Bound-state problem in quantum field theory: Linear and nonlinear dynamics

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We study bound-state solutions of a simple Lagrangian containing two coupled scalar fields (the Wick-Cutkosky model). In this work we obtain (nontopological) soliton solutions which are nonperturbative in character and which can be studied for large values of the coupling constant. We provide a unified approach to this problem by starting with a Bethe-Salpeter equation and demonstrating that different choices of the kernel will lead to either the usual "ladder approximation" or to the nonlinear equations of the soliton analysis.

I. INTRODUCTION

A problem which has received a good deal of attention over the last decade is the construction of models to describe hadron structure in quantum chromodynamics (QCD). There are a large number of competing models. These include bag models, chiral bag models, nonrelativistic potential models, nontopological and topological soliton models, and models based upon the use of the Bethe-Salpeter equation. Usually these models involve the introduction of an effective Lagrangian which is written in terms of a number of effective fields. (The only model which attempts to use the degrees of freedom which appear in the QCD Lagrangian is that based upon an explicit construction of the hadronic Fock-space wave function in terms of operators defined using light-cone variables.[1])

On the whole, the relation between these various approaches to the bound-state problem is obscure. In this work we do not wish to choose between the various models of hadron structure. We do, however, wish to discuss some relation between the theory of nontopological solitons and Bethe-Salpeter dynamics. More precisely, we will consider a specific Lagrangian involving two scalar fields (the Wick-Cutkosky model[2]) and approach the problem of constructing bound-state solutions using two distinct factorizations of the field equations of the model. [Ultimately we are interested in discussing hadron structure. For example, we might consider a valence quark and antiquark bound by some field to produce a meson. In this work we wish to avoid the complications introduced by the quark spin. Therefore we limit our considerations to the interaction of two scalar fields: $\phi(x)$ and $\chi(x)$. Our interest in fermion dynamics leads us to consider a Hilbert space in which two quanta of the ϕ field are present at all times. The role of this constraint in deriving the dynamical equations of the model will become clear as we proceed.]

As we will see, one factorization of the matrix elements of the field equations leads to a *linear* equation, which is essentially the Bethe-Salpeter equation[3] (in the ladder approximation) with one particle kept on its mass shell. The other approach leads to a *nonlinear* version of the Bethe-Salpeter equation in which the kernel of the equation is a functional of the solution. An equation of the latter type can be solved by iteration and provides a *mean-field approach* to the bound-state problem.[4] This latter approach is nonperturbative and therefore can have a different range of validity when compared to the linear Bethe-Salpeter analysis.

The model considered here is described in detail in Sec. II. The result of our analysis can be given a pictorial representation, which we believe to be instructive. In Fig. 1(a) we represent the Bethe-Salpeter equation with an arbitrary kernel K. In Fig. 1(b) we exhibit that choice for K which leads to the equation depicted in Fig. 1(d). (This choice is what is usually termed the "ladder approximation" and may be considered to be a weak-coupling approximation which involves a selected summation of diagrams of perturbation theory.) In Fig. 1(c) we denote an approximation for the kernel which depends upon the vertex function, represented by the shaded triangle. The motivation for that choice will become clear as we proceed. The resulting equation may be depicted as in Fig. 1(e), where the nonlinear character of the equation is apparent. An alternate representation of the nonlinear equation is given in Fig. 2(a). There the open circle denotes a form factor of the bound state. The form factor is defined pictorially in Fig. 2(b). We can denote the binding field (wavy line) by χ and consider the bound particles to be quanta of a field $\phi(x)$. In Fig. 2 we see that the hadron itself is the source of the binding field $\chi(x)$ in which the particles of the ϕ field move.[4]

