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Systematic search for anomaly-free theories

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Many new anomaly-free configurations are found group theoretically for space-time dimensions of $D=6,10,14,18$. They contain $N=1$ supergravitylike constituents, Yang-Mills matter for a simple group, and many Yang-Mills singlets (shadow matter). We also discuss constraints for theories without supergravity, chiral four-dimensional theories, and the absence of global gauge anomalies.

I. INTRODUCTION AND SUMMARY

Recent excitement about superstring theories started from the analysis of anomaly-free theories. Alvarez-Gaume and Witten initiated a search for such theories.[1] Green and Schwarz extended the analysis by introducing the so-called Green-Schwarz mechanism for canceling anomalies.[2] They have shown that $N=1$ supergravity theory coupled to the Yang-Mills (YM) gauge multiplet is anomaly-free for the gauge groups SO(32) and $E_8 \times E_8$, at dimension $D=10$. Both of these theories turned out to be derivable as the zero-mass sector of superstring theories. Thierry-Mieg[3] has extended the search into higher dimensions while still assuming that the chiral YM matter is in the adjoint representation (rep) and only one auxiliary spin-$\frac{1}{2}$ field is used with an opposite chirality. He has found that $D=2$, 10, 18, and 26 have solutions with the YM structures, $U(1)^{24}$, SO(32) with **496** or $E_8 \times E_8$ with $(\mathbf{248},1)+(1,\mathbf{248})$, E_8 with **248**, and no Yang-Mills matter, respectively. He claims that all of them have the possibility of being derivable from superstrings. Schellekens[4] has discussed a new way of obtaining anomaly-free theories from known ones and has found some new solutions (in particular, $D=14$ has a solution whose YM rep consists of $\Lambda=\mathbf{133}-10\,\mathbf{56}-66$ singlets for the gauge group E_7). Others have also discussed ways of obtaining new string theories from known ones and attempted to formulate string theories in noncanonical dimensions ($D\neq 10$ or 26) (Ref. 5).

In this paper we reexamine the search for anomaly-free theories and find many *new* solutions to the *local* anomaly-free conditions at dimension $D=4k-2$. We give a detailed account of how to find these solutions, which were reported in a Letter.[6] We are not generating new theories from known ones. Because their structures are different, we believe that some of the solutions may lead to new kinds of theories, such as supermembranes. Our search is a systematic one, which employs Lie-algebra trace identities as much as possible. Thus, our search needs no fluke, but patience. Unfortunately, the anomaly-free conditions give information only on the fermionic massless sector of a theory. (A self-dual totally antisymmetric tensor can be considered as a symmetric product of two spinors.) Thus, the solutions we find are only a clue to a new theory.

We assume that the massless fermionic content imitates the $N=1$, $D=10$ supergravity theory closely, i.e., one gravitino, and chiral spin-$\frac{1}{2}$ YM matter field with *not only* one adjoint rep *but also* multiple copies of nontrivial irreducible representations (irreps) (different from adjoint) of a *simple* group. The presence of additional irreps besides an adjoint rep is the departure from the $N=1$, $D=10$ supergravity theories. If a nontrivial irrep is complex, multiple copies of its conjugate are also allowed. For $SO(2n)$, two different kinds of spinors (λ_n and λ_{n-1}) are allowed. We also find solutions which do not satisfy all the assumptions on the representation.

The number of pure gravitational spin-$\frac{1}{2}$ fields is not fixed, since we do not know any systematic method of finding out how many we need.[7] In the case of the well-known $N=1$, $D=10$ supergravity, one needs one negative chirality spin-$\frac{1}{2}$ field. Because these fields behave as YM gauge singlets and we take the Schellekens solution as a clue, *we allow as many YM gauge singlets as required to cancel the pure gravitational anomaly.* The allowance of many singlets is another difference between this work and related papers.

Thus, our (conservative) assumptions are set up as generally as possible for new theories. The assumption of a simple group is the most restrictive one (or the weakest, depending upon a point of view). Various new superstring theories use semisimple groups.[5] Note that our method is local and thus we cannot, for example, distinguish SO(32) from $Spin(32)/Z_2$.

The immediate consequence of our assumption on the fermionic content is that *no theories at* $D=22$ *and beyond* $D>26$ *exist* in order to have the leading pure gravitational anomaly canceled. This fact is discussed in Secs. II and VIII.

