Bifurcation in the Yang-Mills field equations with static sources

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We argue that the bifurcation phenomenon in the Yang-Mills field equations can be distinguished into weak and strong forms. For the weak form we demonstrate explicitly that there are an infinite number of bifurcating branches emanating from a bifurcation point.

I. INTRODUCTION

Several years ago Mandula[1] showed that the Abelian Coulomb solution of the Yang-Mills (YM) field equations[2] in the presence of a spherically symmetric source is unstable when the external source strength exceeds a certain critical magnitude. Since then there has been interest in the classical solutions of the YM equations with external sources.[3–12] This is due to the fact that quantum chromodynamics is hard to solve while the nonlinear aspect of the YM fields can be readily studied at the classical level. Furthermore stable classical solutions may provide, in the semiclassical approximation, some insight into the full quantized theory. Among the classical solutions so far constructed, the bifurcating solutions, which were first discovered by Jacobs, Jackiw, and Rebbi,[4] are the most fascinating since they are gauge nonequivalent and are sustained by the same nonvanishing external source strength. Multifurcation phenomenon has also been found in the classical YM mechanics, the solutions in fact exhibit a chaotic behavior.[11]

So far, bifurcating solutions of the YM field equations fall into two types. (a) When the external source is specified in the radial gauge frame with Kronecker index equal to one, bifurcating solutions can exist only if the source strength exceeds a nonzero critical value. Apart from the numerical solutions,[4,8] analytic solutions have also been obtained.[9,10] (b) For the external source specified in the Abelian gauge frame with vanishing Kronecker index, the Abelian Coulomb solution exists for all values of the source strength while the stable magnetic multipole solution[3] emerges and branches out only after the source strength acquires some critical nonzero value. This is a pitchforklike bifurcation.[12]

We now examine more closely the analytic bifurcating solutions of type (a). For the analytic solutions of Ref. 9, the energy ξ and the total charge Q reach their respective minimum at the same parametric value. However, at this parametric value, the gauge field is on the verge of becoming imaginary so that one cannot further vary the parameters of the solutions in order to obtain the second branch. The second branch, if it exists, must be due to a different parametrization. In contrast, the analytic solutions of Ref. 10 display explicitly the two branches in the energy ξ versus the total charge Q plot. When the parameters of the solutions are altered, both ξ and Q first decrease and reach their respective minimum at the same parametric value, they finally increase on further varying the parameters and as a result cusplike behavior is exhibited. However, we emphasize that the resulting two branches emanating from the same bifurcation point are in fact due to different charge-density distributions although their total charge is the same. Note that the two branches of Ref. 4 have the same behavior but they correspond to the same charge-density distribution and hence the same total charge because of the choice of a δ function as the external source.

From the above, one naturally raises the question about the meaning of bifurcation: should the two solution branches correspond to the same charge-density distribution or the same total charge? If one adopts the view that when the parameter of a nonlinear differential equation exceeds a certain value, two or more solutions emerge (more details in Sec. IV), then the solutions of Ref. 10 are bifurcating solutions. We shall call these branching solutions with the same total charge but different charge densities as weakly bifurcating solutions, whereas the solutions of the type (b) (Ref. 12) and Ref. 4 associated with the same charge density and necessarily the same total charge we shall call the strongly bifurcating solutions. Thus as a working definition, bifurcation means that the respective minimum of the energy and the total charge occurs at the same parametric value of the solutions. Consequently there can be two branches emanating from the minimum point in the ξ vs Q diagram.

In this paper we present a family of analytic solutions that exhibit the weak bifurcation. We only consider the case when the external source current $j_i^a(x)$ is zero since for $j_i^a(x)\neq 0$, the energy ξ is then gauge dependent and, in addition, the physical origin of $j_i^a(x)$ is not clear.[7] We demonstrate explicitly that from a single bifurcation point it is possible to construct as many pairs of bifurcating branches as one wishes. This leads us to believe that strongly bifurcating solutions may be physically more relevant. In the following section we introduce our notation and our analytic solutions are presented in Sec. III. We discuss bifurcation briefly in Sec. IV and in Sec. V we explicitly construct an infinite number of pairs of bifurcating branches emanating from a single bifurcation point. We end with some remarks in the last section.