It should be clear that the "physics" described by the linear and nonlinear equations is quite different. For the linear case one considers the χ field to be only weakly excited. The ϕ particles exchange the quanta of that field. In the nonlinear model, which here represents one application of the theory of nontopological solitons, one can consider the χ field to be semiclassical in nature with *many* χ quanta playing a role in the dynamics. We recall that in the usual formulation of the theory of solitons, one studies the *classical* Euler-Lagrange equations. These equations may or may not have soliton solutions of topological or nontopological character.[5] Our *non-*

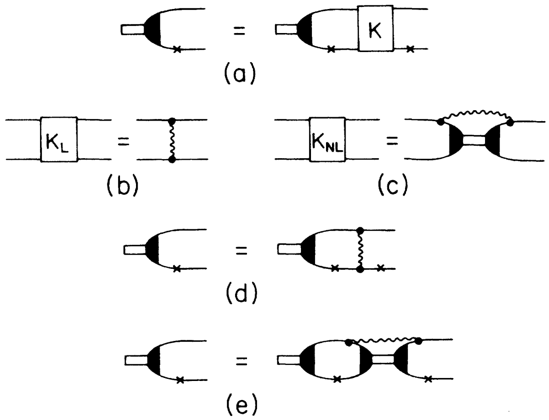

FIG. 1. Diagrammatic representation of the linear and nonlinear equations considered in this work. (a) The Bethe-Salpeter equation for bound states. Here K denotes the kernel and the solid triangle is the vertex function. (The cross denotes an on-mass-shell particle.) (b) The choice for K which defines the "ladder approximation." (c) The choice for K which leads to the nonlinear equations of this work. (d) The Bethe-Salpeter equation in the ladder approximation. (e) Nonlinear equation obtained from (a) when the kernel of (c) is used.

linear equation is rather closely related to the classical Euler-Lagrange equations of the theory; however, to exhibit that correspondence one has to consider a static limit of our formalism.

A general comment is in order here. If the coupling constant is small, the Bethe-Salpeter equation in the ladder approximation can have meaning within the context of perturbation theory. However, there is nothing to prevent one from solving the Bethe-Salpeter equation for large values of the coupling constant and that has often been done.[3] In this work we discuss an alternative solution for large values of the coupling based upon a nonlinear formalism. One virtue of our approach is that it is based upon semiclassical approximations which may be valid at strong coupling. In principle, one can improve upon our approach by calculating corrections to the "no-loop" approximation in the theory of soliton dynamics.[6]

The plan of our work is as follows. In Sec. II we derive both the linear and nonlinear versions of our equations. In Sec. III these equations are reexpressed in terms of dimensionless variables and in Sec. IV we describe the results of numerical computation. Section V contains some concluding remarks.

II. LINEAR AND NONLINEAR BOUND-STATE DYNAMICS

We consider the Lagrangian

$$\mathcal{L}(x)=\tfrac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x)-\frac{m^2}{2}\phi^2(x)+\tfrac{1}{2}\partial_\mu\chi(x)\partial^\mu\chi(x)-\frac{\lambda^2}{2}\chi^2(x)+\frac{g}{2!}\phi^2(x)\chi(x) \tag{2.1}$$

which describes the coupling of two scalar fields $\phi(x)$ and $\chi(x)$. As in the work of Wick and Cutkosky,[2] we will limit our considerations to a Hilbert space in which two particles of the ϕ field are present. These particles can form a bound state via their coupling to the field $\chi(x)$. As we discussed in the Introduction, a central goal of this work is to compare two different approaches to the bound-state problem: (1) linear Bethe-Salpeter dynamics; (2) nontopological-soliton dynamics. In order to minimize the complexity of the analysis, we will simplify the model by keeping one of the constituents on mass shell. This can be achieved by considering a matrix element of the scalar field $\phi(x)$ at $x_\mu=0$:

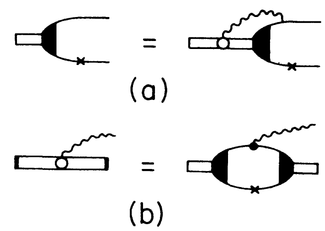

FIG. 2. Alternate diagrammatic representation of the nonlinear equation depicted in Fig. 1(e). (a) The solid triangles are vertex functions and the open circle is a form factor. The wavy line represents the propagator of the field χ. (b) Diagrammatic representation of the form factor of (a) expressed in terms of the vertex functions (solid triangles).

$$\langle \mathbf{k}\,|\,\phi(0)\,|\,\mathbf{p}\rangle=\left[\frac{1}{(2\pi)^3 2E(\mathbf{p})}\right]^{1/2}\left[\frac{1}{(2\pi)^3 2\omega(\mathbf{p})}\right]^{1/2}\times A(k\cdot p)\ , \tag{2.2}$$

where the normalization of the (on-mass-shell) constituent state $|\,\mathbf{k}\rangle$ and the bound state $|\,\mathbf{p}\rangle$ are

$$\langle \mathbf{k}'\,|\,\mathbf{k}\rangle=\delta(\mathbf{k}-\mathbf{k}') \tag{2.3a}$$

and

$$\langle \mathbf{p}'\,|\,\mathbf{p}\rangle=\delta(\mathbf{p}'-\mathbf{p})\ . \tag{2.3b}$$

The energies of the bound state (with mass M) and the on-mass-shell constituent are given by

$$E(\mathbf{p})=\sqrt{\mathbf{p}^2+M^2} \tag{2.4a}$$

and

$$\omega(\mathbf{k})=\sqrt{\mathbf{k}^2+m^2}\ . \tag{2.4b}$$

Since one of the constituents is on mass shell, the bound-state wave function $A(p\cdot k)$ is a function of a single variable $p\cdot k$. In order to find an equation of motion for $A(p\cdot k)$, one may consider the operator equations

$$(\partial^2+m^2)\phi(x)=g\phi(x)\chi(x)\ , \tag{2.5a}$$

$$(\partial^2+\lambda^2)\chi(x)=\frac{g}{2}\phi^2(x)\ , \tag{2.5b}$$

and take matrix elements of these equations using the states $|\,\mathbf{p}\rangle$ and $|\,\mathbf{k}\rangle$. Then, Eq. (2.5a) becomes

$$[-(p-k)^2+m^2]\langle \mathbf{k}\,|\,\phi(0)\,|\,\mathbf{p}\rangle=g\langle \mathbf{k}\,|\,\chi(0)\phi(0)\,|\,\mathbf{p}\rangle\ , \tag{2.6a}$$

or

$$[-(p-k)^2+m^2]\langle \mathbf{k} \mid \phi(0) \mid \mathbf{p}\rangle = g\langle \mathbf{k} \mid \phi(0)\chi(0) \mid \mathbf{p}\rangle \; . \tag{2.6b}$$

In the following these equations will be analyzed using different approximations. For example, we may write

$$\langle \mathbf{k} \mid \chi(0)\phi(0) \mid \mathbf{p}\rangle \simeq \int \langle \mathbf{k} \mid \chi(0) \mid \mathbf{k}'\rangle d\mathbf{k}'\langle \mathbf{k}' \mid \phi(0) \mid \mathbf{p}\rangle \tag{2.7a}$$

or

$$\langle \mathbf{k} \mid \phi(0)\chi(0) \mid \mathbf{p}\rangle \simeq \int \langle \mathbf{k} \mid \phi(0) \mid \mathbf{p}'\rangle d\mathbf{p}'\langle \mathbf{p}' \mid \chi(0) \mid \mathbf{p}\rangle \; , \tag{2.7b}$$

where $|\mathbf{p}'\rangle$ and $|\mathbf{k}'\rangle$ are intermediate states of the bound system and the on-mass-shell constituent, respectively. As will become clear, these two approximations lead to different dynamical descriptions of bound states of the same Lagrangian.