The allowance of many shadow matter fields leads us to many new anomaly-free theories. First of all, we show in Sec. II that by adjusting the number of YM singlets, all solutions at $D=18$ are automatically solutions at $D=10$. Similarly, all solutions at $D=14$ and 18 are solutions at $D=6$, but not vice versa. What we mean is that a YM matter rep of a group in one dimension may be used in a different dimension and that theory will also be anomaly-free. In addition to these general statements, we have found many new configurations by investigating the constraints on YM matter fields dimension by dimension.

In order to make our results concise, we hereafter use the following notation. Λ denotes the rep and the symbol $[a]$ denotes the largest integer which does not exceed a. The integers m and m' can be negative, unless otherwise specified. Negative integers imply that the chirality for those fields is negative. The symbol λ_j $(1 \le j \le n)$ refers to the n fundamental weight system for a simple Lie algebra of rank n. Bold numbers denote the dimensions of irreps.

The list of our results for a nontrivial part of YM irreps is the following.

$D=6$ (all solutions are so-called *regular*).

(i) Λ=adjoint + (any number, any chirality of lowest-dimensional reps) of G_2,F_4,E_6,E_7.

(ii) Λ=adjoint + m vectors$-m'$ spinors of SO(N), where $m+2^{[(N-9)/2]}m'=8-N$. For $N=2n$, m' is the sum of the numbers of two kinds of spinors, λ_n and λ_{n-1}, except $N=8$ where their numbers must be equal.

(iii) Λ=adjoint$-m\lambda_j$ of Sp($2n$), where n and m are integer solutions of the following:

$$m=\frac{(n+4)(n-1)(j-1)!(2n+1-j)!}{(n+1-j)[2n^2+3n+4-3j(2n+2-j)](2n-2)!} .$$

The case where $j=1$ is always a solution for an arbitrary n with $m=2(n+4)$. The largest rank solution with $j\neq 1$ up to Sp(200) is given by $(j=2, m=2)$ of Sp(24).

(iv) Λ=adjoint$-m\lambda_j-m'\lambda_j^*$ of SU(N) ($j\neq 1$ or $N-1$ and $N\ge 4$), where N and $(m+m')$ are integer solutions of the following:

$$m+m'=\frac{2N(j-1)!(N-j-1)!}{[N(N+1)-6j(N-j)](N-4)!} .$$

The largest group with a solution up to SU(100) is SU(24) with $j=2$. We do not know whether there exists a solution beyond SU(100).

(v) Λ=adjoint + (any number, any chirality)(**3** and **3***) of SU(3).

$D=10$ (all solutions are regular except those with a # superscript).

(i) Λ=adjoint$+m$ vectors$+m'$ spinors of SO(N), where $m+2^{[(N-7)/2]}m'=32-N$. For $N=2n$, m' is the sum of the numbers of spinors, λ_n and λ_{n-1}, except $N=8$ and 12 where their numbers must be equal. For $m=m'=0$, N must be 32, which is the Green-Schwarz solution.

(ii) $\Lambda^{\#}$=adjoint$-(N+32)$ vectors of Sp(N) (arbitrary even N).

(iii) Λ=adjoint of E_8 (the Green-Schwarz solution).

(iv) $\Lambda^{\#}=-\mathbf{78}$(adjoint)$-\mathbf{27}+\mathbf{351}(\lambda_2)$ of E_6.

(v) $\Lambda=\mathbf{78}+m\,\mathbf{27}+m'\mathbf{27^*}(m+m'=6)$ of E_6.

$D=14$ (all regular solutions).

(i) $\Lambda=\mathbf{153}-26\,\mathbf{18}-\mathbf{256}$ of SO(18).

(ii) $\Lambda=\mathbf{91}-22\,\mathbf{14}-m\,\mathbf{64}-m'\mathbf{64^*}(m+m'=4)$ of SO(14).

(iii) $\Lambda=\mathbf{45}-18\,\mathbf{10}-m\,\mathbf{16}-m'\mathbf{16^*}(m+m'=16)$ of SO(10).

(iv) $\Lambda=\mathbf{133}-10\,\mathbf{56}$ of E_7 (the Schellekens solution).

(v) $\Lambda=\mathbf{78}-m\,\mathbf{27}-m'\mathbf{27^*}(m+m'=18)$ of E_6.

$D=18$ (regular solution).