II. THE YANG-MILLS EQUATIONS

The SU(2) YM equations in the presence of an external static source are

$$(D_\mu F^{\mu\nu})_a = j_a^\nu = \delta_0^\nu \rho_a \ , \tag{1a}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c \ , \tag{1b}$$

where ρ_a is the external charge density and our metric is $g_{ii} = -g_{00} = 1$. The following radial ansatz[4] is employed

$$A_a^0(\mathbf{r}) = n^a f(x)/(gr), \quad n^a = x^a/r \ , \tag{2a}$$

$$A_i^a(\mathbf{r}) = \epsilon_{iaj} n^j [a(x) - 1]/(gr) \ , \tag{2b}$$

$$\rho_a(\mathbf{r}) = n_a q(x)/(g r_0^3) \ , \tag{2c}$$

where g is the coupling constant of the gauge field, r_0 is a length scale, and $x = r/r_0$. The above ansatz simplifies Eq. (1) to a pair of coupled nonlinear differential equations

$$f^2 = (a^2 - 1) - a'' x^2 / a \ , \tag{3a}$$

$$q = -f''/x + 2a^2 f/x^3 \ . \tag{3b}$$

The prime here means differentiation with respect to x. The total energy and the gauge-invariant total charge are given by, respectively,

$$\xi = \frac{4\pi}{g^2 r_0} \int_0^\infty dx \left[(a')^2 + \frac{1}{2x^2}(a^2-1)^2 + \tfrac{1}{2}(f')^2 + \frac{1}{x^2} f^2 a^2 \right] , \tag{4}$$

$$Q = \int d^3 r \, \eta_a(\mathbf{r}) \rho_a(\mathbf{r}) = \frac{4\pi}{g} \int_0^\infty dx \, x^2 |q(x)| \ , \tag{5}$$

with $\eta^a(\mathbf{r}) = \rho^a(\mathbf{r})/|\rho^a \rho^a|$. Note that if $j_i^a(x) \neq 0$, ξ will be gauge dependent.

We seek solutions with finite total energy ξ and finite total charge Q. For the type-II solutions[4] we demand the following asymptotic behavior:

$$a(x) \approx -1 + a_1/x \ , \tag{6a}$$

$$f(x) \approx f_1/x \ , \tag{6b}$$

at large x, while for small x,

$$a(x) \approx 1 + a_0 x^2 \ , \tag{7a}$$

$$f(x) \approx f_0 x^{3/2} \ . \tag{7b}$$

Here a_i and f_i $(i = 0, 1)$ are constants. Observe that in the asymptotic behavior (7b) we do not require $f(x) \approx x^2$ as in Ref. 4 since we do not require $xq(x)$ to vanish near $x = 0$.

III. EXPLICIT TYPE-II SOLUTIONS

Following the method of Ref. 9, it is not difficult to construct analytic type-II solutions. For our purpose we give the solution

$$a(x) = \frac{-x^n - b_1 x^{n-1} + b_2}{x^n + b_2} \ , \tag{8}$$

where b_1, b_2, and n are real positive parameters. The boundary condition (7a) demands that $n \geq 3$. Since expression (8) has exactly one zero at $x = z$, then the continuity of $f(x)$ at z demands that

$$a''(z) = a(z) = 0 \ . \tag{9}$$

Condition (9) can be used to determine b_1 and b_2 in terms of n and z; we find

$$b_1 = rz \ , \tag{10a}$$

$$b_2 = s z^n \ , \tag{10b}$$

where

$$s = \frac{n(n-1) + [n^2(n-1)^2 + 8(n-1)(n-2)]^{1/2}}{2(n-1)(n-2)} \ , \tag{10c}$$

$$r = s - 1 \ . \tag{10d}$$

Employing Eqs. (8) and (10) in (3a), one can show that $f(x)$ is a real function for $z > 0$ and $n \geq 3$.

Trading the variable x for $y = x/z$, Eqs. (4) and (5) can be written as

$$\xi = \frac{1}{z} \xi_0(n) \ , \tag{11}$$

$$Q = Q_0(n) \ , \tag{12}$$

where $\xi_0(n)$ and $Q_0(n)$ are each a function of n only. From expression (12) we observe that for the type-II solution (8), the parameter n determines the total charge while the charge-density distribution is controlled by the parameter z as well. Thus Eq. (11) says that the energy depends on both the total charge and its distribution, as should be the case. We find by numerical computation that $\xi_0(n)$ is a monotonically increasing function of n $(n \geq 3)$ while $Q_0(n)$ has a minimum value $(4\pi/g)8.9015$ at $n = 3.754$.

In passing, we write down the asymptotic expressions of $f(x)$ and $q(x)$ at small x:

$$f(x) \approx d_1 x^p + d_2 x^{p+1} \ , \tag{13a}$$

$$q(x) \approx [2 - p(p-1)] d_1 x^{p-3} + [2 - p(p+1)] d_2 x^{p-2} \ , \tag{13b}$$

with $p = (n-1)/2$, $d_1{}^2 = (n^3 - 3n) b_1 / b_2$, and $d_2 = (n^2 - n - 2)/[(n^3 - 3n)\, b_1 b_2]^{1/2}$.