A. Bethe-Salpeter dynamics: Linear model

If we use Eq. (2.7a) in Eq. (2.6a), we find

$$[-(p-k)^2+m^2]\langle \mathbf{k} \mid \phi(0) \mid \mathbf{p}\rangle = g\int d\mathbf{k}'\langle \mathbf{k} \mid \chi(0) \mid \mathbf{k}'\rangle\langle \mathbf{k}' \mid \phi(0) \mid \mathbf{p}\rangle \; . \tag{2.8}$$

Utilizing Eq. (2.5b), one can obtain $\langle \mathbf{k} \mid \chi(0) \mid \mathbf{k}'\rangle$:

$$\langle \mathbf{k} \mid \chi(0) \mid \mathbf{k}'\rangle = \frac{g}{-(k-k')^2+\lambda^2}\left[\frac{1}{(2\pi)^3 2\omega(\mathbf{k})}\right]^{1/2} \times \left[\frac{1}{(2\pi)^3 2\omega(\mathbf{k}')}\right]^{1/2} . \tag{2.9}$$

Therefore, the equation of motion for $A(p\cdot k)$ is found to be

$$[-(p-k)^2+m^2]A(p\cdot k) = \frac{g^2}{(2\pi)^3}\int d^4l \frac{\delta(l^2-m^2)\theta(l^0)}{-(k-l)^2+\lambda^2}A(p\cdot l) \; . \tag{2.10}$$

One can easily prove that this equation is the same as the ladder approximation to the Bethe-Salpeter equation, when one constituent is placed on mass shell. Furthermore, in the nonrelativistic limit, Eq. (2.10) becomes, in the frame where $\mathbf{p}=0$,

$$\left[\frac{\mathbf{k}^2}{2\mu}+\epsilon_B\right]A(\mathbf{k}) = 4\pi\alpha\int \frac{d\mathbf{l}}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{l})^2+\lambda^2}A(\mathbf{l}) \; . \tag{2.11}$$

Here $\mu=m/2$ is the reduced mass; the binding energy is $\epsilon_B=2m-M$, and $\alpha=g^2/(16\pi m^2)$. We see that Eq. (2.11) is a momentum-space Schrödinger equation where the interaction is a Yukawa potential.

B. Nonlinear model (soliton dynamics)

If we use the factorization of Eq. (2.7b) in Eq. (2.6b), we find

$$[-(p-k)^2+m^2]\langle \mathbf{k} \mid \phi(0) \mid \mathbf{p}\rangle = g\int d\mathbf{p}'\langle \mathbf{k} \mid \phi(0) \mid \mathbf{p}'\rangle\langle \mathbf{p}' \mid \chi(0) \mid \mathbf{p}\rangle \; . \tag{2.12}$$

Utilizing Eq. (2.5b), one can obtain

$$\langle \mathbf{p}' \mid \chi(0) \mid \mathbf{p}\rangle = \frac{g/2}{-q^2+\lambda^2}\left[\frac{1}{(2\pi)^3 2E(\mathbf{p})}\right]^{1/2} \times \left[\frac{1}{(2\pi)^3 2E(\mathbf{p}')}\right]^{1/2} 8F(q^2) \; , \tag{2.13}$$

where $q^2 \equiv (p-p')^2$; the form factor $F(q^2)$ is defined by

$$\langle \mathbf{p}' \mid :\phi(0)\phi(0): \mid \mathbf{p}\rangle \equiv \frac{8}{(2\pi)^3 2\sqrt{E(\mathbf{p})E(\mathbf{p}')}}F(q^2) \; . \tag{2.14}$$

When $q^2=0$, we find that, in the nonrelativistic limit, we have

$$F(0)=1 \; . \tag{2.15}$$

We can normalize the wave function $A(k\cdot p)$ by making use of the number operator

$$N_{\rm op} = \int d\mathbf{x}\, \phi^{(-)}(x) i\overleftrightarrow{\partial}_0 \phi^{(+)}(x) \; , \tag{2.16}$$

and its matrix element

$$\langle \mathbf{p}' \mid N_{\rm op} \mid \mathbf{p}\rangle = 2\delta(\mathbf{p}-\mathbf{p}') \; . \tag{2.17}$$

