEV's) of Wilson loops are carried out by REDUCE. The VEV of planar Wilson loops with length l_i and l_j are written as

$$W(l_i,l_j)=1+g_a{}^2\left[\frac{N^2-1}{N}\right]W^{(2)}(l_i,l_j)+g_a{}^4(N^2-1)\left[W_1^{(4)}(l_i,l_j)+\frac{1}{N^2}W_2^{(4)}(l_i,l_j)\right]+O(g_a{}^6) . \tag{2.5}$$

Algebraic expressions of each coefficient of g_a are written in momentum space, which are numerically calculated on finite lattices with periodic boundary conditions by FORTRAN. The computation on a 16^4 lattice takes $\sim$2.5 CPU hours for $S_{\rm pq}$ and $\sim$74 h for $S_{\rm SW}$ and $S_{\rm Iw}$ on a FACOM M380 at the University of Tsukuba.

We have completely confirmed the results of Heller and Karsch[13] on 8^4 and 16^4 lattices for $S_{\rm pq}$.

On a finite lattice L, there occur divergences at $p=0$, because of the propagator $D_{\mu\nu}(p)$, which is called a toron in Ref. 14. The toron contribution at $O(g^2)$ is obtained for $S_{\rm pq}$, as

$$W^{(2)}_{\rm toron}(l_i,l_j)=-\left[\frac{N^2-1}{N}\right]\frac{(l_il_j)^2}{12L^4} . \tag{2.6}$$

This term is common to all improved actions and we have included it in the $O(g^2)$ term. In $O(g^4)$ the toron contributions have not been obtained yet. They are much more complicated because we must expand both the actions and Wilson loops up to $O(A^8)$. In this paper, we have simply dropped its contributions in $O(g^4)$. $O(g^4)$.

In the previous paper,[12] we have reported the calculation on $L\le 16$ for $S_{\rm pq}$ and $L\le 12$ for other improved actions, which had been the largest lattices where MC results for $W(l_i,l_j)$ were available. However, these lattice sizes are too small to get the continuum limit of Creutz ratios. In this paper we continue the calculations on larger lattices in order to study the continuum limit of lattice calculations; $L\le 20$ for $S_{\rm pq}$, $L\le 16$ for $S_{\rm SW}$, and $L\le 18$ for $S_{\rm Iw}$.

C. Lattice versus continuum calculation

The vacuum expectation value of a Wilson loop does not have a continuum limit, while a Creutz ratio does. It is defined by[2]

$$\chi(l_i,l_j)=-\ln\left[\frac{W(l_i,l_j)W(l_i-\delta,l_j-\delta)}{W(l_i-\delta,l_j)W(l_i,l_j-\delta)}\right] , \tag{2.7}$$

where δ is arbitrary but usually taken to be a.

Creutz ratios have been perturbatively calculated on continuous space from the beginning.[15] They are written as

$$\chi(l_i,l_j)=g_R{}^2\chi_R^{(2)}(l_i,l_j)+g_R{}^4\chi_R^{(4)}(l_i,l_j)+O(g_R{}^6) , \tag{2.8}$$

where g_R is a renormalized coupling constant which contains all the divergences,[16] in the Creutz ratio. In Ref. 15, a kind of dimensional regularization has been used.

$$\chi(l_i,l_j)=g_a{}^2\chi_a^{(2)}(l_i,l_j)+g_a{}^4\chi_a^{(4)}(l_i,l_j)+O(g_a{}^6) , \tag{2.9}$$

where g_a represents the coupling constant of some lattice action (S_{pq}, S_{SW}, and S_{Iw}).