IV. BIFURCATION AND STABILITY

Let M be a nonlinear mapping $M: W x \Lambda \to Y$, where W, Λ, and Y are each Banach spaces. Consider the equation

$$M(w, \lambda) = 0, \quad w \in W, \quad \lambda \in \Lambda \ , \tag{14}$$

with the solution set $S \subset W \times \Lambda$ and let $S_\lambda \equiv \{w \in W : (w,\lambda) \in S\}$; we then say that[13] $\lambda = \lambda_0$ is a bifurcation point if $S_{\lambda_0} \neq \phi$ and there exists $w_0 \in S_{\lambda_0}$ such that for any neighborhood U of (w_0, λ_0), there are two distinct solutions $(w_1,\lambda)(w_2,\lambda) \in U$.

In our context, Eq. (14) is to be identified with Eq. (3) where w represents the functions $a(x)$ and $f(x)$ while λ represents the parameters z and n. For easy interpretation we would like to obtain a criterion for bifurcation in terms of the physical quantities ξ and Q, which in our case are constants of motion.

Denoting the linear part of M as L, i.e., the Fréchet derivative of M, then in order for bifurcation to occur, L must be noninvertible and there exists a subspace V of W such that

$$Lv = 0, \quad v \in V \ . \tag{15}$$

In other words L is the Fredholm operator of index zero. If we perform a stability analysis of a solution w_0 then the small fluctuations Δw around w_0 satisfy[5]

$$L\Delta w = \omega^2 \Delta w \ , \tag{16}$$

when we vary the parameter λ, a stable fluctuation ($\omega^2 > 0$) may become unstable ($\omega^2 < 0$). At the critical point the stability equation yields a zero-eigenvalue mode ($\omega^2 = 0$) and Eq. (16) becomes Eq. (15). The total energy and charge then assume their respective minimum value at this point. We make use of this fact to search for the bifurcation point of the solutions.

V. BIFURCATING SOLUTIONS

We now apply the result of the previous section to show that the type-II solution of Sec. III is weakly bifurcating. We parametrize z in terms of n, that is $z = z(n)$. Then from expressions (11) and (12), the occurrence of the common minimum of ξ and Q demands a nontrivial solution for

$$z'(n) \equiv \frac{dz(n)}{dn} = \frac{z(n)}{\xi_0(n)} \frac{d\xi_0(n)}{dn} \ , \tag{17a}$$

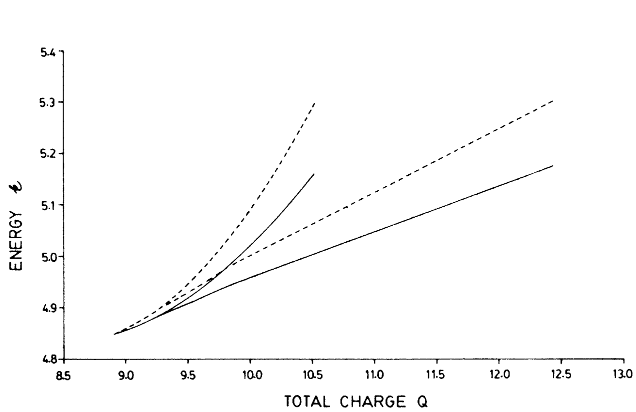

FIG. 1. Energy vs total external charge Q. The solid curve corresponds to the parametrization (18) while the dashed curve has the constraint (21). Here $m = 1$.

$$\frac{dQ_0(n)}{dn} = 0 \ . \tag{17b}$$

From Sec. III, we know that (17b) has the unique solution $n = \bar{n} \equiv 3.754$ and that the charge is a minimum there. Thus (17a) gives us the critical slope

$$z'(\bar{n}) = kz(\bar{n}) \ , \quad k = \frac{1}{\xi_0(\bar{n})} \frac{d\xi_0(\bar{n})}{dn} = 0.4164 \ . \tag{17c}$$