Substituting Eq. (2.2) into Eqs. (2.12) and (2.14), and combining Eqs. (2.12) and (2.13), one finds an equation of motion for $A(p\cdot k)$:

$$[-(p-k)^2+m^2]A(p\cdot k) = \frac{g^2}{4\pi^3}\int d^4q\, \delta(2p\cdot q+q^2)\theta(p^0+q^0)\frac{F(q^2)}{\lambda^2-q^2}A((p+q)\cdot k) \; , \tag{2.18}$$

where

$$F(q^2) = \frac{1}{16\pi^3}\int d^4k\, \delta(k^2-m^2)\theta(k^0) \times A((p+q)\cdot k)A(p\cdot k) \; . \tag{2.19}$$

Using Eqs. (2.16) and (2.17), we see that, in the nonrelativistic limit $[\omega(k)\simeq m \simeq M/2]$,

$$\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega(\mathbf{k})}\left[1-\frac{\omega(\mathbf{k})}{E(\mathbf{p})}\right] |A(p\cdot k)|^2 \simeq \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega(\mathbf{k})}|A(p\cdot k)|^2 = 2 \; . \tag{2.20}$$

We have constructed a nonlinear integral equation [see Eqs. (2.18) and (2.19)] which may be compared to the linear equation obtained previously [Eq. (2.10)].

In the nonrelativistic limit, Eqs. (2.18) and (2.19) become

$$\left[\frac{\mathbf{k}^2}{2\mu}+\epsilon_B\right]A(\mathbf{k})=8\pi\alpha\int\frac{d\mathbf{q}}{(2\pi)^3}\frac{F(-q^2)}{\mathbf{q}^2+\lambda^2}A\left[\mathbf{k}-\frac{\mathbf{q}}{2}\right] \tag{2.21}$$

and

$$F(-\mathbf{q}^2)=\frac{1}{8m}\int\frac{d\mathbf{k}}{(2\pi)^3}A(\mathbf{k})A\left[\mathbf{k}-\frac{\mathbf{q}}{2}\right], \tag{2.22}$$

in the frame where $\mathbf{p}=0$.

III. DIMENSIONLESS VARIABLES

It is useful to write the linear and nonlinear equations, obtained in the last section, in terms of dimensionless variables. We define $x=|\mathbf{k}|/m$, $y=|l|/m$, $r=\lambda/m$, and $\epsilon=\epsilon_B/m\equiv 2-M/m$. We now present the equations of motion in terms of these variables for several cases.

A. Linear model

1. Nonrelativistic limit of the linear model

After performing the angular integration, Eq. (2.11) becomes

$$(x^2+\epsilon)xA(x)=\frac{\alpha}{2\pi}\int dy\,\ln\left|\frac{(x+y)^2+r^2}{(x-y)^2+r^2}\right|yA(y)\,. \tag{3.1}$$

2. One constituent on mass shell (Bethe-Salpeter equation)

The equation

$$[2(\sqrt{1+x^2}-1)+\epsilon]xA(x)=\frac{\alpha}{(2-\epsilon)\pi}\int dy\frac{y}{(1+y^2)^{1/2}}\ln\left|\frac{r^2-2+2\sqrt{1+x^2}\sqrt{1+y^2}+2xy}{r^2-2+2\sqrt{1+x^2}\sqrt{1+y^2}-2xy}\right|A(y) \tag{3.2}$$

is obtained after performing the angular integration in Eq. (2.10).