In two-loop order and in the $L \to \infty$ and $a \to 0$ limit, g_R and g_a are expressed by scale parameters Λ:

$$\frac{1}{g_R{}^2(l)} = -2\beta_0 \ln l\Lambda_R + \frac{\beta_1}{\beta_0}\ln\left[\ln\left[\frac{1}{l\Lambda_R}\right]^2\right] , \quad (2.10)$$

$$\frac{1}{g_a{}^2} = -2\beta_0 \ln a\Lambda_a + \frac{\beta_1}{\beta_0}\ln\left[\ln\left[\frac{1}{a\Lambda_a}\right]^2\right] , \quad (2.11)$$

where $\beta_0 = 11N/48\pi^2$, $\beta_1 = 34N^2/3(16\pi)^2$, and l is a characteristic length of $\chi(l_i,l_j)$ which is defined as $l=(l_il_j)^{1/2}$. From (2.10) and (2.11), we obtain

$$\frac{1}{g_R{}^2(l)} = \frac{1}{g_a{}^2} - 2\beta_0 \ln\left[\frac{l\Lambda_R}{a\Lambda_a}\right] - \frac{\beta_1}{\beta_0}\ln\left[\frac{g_R{}^2}{g_a{}^2}\right] + O(g_R{}^2, g_a{}^2) . \quad (2.12)$$

The ratios Λ_R/Λ_a were already obtained as[15,17]

$$\Lambda_R/\Lambda_{pq} = 25.7 , \quad (2.13)$$

$$\Lambda_R/\Lambda_{SW} = 4.86 , \quad (2.14)$$

$$\Lambda_R/\Lambda_{Iw} = 0.435 . \quad (2.15)$$

Using (2.12), the Creutz ratio calculated on continuous space is expanded in powers of the lattice coupling constant g_a. Each coefficient of g_a should be equal to those of (2.9) in the $L \to \infty$ and $a \to 0$ ($l/a \to \infty$) limit

$$\chi_a^{(2)}(l_i,l_j) = \chi_R^{(2)}(l_i,l_j) , \quad (2.16)$$

$$\chi_a^{(4)}(l_i,l_j) = \chi_R^{(4)}(l_i,l_j) + \chi_R^{(2)}(l_i,l_j) 2\beta_0 \ln\left[\frac{l\Lambda_R}{a\Lambda_a}\right] . \quad (2.17)$$

This is the meaning of universality for perturbative Creutz ratios.[15]

We get the left- and right-hand side of (2.16) and

TABLE I. Coefficients of perturbative expansion of the Creutz ratio defined by (2.9) on various lattices. I(a) corresponds to $\chi_a^{(2)}(l,L)10^2$ and I(b) to $\chi_a^{(4)}(l,L)10^3$.

l \ L	12	14	16	18	20	∞ Extra.	∞ WW (Ref. 19)
				(a)			
				S_{pq}			
3	3.4315	3.4423	3.4470	3.4492	3.4504	3.4524	3.4520
4	1.5601	1.5901	1.6023	1.6080	1.6109	1.6153	1.6153
5	0.80083	0.87028	0.89735	0.90955	0.91568	0.92422	0.92480
6	0.35167	0.49640	0.55022	0.57375	0.58527	0.60042	0.60200
				S_{SW}			
3	3.1586	3.1693	3.1739			3.1791	
4	1.4362	1.4653	1.4773			1.4895	
5	0.75610	0.82268	0.84903			0.87408	
6	0.34053	0.47843	0.53058			0.57660	
				S_{Iw}			
3	2.8336	2.8441	2.8487	2.8510		2.8543	
4	1.2967	1.3239	1.3354	1.3410		1.3482	
5	0.69608	0.75702	0.78182	0.79335		0.80750	
6	0.32395	0.44942	0.49792	0.51973		0.54497	
				(b)			
				S_{pq}			
3	19.599	19.738	19.803	19.837	19.856	19.861	20.04±0.08
4	9.4039	9.7444	9.8970	9.9740	10.017	10.091	10.37±0.17
5	4.7483	5.4615	5.7688	5.9194	6.0005	6.1333	6.42±0.17
6	1.6349	2.9801	3.5424	3.8104	3.9515	4.1698	4.6 ±0.8
				S_{SW}			
3	10.706	10.791	10.831			10.885	
4	5.4534	5.6671	5.7637			5.8817	
5	2.9228	3.3768	3.5748			3.7993	
6	1.0579	1.9149	2.2809			2.6797	
				S_{Iw}			
3	0.65739	0.66128	0.66314	0.66425		0.66711	
4	0.68858	0.72169	0.73569	0.74264		0.75244	
5	0.58494	0.68250	0.72411	0.74397		0.76972	
6	0.28542	0.48261	0.57333	0.61686		0.67394	

(2.17) independently, then by comparing them, we can study the size effects and artifact without ambiguity.