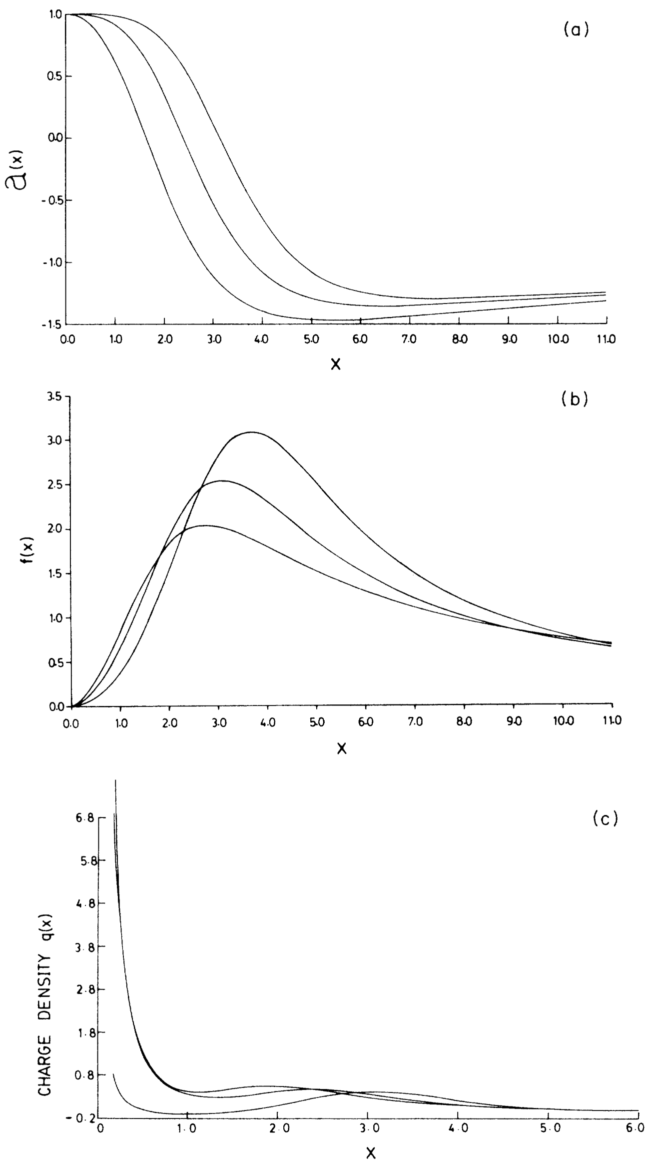

FIG. 2. (a) The function $a(x)$ for the solution (8) with the relation (18) and $m = 1$. Starting from the curve with the lowest value at $x = 3$, these correspond to $n = 3$, 3.754, and 4.5. (b) The function $f(x)$ for the solution (8) with the relation (18) and $m = 1$. Starting from the curve with the lowest peak value, these correspond to $n = 3$, 3.754, and 4.5. (c) The charge density $q(x)$ for the solution (8) with the relation (18) and $m = 1$. Starting from the curve with the highest value at $x = 2$, these correspond to $n = 3$, 3.754, and 4.5.

To satisfy Eqs. (17) it is sufficient to impose a linear relation of the form

$$z_{\rm I}(n)=mn+c \ . \tag{18}$$

Using Eq. (18) in (17c), we determine c in terms of m:

$$c=m\left[\frac{1}{k}-\bar{n}\right]=-1.35m \ . \tag{19}$$

In order for the energy to assume a minimum value at $n=\bar{n}$, we require that $\xi''(\bar{n})>0$. Using Eqs. (11), (18), and (19) together with the fact that k and $\xi_0''(\bar{n})$ are positive, one easily shows that $\xi''(\bar{n})>0$ if and only if $m>0$. The minimum value of the energy is

$$\xi(\bar{n})=\xi_0(\bar{n})/z_{\rm I}(\bar{n})=\xi_0'(\bar{n})/m \ . \tag{20}$$

Hence each positive value of m provides us a relation (18) between z and n that causes the solution (8) to have a bifurcation point at $n=\bar{n}$. Equation (20) shows that the bifurcation point in the ξ-Q plane depends on the value of m. Figure 1 shows a plot of ξ vs Q for the case $m=1$. The corresponding solutions $a(x)$ and $f(x)$ and the external charge density $q(x)$ near the bifurcation point are displayed in Figs. 2(a), 2(b), and 2(c), respectively.

Let us now fix our attention on the single bifurcation point determined by $m=1$ in (18) and (19). We want to show that there are infinitely many distinct relations between z and n that will cause the solution (8) to bifurcate from the same bifurcation point. Let us choose

$$z_{\rm II}(n)=\alpha n^2+\beta n+\gamma \ , \tag{21}$$

where α, β, and γ are constants. In order for the relation (21) to produce bifurcation at $n=\bar{n}$ with the same value of energy and total charge as relation (18) with $m=1$, it is clear from Eqs. (11) and (17a) that the necessary conditions are