B. Nonlinear model

1. Nonrelativistic limit of the nonlinear model

We have

$$(x^2+\epsilon)A(x)=\frac{16\alpha}{\pi}\int_0^\infty dy\int_{-1}^{1}dz\frac{F(-4\xi^2(x,y,z))}{4\xi^2(x,y,z)+r^2}y^2A(y)\,, \tag{3.3}$$

where

$$\xi(x,y,z)=\sqrt{x^2-2xyz+y^2} \tag{3.4}$$

and

$$F(-4\xi^2)=\frac{1}{32\pi^2}\int_0^\infty d\omega\int_{-1}^{1}d\zeta\,\omega^2A(\omega)A(\sqrt{\omega^2-2\omega\xi\zeta+\xi^2})\,. \tag{3.5}$$

2. One constituent on mass shell (nonlinear model)

We find

$$[2(\sqrt{1+x^2}-1)+\epsilon]A(x)=\frac{8\alpha}{\pi}(2-\epsilon)\int_0^\infty dy\frac{y^2}{\sqrt{1+y^2}}\int_{-1}^{1}dz\left[\frac{\sqrt{1+x^2}+\sqrt{1+y^2}}{\sqrt{1+x^2}\sqrt{1+y^2}+1-xyz}\right]^2 \times\frac{F(-(2-\epsilon)^2\xi_r^2(x,y,z))}{(2-\epsilon)^2\xi_r^2(x,y,z)+r^2}A(y)\,, \tag{3.6}$$

where

$$\xi_r(x,y,z)=\left[\frac{2(\sqrt{1+x^2}+\sqrt{1+y^2})^2}{\sqrt{1+x^2}\sqrt{1+y^2}+1-xyz}-4\right]^{1/2}, \tag{3.7}$$

and

$$F(-(2-\epsilon)^2\xi_r{}^2(x,y,z))=\frac{1}{32\pi^2}\int_0^\infty d\omega\frac{\omega^2}{\sqrt{1+\omega^2}}A(\omega)\int_{-1}^{1}d\zeta\,A\left[\left\{\left[1+\frac{\xi_r{}^2}{2}\right]\sqrt{1+\omega^2}-\left[\left[1+\frac{\xi_r{}^2}{2}\right]^2-1\right]^{1/2}(\omega\zeta)\right\}^2-1\right]^{1/2}. \tag{3.8}$$

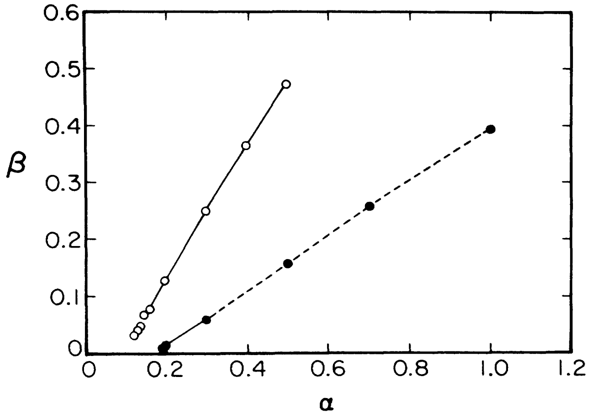

FIG. 3. The quantity $\beta=[1-(M/2m)^2]^{1/2}$, for various values of α. Results for the (linear) Schrödinger equation are given as solid circles, while the results for the nonlinear (soliton) equation are given as open circles. (All calculations are for $r=\lambda/m=0.1$.) For *small binding*, β is the square root of the binding energy, $\epsilon_B=2m-M$, in units of m.