III. SIZE DEPENDENCE AND LATTICE ARTIFACT

A. Size dependence

In the following, as we consider only Creutz ratios with $l_i = l_j = l$ on finite lattices L, we denote it just as $\chi(l,L)$. We take the lattice spacing to be unity $a=1$, and present the results only for SU(3).

In order to show the size dependence of Creutz ratios, we show a few $\chi^{(i)}(l,L)$ for the three kinds of actions. The size dependence of $W^{(i)}(l_i,l_j)$ is shown in Ref. 13 for $S_{\rm pq}$. It is found that Creutz ratios have much larger size dependences. This is expected because $W(l_i,l_j)$ contains the perimeter term, which corresponds to mass renormalization of the test quark line. This term is the main contribution to $W(l_i,l_j)$ and has minimum L dependences.

As the size dependences are rather large on these lattice sizes, we get the $L\to\infty$ limit by the following formula using $\chi(l,L)$ of the largest three lattices:

$$\chi_a^{(i)}(l,L) = c_a^{(i)}(l) L^{-n^{(i)}(l)} + \chi_a^{(i)}(l,\infty) \quad (i=2,4) \ . \quad (3.1)$$

The results for $\chi^{(i)}(l,\infty)$ are shown in Table I.

In Fig. 1 we plot the ratio

$$R_a^{(i)}(l,L) = \frac{\chi_a^{(i)}(l,L) - \chi_a^{(i)}(l,\infty)}{\chi_a^{(i)}(l,\infty)} \ . \quad (3.2)$$

We find that $R_a^{(i)}(l,L)$ are almost equal for all three kinds of actions. It is seen from Figs. 1(a) and 1(b) that the $O(g^4)$ coefficients have larger size dependences than the $O(g^2)$ coefficients. We study whether these differences could be explained by toron contributions.

In Fig. 1(b) we have also plotted $R_a^{(2)}(l,L)$ without the toron contribution for $S_{\rm Iw}$. For other actions the results fall in the dotted regions of Fig. 1(b). It is seen that they are very close to $R_a^{(4)}(l,L)$. We have studied how $c_a^{(i)}$, $n^{(i)}(l)$, and $\chi_a^{(i)}(l,\infty)$ depend on the lattice size L used to make extrapolations. It is seen that all $n^{(i)}(l)$ are approaching 4 as L increases.[18] Then we find the following universal size dependences: $c_a^{(i)}(l)/\chi_a^{(i)}(l,\infty)$ and $n^{(i)}(l)$ become independent of the action and order of g as L increases, and in the $L\to\infty$ limit, $n^{(i)}(l)$ approaches 4.

B. Lattice artifact

We compare the $\chi^{(i)}(l,\infty)$ with the corresponding calculation on continuous space.[15] In $O(g^4)$, instead of comparing $\chi^{(4)}(l,\infty)$ directly, we define Λ parameters of the actions for finite L and l by (2.17) and study how they approach the predicated values given by (2.13)–(2.15). We solve (2.17) to get Λ_R/Λ_a on a finite lattice:

$$\frac{\Lambda_R}{\Lambda_a(l,L)} = \frac{1}{l}\exp\left[\frac{\chi_a^{(4)}(l,L) - \chi_R^{(4)}(l)}{2\beta_0 \chi_R^{(2)}(l)}\right] . \quad (3.3)$$

In Fig. 2 we show the following ratio to study how $\chi_a^{(2)}$ and $\Lambda_a(l,\infty)$ approach their continuum limit:

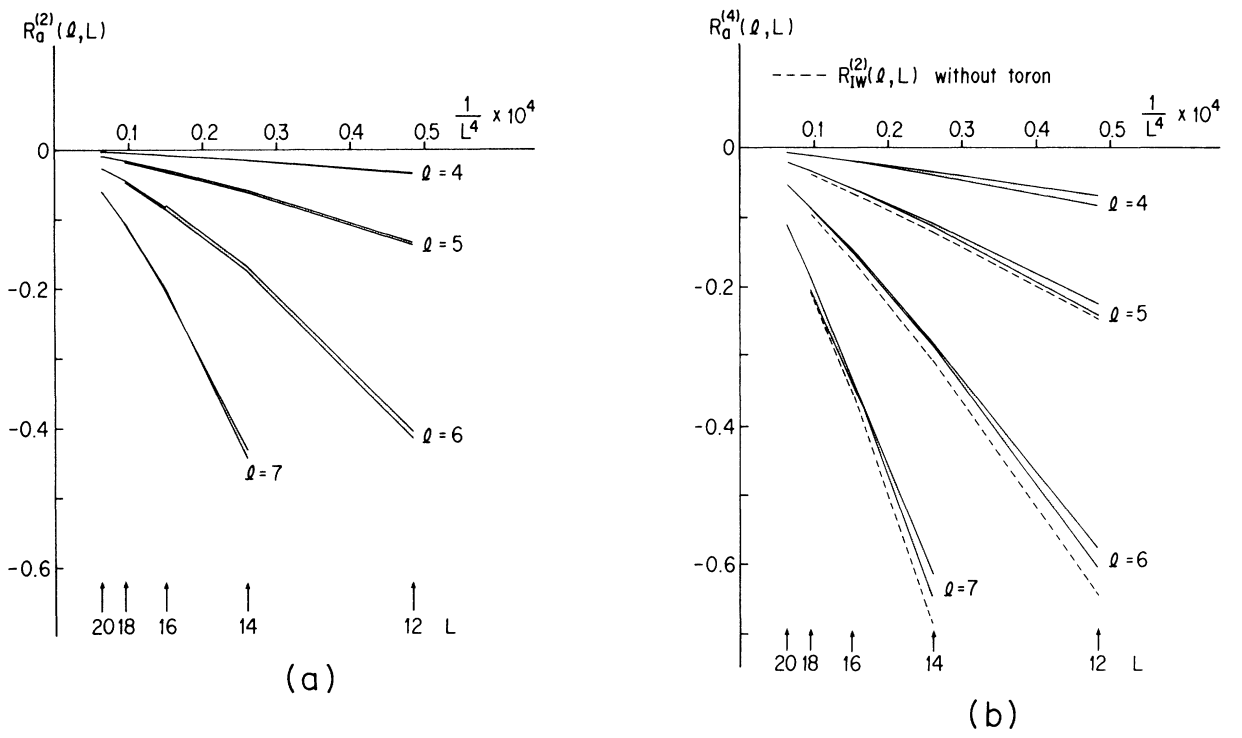

FIG. 1. The lattice size dependences of the coefficients of the Creutz ratio where we have shown the ratio defined by (3.2).