$$z_{\rm II}(\bar{n})=z_{\rm I}(\bar{n})=1/k \ , \tag{22a}$$

$$z_{\rm II}'(\bar{n})=z_{\rm I}'(\bar{n})=1 \ . \tag{22b}$$

The above conditions allow us to express α and β in terms of γ: namely,

$$\alpha=\frac{1}{\bar{n}}-\frac{1}{\bar{n}^2}\left[\frac{1}{k}-\gamma\right] , \tag{23a}$$

$$\beta=\frac{2}{\bar{n}}\left[\frac{1}{k}-\gamma\right]-1 \ . \tag{23b}$$

We also evaluate $\xi''(\bar{n})$ for the constraint (21):

$$\xi''(\bar{n})=k[\xi_0''(\bar{n})-2\alpha k\xi_0(\bar{n})] \ . \tag{24}$$

For the energy to be a minimum at $n=\bar{n}$, we require that $\xi''(\bar{n})>0$. From (24), we see that it is sufficient to choose $\alpha\le 0$ or equivalently from (23a) to choose

$$\gamma\le\left[\frac{1}{k}-\bar{n}\right]=-1.35 \ . \tag{25}$$

In fact the choice $\alpha=0$, $\gamma=1/k-\bar{n}$ reduces (21) to (18) with $m=1$. Therefore the original relation (18) is included in the infinite family of relations given by Eqs. (21), (23), and (25). Each relation of the form (21) is labeled by the continuous index γ satisfying (25), and it renders the solution (8) to weakly bifurcate at $n=\bar{n}$ with $Q_{\min}=Q_0(\bar{n})$ and $\xi_{\min}=\xi_0'(\bar{n})$. In Fig. 1, the dashed curve corresponds to the case $\gamma=-1/k$. The curves for $a(x)$, $f(x)$, and $q(x)$ are very similar to those for Figs. 2(a)–2(c) and are not shown. We remark that since for the type-II solution (8) we must have $z>0$, the range of allowed values of n will be constrained for each of the relations (21).

Equations (21), (23), and (25) refer to the case when $m=1$ in Eqs. (18) and (19). For the general case $m>0$ in (18) and (19) expressions (23) and (25) are replaced by

$$\alpha=\frac{m}{\bar{n}}-\frac{1}{\bar{n}^2}\left[\frac{m}{k}-\gamma\right] , \tag{26a}$$

$$\beta=\frac{2}{\bar{n}}\left[\frac{m}{k}-\gamma\right]-m \ , \tag{26b}$$

$$\gamma\le m\left[\frac{1}{k}-\bar{n}\right] . \tag{26c}$$

We summarize the results. The type-II solution (8) has infinitely many weak-bifurcation points. The bifurcation points can be labeled by the continuous index $m>0$ with the energy and total charge at the bifurcation point given by $Q_{\min}=Q_0(\bar{n})$, $\xi_{\min}=\xi_0'(\bar{n})/m$. Each bifurcation point (m fixed) can be arrived at by infinitely many distinct relations between z and n, labeled by the continuous index γ and given by Eqs. (21) and (26). Each of the distinct relations produces a different pair of curves $\xi(Q)$ in the ξ-Q plane.

VI. COMMENTS

(a) It is possible that even though the energy ξ and total charge Q have their local minima at the same values of the parameters, a plot of ξ vs Q shows only one branch. This can happen when there is a one to one relationship between the energy ξ and Q. The two branches are then actually degenerate. As an explicit example, consider the following monotonic relation imposed between the energy and total charge as given by Eqs. (11) and (12):

$$\xi=\frac{1}{z}\xi_0(n)\equiv Q_0(n) \ . \tag{27}$$

Equation (27) induces a relation between z and n

$$z=\frac{\xi_0(n)}{Q_0(n)}, \quad n\ge 3 \ . \tag{28}$$

Thus the relation (28) produces a minimum of energy and total charge at $n=\bar{n}\equiv 3.754$ and above this bifurcation point the two branches of the ξ vs Q curve are degenerate because of the one to one relation between ξ and Q given by (27).

(b) The procedures outlined and demonstrated in Secs. IV and V can be used for expressions of $a(x)$ other than solution (8). However, in general, one may not be able to obtain closed-form expressions of the ξ and Q in terms of the parameters, or even a simplification as in expression Eqs. (11) and (12). Nevertheless one can still search for the bifurcation point numerically by trying a relation between the parameters.

(c) The construction of weakly bifurcating solutions as described in Secs. IV and V depends crucially on the imposition of a relation between the originally independent parameters. In Sec. III we stated that for the solution (8) the parameter n determines the total charge and for a fixed total charge, z determines how that charge is distributed. Choosing a relation between z and n as in Sec. V to obtain weak bifurcation then means that we are imposing a relation between the total charge and the charge density.

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