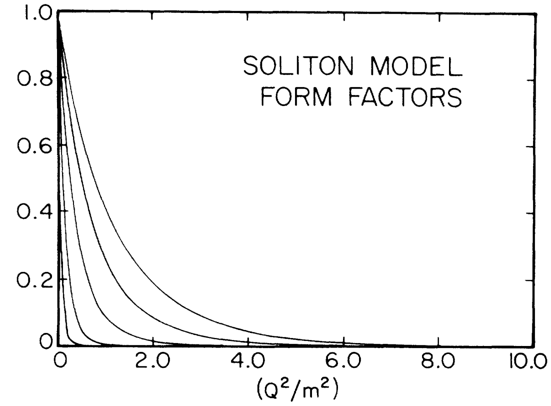

FIG. 5. Form factors calculated using the solutions of the nonlinear (soliton) model. Curves at $Q^2=0$, having progressively small magnitudes for their slopes, correspond to $\alpha=0.15$, 0.2, 0.3, 0.4, and 0.5, and to the wave functions shown in Fig. 4.

One can easily see that $\xi_r(x,y,z)\rightarrow\xi(x,y,z)$ in the nonrelativistic limit.

IV. RESULTS OF NUMERICAL CALCULATIONS

In this work we will present results of calculations for the nonrelativistic version of our dynamical equations. We will contrast the results obtained in the linear and nonlinear models.

In Fig. 3 we present values of the quantity

$$\beta=\left[1-\left[\frac{M}{2m}\right]^2\right]^{1/2} \tag{4.1}$$

plotted versus the coupling constant α. (Note that for small binding, β is the square root of the binding energy in units of m.) The calculation was carried out for $r=\lambda/m=0.1$. The solid circles are the results for the linear model and the open circles are the results for the soliton model. (The lines are drawn as a guide.) For $\alpha>0.3$, the line passing through the solid circles is drawn as a dashed line. The solid portion of the line might be considered to be the result of a perturbative analysis. However, as the coupling constant is made larger, perturbation theory is not applicable. We can still solve the Bethe-Salpeter equation for large values of the coupling constants, and the results are shown in the figure.

In Fig. 4 we present the soliton coordinate-space wave functions as a function of the dimensionless variable $\eta=m\rho$, where ρ is the radial coordinate. These wave functions are presented for $\alpha=0.15$, 0.2, 0.3, 0.4, and 0.5. It is clear that for fixed m, the bound-state radius decreases as α is increased.

In Fig. 5 we show the form factors $F(Q^2/m^2)$ of the nonlinear model for $\alpha=0.15$, 0.2, 0.3, 0.4, and 0.5. As expected, the form factors for the larger objects (i.e., smaller α) decrease more rapidly with increasing momentum transfer. The form factors have been set

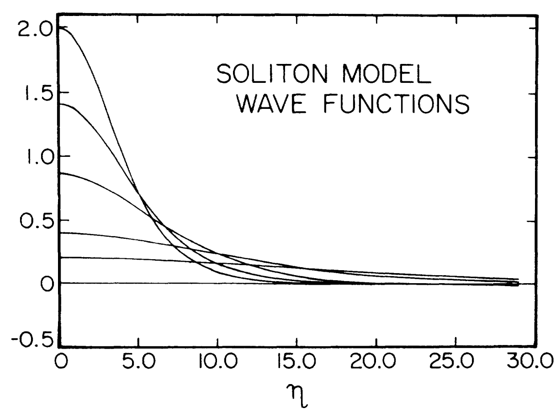

FIG. 4. Radial wave functions for the nonlinear (soliton) model for various values of α, plotted against a dimensionless radial coordinate $\eta=(m\rho)$. For $\eta=0$, curves with progressively larger values at the origin are calculated for $\alpha=0.15$, 0.2, 0.3, 0.4, and 0.5. (The vertical scale is chosen arbitrarily.)