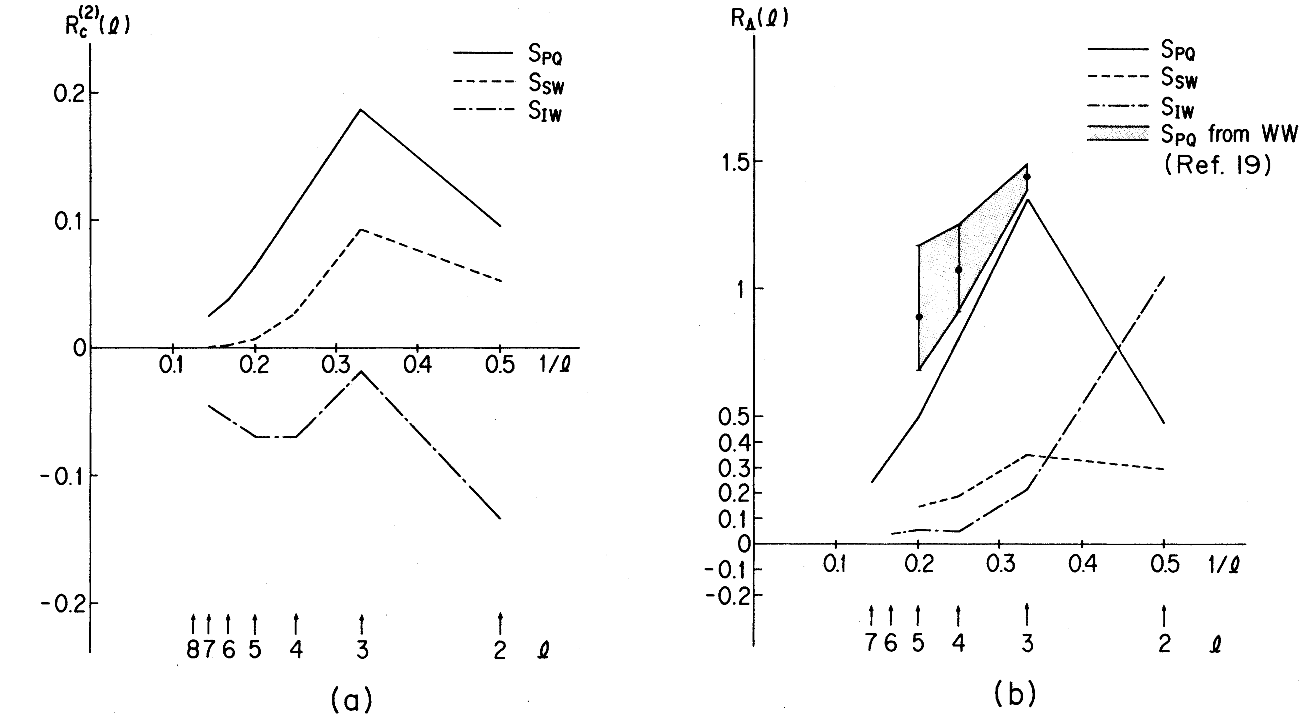

FIG. 2. The lattice artifacts of the Creutz ratio on $L=\infty$ lattice. In order to clarify the lattice artifacts the normalized quantities defined by (3.4) and (3.5) are shown in (a) and (b), respectively.

$$R_c^{(2)}(l)=\frac{\chi_a^{(2)}(l,\infty)-\chi_R^{(2)}(l)}{\chi_R^{(2)}(l)}\ , \tag{3.4}$$

$$R_\Lambda(l)=\frac{\Lambda_a(l,\infty)-\Lambda_a}{\Lambda_a}\ , \tag{3.5}$$

where Λ_a is given by (2.13)–(2.15).

It is observed that they approach zero as l increases as expected. This is another indication that our perturbative calculations are correct.

From Fig. 2(a) we find that $R_c^{(2)}(l)$ of $S_{\rm Iw}$ has different l dependences from those of $S_{\rm pq}$ and $S_{\rm SW}$, and $S_{\rm SW}$ is the most improved action at $O(g^2)$. In Fig. 2(b) we see that all $R_\Lambda(\infty)$ approach zero from above as l increases, and in this case $S_{\rm Iw}$ is the best action. We list the minimum size of a Creutz ratio l_m where the lattice artifacts becomes less than 10% on the $L=\infty$ lattice:

	$\chi_a^{(2)}(l,\infty)$	$\Lambda_a(l,\infty)$	
	5	$\gtrsim 8$	for $S_{\rm pq}$
l_m	2	$\gtrsim 6$	for $S_{\rm SW}$
	3	4	for $S_{\rm Iw}$

(3.6)

From (3.6) we clearly find the effects of improvement of the action. We find that l_m is not so large. However, in order to get rid of size dependences, the calculations on larger L and suitable extrapolation are necessary.