$\Lambda=\mathbf{248}$ of E_8 (the Thierry-Mieg solution).

We have eliminated trivial subgroup solutions from the list. Note that we must add an appropriate number of singlets to cancel the pure gravitational anomaly. We could not find any theories with just one antichiral spin-$\frac{1}{2}$ field, except those found by Green and Schwarz and Thierry-Mieg. Most of the solutions cannot be obtained from higher-dimensional theories, although some of them may be, just like the Schellekens solution at $D=14$ which comes from $D=18$ by compactifying four dimensions as K_3. We have not investigated this possibility. We do not know at this stage whether these new anomaly-free configurations could be derivable from string theories or not.

We have also found few solutions which do not contain an adjoint rep.

$D=6$.

$\Lambda=\lambda_2+m\lambda_1(N+m=8)$ for Sp(N) (Neven) and SU(N).

$D=10$.

(i) $\Lambda=\lambda_4(\mathbf{495})$ of SO(12).

(ii) $\Lambda=\lambda_2(\mathbf{495})$ of Sp(32).

(iii) $\Lambda=\lambda_2+\lambda_4(\mathbf{65}+\mathbf{429})$ of Sp(12).

(iv) $\Lambda=\lambda_2+(32-N)\lambda_1$ of Sp(N) (N even).

(v) $\Lambda=(2\lambda_1)-(N+32)\lambda_1$ of SO(N).

$D=18$.

$\Lambda=\lambda_3-\lambda_1(\mathbf{273}-\mathbf{26})$ of F_4.

Note that none of these is a regular solution. Although we regard shadow matter fields as gauge singlets, they can be gauged, provided that (1) we assign another group *only* to the shadow world, (2) only shadow particles have quantum numbers for the shadow gauge group, and (3) the YM constraint is satisfied by themselves. The simplest well-known example is the E_8 case at $D=10$. The adjoint **248** of E_8 does satisfy the $D=10$ YM constraint, but we need $l=-247$ shadow particles. When we assign the shadow group E_8 to 248 of them, then the whole theory becomes the $E_8\times E_8$ theory with the rep $(\mathbf{248},\mathbf{1})+(\mathbf{1},\mathbf{248})$ and only one auxiliary field. In this paper, we do not discuss the possibility of the shadow gauge groups and leave it as a future project.

In Secs. II–V we show how we found these solutions. In Sec. II A we derive anomaly-cancellation constraints for dimensions $D=4k-2$. Then, we rewrite the YM part in terms of so-called indices of reps in Sec. II B. Therefore, finding solutions becomes a group-theoretical problem. The next three sections present the systematic search for solutions of YM constraints. Section III is for solutions with a single irrep. In III A solutions with only an adjoint rep are discussed, while in III B we discuss the uniqueness of solutions with a single irrep without restricting ourselves to an adjoint rep. Section IV is for solutions with two irreps (one of them is adjoint). We examine exceptional groups in IV A and classical groups in IV B. Section V is for solutions with three irreps of SO(N).

In Sec. VI we briefly discuss the constraints for a nonsupergravity theory. Again, we find a new solution with two irreps at $D=10$: $\Lambda=m$ spinors $-m2^{[(N-7)/2]}$ vectors of SO(N), where m and N are arbitrary integers and m can be negative. For even N, one can use any mixture of two kinds of spinors, except for $N=8$ and 12 where one must use the same numbers of spinors and another spinors.

Having found many anomaly-free theories, we investigate two ways of further restricting anomaly-free theories in Sec. VII. One constraint is that we should get a chiral four-dimensional theory. The other constraint is the absence of global anomalies. In both cases, we list allowed gauge groups.

In Sec. VIII we discuss theories with more than one gravitino. We show that no theories exist still at $D=22$ or $D>26$, provided that the number of gravitinos is less than 691 and the fermionic content is limited to those used in the one gravitino case.

We realize that our findings are incomplete in the sense that we have not yet constructed full field theories for new solutions. Also, some work to be done is indicated above. However, by finding these solutions, we hope to bring open mindedness to the search for new physical theories.