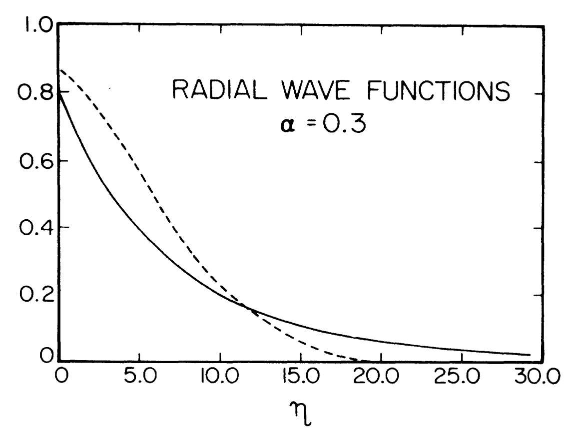

FIG. 6. Radial wave functions for the case $\alpha=0.3$ are compared for the linear Schrödinger equation (solid line) and the nonlinear soliton model (dashed line). (The vertical scale is chosen arbitrarily.)

equal to unity at the origin, which is the correct normalization in the nonrelativistic limit, $\omega(\mathbf{k}) \simeq m \simeq M/2$.

In Fig. 6 we compare the wave functions obtained from the linear Schrödinger equation with the soliton radial wave function for the case $\alpha = 0.3$. Here we see that for a fixed value of α, the radius of the bound state in the case of the soliton solution is significantly larger than the bound-state radius calculated from the solution to the linear equation.

In Table I we present the root-mean-square value of $|\mathbf{k}|/m$ for the momentum-space wave functions of the linear and nonlinear model. It may be seen that, for the same value of α, the solution in the case of the nonlinear model has a larger value for the rms value of $|\mathbf{k}|/m$. It is also clear that for the larger values of α relativistic corrections to these results will become more significant.

TABLE I. The root-mean-square value of $(\mathbf{k}/m)$ for the bound-state wave functions obtained for the nonlinear (soliton) model and for the linear (Schrödinger) equation.

	α	$\langle (|\mathbf{k}|/m)^2 \rangle^{1/2}$
Soliton	0.125	0.043
equation	0.15	0.068
	0.20	0.109
	0.30	0.181
	0.40	0.252
	0.50	0.319
Linear	0.20	0.049
equation	0.30	0.117
	0.50	0.232
	0.70	0.348
	1.00	0.488

V. DISCUSSION

We have presented a unified approach to the solution of the Bethe-Salpeter equation in the weak- and strong-coupling regimes. In general, we should discuss the question of convergence of the solution. When using perturbation theory, we can calculate more complicated forms for the kernel, including terms of order α^2, α^3, etc. In the case of the soliton solutions, which we see as a strong-coupling or mean-field approximation, one can also attempt to calculate corrections in a systematic fashion. (If we were developing the theory in the static limit, we could make a loop expansion, the mean-field analysis being the "no-loop" or "tree approximation.") Single-loop corrections to (static) soliton models of hadrons are presently being studied by a number of researchers.[6] We reserve such investigations for future research.

ACKNOWLEDGMENT

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[1]See, for example, S. Brodsky, in *Quarks and Nuclear Forces,* edited by D. Fries and B. Zeitnitz (Springer, Berlin, 1982).

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[3]There is a very large work dealing with the Bethe-Salpeter equation: E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951). For a review up to 1969, see N. Nakanishi, Prog. Theor. Phys. Suppl. **43**, 1 (1969). Further developments may be found in E. Brezin, C. Itzykson, and J. Zinn-Justin, Phys. Rev. D **1**, 2349 (1970); G. Feldman, T. Fulton, and J. Townsend, *ibid.* **7**, 1814 (1973); P. Danielewicz and J. M. Namyskowski, Phys. Lett. **81B**, 110 (1979); M. Sawicki, Phys. Rev. D **33**, 1103 (1986); **32**, 2666 (1985). For references to the applications to the study of positronium, see C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), Chap. 10.

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[5]Various review articles may be found in *Solitons in Nuclear and Elementary Particle Physics,* edited by A. Chodos, E. Hadjimichael, and C. Tze (World Scientific, Singapore, 1984).

[6]See, for example, Ming Li, R. J. Perry, and L. Wilets, Phys. Rev. D **36**, 596 (1987); R. J. Perry and Ming Li, Mod. Phys. Lett. **A2**, 353 (1987).