C. Estimation of errors in the extrapolation

In the case of the $O(g^2)$ coefficient, we can carry out the calculation on larger lattices and make extrapolations with these results to get $\chi_a^{(2)}(l,\infty)$, as the CPU time of the calculation is small. The extrapolated results with calculations[11] on $L=20$, 26, and 32 are slightly larger than those of Table I; however, the differences, are less than 1% for $l\le 7$.

Furthermore, there is the calculation of $\chi^{(i)}(l)$ on an $L=\infty$ lattice for $S_{\rm pq}$ (Ref. 19). The results from Wohlert and Weisz[19] (WW) are shown in Tables I(a) and I(b). In $O(g^2)$, the difference is also less than 1% for $l\le 7$ and we therefore think that the error in Table I(a) is less than 1%.

In $O(g^4)$, we estimate the $L\to\infty$ limit in another way. We use the universal L dependences of $R_a^{(i)}(l,L)$ of (3.2) for $i=2$ and 4, found in Fig. 1(b). We think that $\chi^{(2)}(l,\infty)$ obtained by the extrapolation of the $L=20$, 26, and 32 calculations represents the correct $L\to\infty$ limit; then $\chi_a^{(4)}(l,\infty)$ is estimated by the formula

$$\chi_a^{(4)}(l,\infty)=\frac{\chi_a^{(2)}(l,\infty)}{\chi_a^{(2)}(l,L)}\chi_a^{(4)}(l,L)\ . \tag{3.7}$$

The results obtained by (3.7) are equal to those of Table I(b) within 1% for $l\le 7$. Thus we would like to think that in $O(g^4)$ the errors of our extrapolation are less than a few percent at most.

For $S_{\rm pq}$, we compare our $\chi_{\rm pq}^{(4)}(l,\infty)$ obtained by (3.1) with those of WW (Ref. 19). It is found that our results are somewhat smaller than theirs. The difference is about 7.3% at $l=5$, which is much larger than our error estimation. As our results completely agree with those of Heller and Karsch,[13] we think that our results on finite lattices are correct. Then we think of two possibilities for the discrepancy of $\sim 7\%$ stated above: (i) the size dependences are so complicated that the lattice sizes used in the extrapolation formula (3.1) are too small to get the correct results at $L=\infty$; (ii) there are some errors in the results of WW (Ref. 19) or in their error estimations.

In Ref. 19, the authors say that their result for the expectation value of the plaquette is in agreement with di Giacomo and Rossi.[20] But their result is different from the more precise value of di Giacomo and Paffuti,[21] and the latter result is completely equal to ours but the former is not.

In Fig. 2(b) we have also shown the $R_\Lambda(l)$, defined by (3.5), obtained by using $\chi_{\rm pq}^{(4)}(l)$ of WW. It approaches the continuum limit much slower than our results as l increases. Then the limit l_m given by (3.6) becomes larger.

IV. DISCUSSIONS AND COMMENTS

A. Scale transformation on finite lattices

The Creutz ratios calculated on continuous space satisfy a renormalization-group equation; the Creutz ratios are invariant if the scale and the coupling constant is changed according to the renormalization-group equation. On a finite lattice, a natural extension of this transformation may be[13]

$$\chi(l,L;g')\,|_{\delta=1}=\chi(2l,2L,g)\,|_{\delta=2}\ , \tag{4.1}$$

where $g'=g$ (2a) and δ is given by (2.7). This kind of relation, combined with block transformations, is used to determine the β function on a finite lattice by Monte Carlo renormalization-group (MCRG) method.[22]

Perturbatively, Eq. (4.1) is expressed in terms of the coefficients $\chi^{(i)}$ of g_a as

$$\chi_a^{(2)}(2l,2L)=\chi_a^{(2)}(l,L)\ , \tag{4.2}$$

$$\chi_a^{(4)}(2l,2L)=\chi_a^{(4)}(l,L)+2\beta_0\ln 2\chi_a^{(2)}(l,L)\ . \tag{4.3}$$

These relations are satisfied in the $L\to\infty$ and $l\to\infty$ limit. We study how they are satisfied or violated at finite L and l.