II. ANOMALY CANCELLATION CONSTRAINTS

We look at theories in $(4k-2)$-dimensional space-time where chirality is naturally defined. As shown by Alvarez-Gaume and Witten, and by Alvarez-Gaume and Ginsparg,[1] anomalies for a Rarita-Schwinger field and spin-$\frac{1}{2}$ chiral fermions at $D=4k-2$ are related to the $4k$-forms in the following polynomials: $\hat{A}\,\mathrm{Tr}[\exp(iR/2\pi)-1]+(4k-3)\hat{A}$ for a Rarita-Schwinger field, $-l\hat{A}$ for l chiral spin-$\frac{1}{2}$ (auxiliary and YM singlet) fields, $\sum\epsilon\hat{A}\,\mathrm{Ch}(F)$ for Yang-Mills spin-$\frac{1}{2}$ fermions with chirality ϵ, where $\hat{A}$ is the Dirac genus and Ch(F) denotes the Chern character. The summation symbol for YM matter fields implies the sum over various chiralities ϵ also. YM singlets can be counted as auxiliary fields. Thus, the total anomaly at $D=4k-2$ is related to the $4k$-forms

$$I_k=\left[\sum\epsilon n_G+4k-3-l\right]\hat{A}_k+\sum_{m=0}^{k-1}\hat{A}_m\left[G_{k-m}+\frac{(-)^{k-m}\sum\epsilon F_{k-m}}{[2(k-m)]!}\right], \tag{2.1}$$

where n_G is the dimension of an irrep with chirality ϵ and

$$G_k=\frac{(-)^k 4k}{B_k}C_kR_k,\quad F_k=\frac{\mathrm{Tr}F^{2k}}{(2\pi)^{2k}},\quad R_k=\frac{\mathrm{Tr}R^{2k}}{(2\pi)^{2k}},$$

$$\sum_{j=0}\hat{A}_jz^j=\exp\left[\sum_{m=1}C_mR_mz^m\right]\quad(\hat{A}_0=1),$$

$$C_k=\frac{B_k}{4k(2k)!},$$

where B_k are the Bernoulli numbers. See Appendix A and Ref. 8 for the derivation. The first eight Bernoulli numbers are $\frac{1}{6},\frac{1}{30},\frac{1}{42},\frac{1}{30},\frac{5}{66},\frac{691}{2730},\frac{7}{6},\frac{3617}{510}$.

The leading $\mathrm{Tr}R^{2k}$ term should vanish. Thus, we must have

$$n+(-)^k\frac{4k}{B_k}=0\quad\text{where } n=\sum\epsilon n_G-l+4k-3\ . \tag{2.2}$$

We show that no solution exists at $D=22$ ($k=6$) and $D>26$ (Ref. 8). For this purpose, we use an approximate expression for a Bernoulli number:

$$B_k>4\sqrt{\pi k}\left[\frac{k}{\pi e}\right]^{2k}$$

(the difference is less than 1% for $k\geq5$) which can be derived from Euler's identity

$$B_k=\frac{2(2k)!}{(2\pi)^{2k}}\zeta(2k)\ ,$$

where $\zeta(m)$ is the Riemann's ζ function with $\zeta(m)>1$ and Stirling's formula

$$m!>\sqrt{2\pi m}\left[\frac{m}{e}\right]^m .$$

Thus, we have

$$|n|<\left[\frac{k}{\pi}\right]^{1/2}\left[\frac{\pi e}{k}\right]^{2k} . \tag{2.3}$$

For $k\geq9$, the right-hand side of Eq. (2.3) is less than 0.66. For $k=8$, $4k/B_k$ is $\frac{16\,320}{3617}$, and for $k=6$, $4k/B_k$ is $\frac{65\,520}{691}$. Hence, the integer solutions for Eq. (2.2) are as shown in Table I. Thierry-Mieg solutions[3] are the last column.

Following Green and Schwarz,[2] we can cancel the remaining anomalies by adding local counterterms, provided that the rest of the $4k$-form terms takes on the form

$$I_k=(R_1+\alpha)X_{k-1}\quad(k\geq2)\ , \tag{2.4}$$

where X_{k-1} is a gauge-invariant $2(k-1)$-form made out of 2-forms F and R. The function α is a gauge-invariant Yang-Mills 4-form, which is not necessarily in proportion to $\sum\epsilon F_1$. (In the case of nonsupergravity theories, it is not as we will show later.) This way of canceling the anomalies imposes trace constraints on F. Note the trivial fact that YM singlets do not contribute to these YM traces, but only contribute to the pure gravitational anomaly. Therefore, by suitably adjusting the number of singlets, *the problem of finding anomaly-free solutions reduces to the problem of finding a rep satisfying trace identities in the Lie algebra.* We show that the trace

TABLE I. Solutions free of gravitational anomalies.