In order to test the matching of (4.2), we take the following ratios and show them in Table II(a):

$$R_s^{(2)}(l,L)=\frac{\chi_a^{(2)}(2l,2L)-\chi_a^{(2)}(l,L)}{\chi_a^{(2)}(l,L)}\ . \tag{4.4}$$

It is seen that the $R_s^{(2)}$ are quite small and size dependences are much smaller than those of the Creutz ratios themselves. In $O(g^2)$ we find that the scaling relation is satisfied to within $\sim 10\%$ for $L\geq 6$.

In $O(g^4)$ we use (4.3) as the definition of β_0 at finite L and l, and study how it approaches to the asymptotic value $\beta_0=11/16\pi^2$. In Table II(b) we have shown the ratio

TABLE II. Matching of the Creutz ratio according to renormalization-group equation shown by (4.2) and (4.3). In II(a) and II(b), $R_s^{(2)}(l,L)$ of (4.4) and $R_\beta(l,L)$ of (4.5) are shown, respectively. We have used the results of Ref. 13 for $L=24$.

l \ L	6	8	10	12	∞
			(a)		
			$S_{\rm pq}$		
2	0.062 07	0.059 87	0.059 68	0.058 65	0.059 53
3	−0.104 98	−0.115 28	−0.118 80	−0.119 77	−0.113 55
4		−0.083 23	−0.086 43	−0.111 63	−0.089 64
			$S_{\rm sw}$		
2	0.017 51	0.017 00			0.016 62
3	−0.089 75	−0.083 40			−0.084 22
			$S_{\rm lw}$		
2	0.115 61	0.114 62			0.113 87
3	−0.071 91	−0.051 87			−0.048 40
			(b)		
			$S_{\rm pq}$		
2	0.384 32	0.371 63	0.369 79	0.369 22	0.367 69
3	−0.972 80	−0.817 67	−0.778 17	−0.768 14	−0.772 34
4		−0.806 50	−0.609 53	−0.561 98	−0.581 74
			$S_{\rm sw}$		
2	−0.090 76	−0.086 50			−0.087 84
3	−0.418 64	−0.410 63			−0.342 65
			$S_{\rm lw}$		
2	−0.884 14	−0.858 19			−0.847 37
3	−0.245 59	−0.193 75			−0.204 46

TABLE III. Universality of the perturbative Creutz ratio between actions.

l	$S_{pq}-S_{SW}$ $P^{(2)}(l,16)$	$S_{pq}-S_{SW}$ $P^{(4)}(l,16)$	$S_{pq}-S_{Iw}$ $P^{(2)}(l,16)$	$S_{pq}-S_{Iw}$ $P^{(4)}(l,16)$
4	−0.0788	−0.1853	−0.1651	−0.0664
5	−0.0555	−0.1345	−0.1328	−0.0156
6	−0.0386	−0.1065	−0.1027	−0.1013
7	−0.0296	−0.0996	−0.0852	−0.1379

$$R_\beta(l,L)=\frac{\beta_0(l,L)-\beta_0}{\beta_0}\ . \tag{4.5}$$

It is found that size dependences are larger in this case than in $R_s^{(2)}$ because we take the exponential to get $\beta_0(l,L)$ from (4.3). For all actions considered in this paper, $\beta_0(l,\infty)$ approaches the asymptotic value from below. We also notice that for S_{pq} and S_{SW}, $\beta_0(l,\infty)$ have dips at $l=3$ and $l\simeq 3$, respectively.