D	k	n	$\sum\epsilon n_G(l=1)$
2	1	24	24
6	2	−240	−244
10	3	504	496
14	4	−480	−492
18	5	264	248
26	7	24	0

identities fix the chirality of Yang-Mills matter for most of the cases of k. Note that in the case of $k=1$ ($D=2$), the method above does not work. The only way out is to use U(1) group(s), in order for $\sum \epsilon F_1$ to factorize, since $\mathrm{Tr}F=0$ for a (semi)simple group.

Note that the Green-Schwarz factorization also imposes further constraints on n. However, n has already been chosen by the requirement of the vanishing coefficient for the leading $\mathrm{Tr}R^{2k}$, Eq. (2.2). As we will see, it is rather remarkable to find n which satisfies all the requirements up to $D=26$ ($k=7$).

For the case of $N=1$ supergravity coupled to YM matter, the coefficient α has a closed expression, which can be found by looking at the coefficients of $R_1 R_{k-1}$ and R_{k-1} terms and is in proportion to $\sum \epsilon F_1$. The result is

$$\alpha = -\frac{6}{(-)^{k-1}\left[\dfrac{k}{B_k}+\dfrac{k-1}{B_{k-1}}\right]-\dfrac{1}{B_1}}\sum \epsilon F_1 = \alpha'\left[\sum \epsilon F_1\right] \quad (k \geq 3)\ . \tag{2.5}$$

We show in Sec. II A that the formula is only valid up to $k=5$. Thus, $\alpha' = -\frac{1}{30}$ ($k=3$), $\frac{1}{42}$ ($k=4$), $-\frac{1}{30}$ ($k=5$), respectively, which happen to be $-B_2$, B_3, $-B_4$. Now, we discuss each case separately. Hereafter we omit the summation symbol in front of ϵF_j.

A. Yang-Mills trace constraints

We derive the YM trace constraints dimension by dimension. We show that no Yang-Mills matter is allowed at $D=26$.

1. D=2 (k=1)

We have $n=\epsilon n_G - l + 1 = 24$ and G must be U(1)'s. We have nothing more to say on this case, because of the reason given earlier.

2. D=6 (k=2)

The total anomaly is given by

$$I_2 = n\hat{A}_2 + \left[G_2 + \frac{\epsilon F_2}{4!} + \hat{A}_1\left[G_1 - \frac{\epsilon F_1}{2}\right]\right] \tag{2.6}$$

which has to be of the form

$$I_2 = (R_1+\alpha)(aR_1+b)\ .$$

Hence, we have

$$0 = n + \frac{8}{B_2}, \quad a = \left[\frac{n}{2} - \frac{4}{B_1}\right]C_1{}^2\ ,$$
$$b + \alpha a = -\frac{C_1}{2}\epsilon F_1, \quad b\alpha = \frac{1}{4!}\epsilon F_2\ , \tag{2.7}$$

which lead to

$$\epsilon n_G - l = -245, \quad \epsilon F_2 = \tfrac{1}{4}\alpha(6\alpha - \epsilon F_1)\ . \tag{2.8}$$

Compared with cases of larger k, we cannot fix α and the constraint on F is weak. Thus, we expect to find many solutions. The choice of $\alpha = \alpha' \epsilon F_1$ and $\alpha' = -\frac{1}{30}$ ($\frac{1}{42}$) leads to the same fourth-order identities one encounters at $D=14$ ($D=18$). Consequently, all the solutions at $D=14$ and 18 are solutions at $D=6$, provided that we adjust the number of singlets for the cancellation of the pure gravitational anomaly. In addition to these solutions, we show the existence of other solutions later.

3. D=10 (k=3)

By identifying I_3 as

$$(R_1+\alpha)(aR_2 + bR_1{}^2 + cR_1 + d)\ , \tag{2.9}$$

we obtain

$$\epsilon n_G - l = 495, \quad \alpha = -\tfrac{1}{30}(\epsilon F_1)\ ,$$
$$\epsilon F_3 = \tfrac{1}{48}(\epsilon F_2)(\epsilon F_1) - \tfrac{1}{14\,400}(\epsilon F_1)^3\ . \tag{2.10}$$

If we use just one irrep for F, we must have $\epsilon=+1$, since F_3 becomes negative otherwise.