B. Universality of Creutz ratio among lattice actions

The universality relations (2.16) and (2.17) are expressed as follows between Creutz ratios of different lattice actions:

$$\chi_A^{(2)}(l,L)=\chi_B^{(2)}(l,L)\ , \tag{4.6}$$

$$\chi_A^{(4)}(l,L)=\chi_B^{(4)}(l,L)+2\beta_0\ln\left[\frac{\Lambda_B}{\Lambda_A}\right]\chi_B^{(2)}(l,L)\ , \tag{4.7}$$

where A and B denote three kinds of actions. These equations are satisfied in the $L\to\infty$ and $l\to\infty$ limit. However, at finite L and l, these universality relations may be satisfied rather well if size effects and artifacts are canceled in both sides of equations.

We take B as S_{pq} and A as S_{SW} and S_{Iw}. In $O(g^2)$ we take the following ratios to see the matching of both sides of Eq. (4.6):

$$P^{(2)}(l,L)=\frac{\chi_A^{(2)}(l,L)-\chi_B^{(2)}(l,L)}{\chi_B^{(2)}(l,L)}\ . \tag{4.8}$$

At $O(g^4)$ we define the ratio (Λ_B/Λ_A) at fixed L and l by (4.7) and also take the ratio in order to see the difference from the asymptotic values, which are given by (2.13)–(2.15):

$$P^{(4)}(l,L)=\frac{(\Lambda_B/\Lambda_A)(l,L)-(\Lambda_B/\Lambda_A)^{\rm asy}}{(\Lambda_B/\Lambda_A)^{\rm asy}}\ . \tag{4.9}$$

In Table III we have shown $P^{(2)}$ and $P^{(4)}$ for $L=16$. It is seen that in $O(g^2)$, the $P^{(2)}$ are smaller for S_{pq} and S_{SW}. This is expected because the $\chi^{(2)}(l,L)$ have similar l and L dependences between these actions. From Table III we notice that the violation of universality between three kinds of actions is much smaller than the difference between lattice and continuum calculations; for example, the difference between $\Lambda_R/\Lambda_{pq}(7,16)$ defined by (3.3) and its asymptotic value of (2.13) extends to 80% contrary to the $\sim 14\%$ of $P^{(4)}(7,16)$ of Table III.

Therefore, the check of universality of physical quantities between different lattice actions is not enough to guarantee the correct continuum limit at least perturbatively. The approximate universality of the Creutz ratio between lattice actions at $L=16$ is realized because the size dependences and artifacts are similar for physical quantities between these actions. Therefore, these lattice effects are largely canceled if we take the ratio of these physical quantities; then the ratios obtained by lattice calculations are close to those of continuous limit even if the physical quantities themselves are largely different from the continuum limit.

Finally we make a few comments. Let us take $R_a^{(4)}(l,L)$ defined by (3.3), as an example. The correct continuum limit corresponds to $R_a^{(4)}(\infty,\infty)=0$. However, we see from Fig. 1(b) that if we take a particular path in (l,L) plane, $l=f(L)$, $R_a^{(4)}(l,L)$ seems to approach an arbitrary negative value even if we limit $L>l$; namely, $R_a^{(4)}(l,L)$ is discontinuous at $l=\infty, L=\infty$. The correct continuum limit is obtained by first taking the limit $L\to\infty$ and second the limit $l/a\to\infty$.

In the case of nonperturbative quantities the limit is generally more complicated because we have one more parameter correlation length ξ. The real continuum limit corresponds to the limit $L\gg l\gg\xi\to\infty$. What is the minimum correlation length ξ_m, from which the physical quantities satisfy asymptotic scaling and the lattice calculations become good approximations to continuous results? Our results on l_m of (3.6) may give an estimation on ξ_m. Because ξ represents the typical size of vacuum fluctuation and if the fluctuation on lattices is as well regarded as those on continuous space, the calculations on the lattice will show scaling and they are good approximations to those on continuum space. The systematic study of size dependences and lattice artifacts on nonperturbative quantities should be carried out to get the estimation on ξ_m.

ACKNOWLEDGMENTS

This work was supported by University of Tsukuba Project Research. One of us (H.I.) would like to thank Iwanami Fujukai for financial support.

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