4. D=14 (k=4)

We obtain two constraints on the pure gravitational anomaly part:

$$0 = n + \frac{16}{B_4} \quad \text{for } \mathrm{Tr}R^8, \quad 0 = \frac{n}{2} + \frac{8}{B_2} \quad \text{for } (\mathrm{Tr}R^4)^2 \tag{2.11}$$

which are simultaneously satisfied by $n=-480$, since $B_2 = B_4 = \frac{1}{30}$. For the Yang-Mills part, we obtain

$$\epsilon F_2 = -\tfrac{1}{196}(\epsilon F_1)^2\ ,$$
$$\epsilon F_4 = -\tfrac{1}{36}(\epsilon F_3)(\epsilon F_1) - \tfrac{5}{24\,192}(\epsilon F_2)(\epsilon F_1)^2 - \tfrac{5}{14\,224\,896}(\epsilon F_1)^4\ . \tag{2.12}$$

Thus, we need at least one YM matter multiplet with opposite chirality.

5. D=18 (k=5)

We obtain two constraints on the pure gravitational anomaly part:

$$0 = n - \frac{20}{B_5} \quad \text{for } \mathrm{Tr}R^{10}\ ,$$
$$0 = n + \frac{8}{B_2} - \frac{12}{B_3} \quad \text{for } \mathrm{Tr}R^4\mathrm{Tr}R^6\ . \tag{2.13}$$

Both of these are consistently satisfied by $n=264$, because of particular values of Bernoulli numbers. For the Yang-Mills part, we have

$$\epsilon F_2 = \tfrac{1}{100}(\epsilon F_1)^2 ,$$
$$\epsilon F_3 = \tfrac{1}{48}(\epsilon F_2)(\epsilon F_1) - \tfrac{1}{14\,400}(\epsilon F_1)^3 , \quad (2.14)$$
$$\epsilon F_5 = \tfrac{1}{16}(\epsilon F_4)(\epsilon F_1) - \tfrac{7}{5760}(\epsilon F_3)(\epsilon F_1)^2 + \tfrac{7}{829\,440}(\epsilon F_2)(\epsilon F_1)^3 - \tfrac{7}{414\,720\,000}(\epsilon F_1)^5 .$$

Again, if we have just one irrep, then we must use $\epsilon = +1$. In this case, we have two independent equations for α: one from comparing the coefficients of $R_4 R_1$ and R_4 and the other from $R_2{}^2 R_1$ and $R_2{}^2$. Fortunately they yield the same value, $\alpha' = -\frac{1}{30}$. Note that the value of α and the constraint on F_3 are exactly the same for the $D = 10$ ($k = 3$) case. Consequently, a solution at $D = 18$ is a solution at $D = 10$ with an adjustment of the number of singlets (but *not* vice versa).

6. $D=26$ $(k=7)$

We have four constraints for the pure gravitational anomaly part:

$$0 = n - \frac{28}{B_7} \quad \text{for } \mathrm{Tr}R^{14} ,$$
$$0 = n - \frac{20}{B_5} + \frac{8}{B_2} \quad \text{for } \mathrm{Tr}R^4\mathrm{Tr}R^{10} ,$$
$$0 = n + \frac{16}{B_4} - \frac{12}{B_3} \quad \text{for } \mathrm{Tr}R^6\mathrm{Tr}R^8 , \quad (2.15)$$
$$0 = \frac{n}{2} - \frac{6}{B_3} + \frac{8}{B_2} \quad \text{for } (\mathrm{Tr}R^4)^2\mathrm{Tr}R^6 .$$

All of them are mysteriously satisfied by $n = 24$. However, if $\epsilon F_1 \neq 0$, then we have four independent relations for α from $R_6, R_2R_4, R_3{}^2, R_2{}^3$, all of which are in conflict to each other. For the case where $\epsilon F_1 = 0$, we have $\alpha = 0$ and $\epsilon F_j = 0$ ($j = 2,3,4,5,7$), whose solutions are hard to find, unless we have a vectorlike theory with equal numbers of both positive and negative chiral matter fields for all the irreps. Consequently, we cannot use YM matter at this dimension, because we assume only one chiral adjoint matter field. That is, at $D = 26$ we may only have a Rarita-Schwinger field and one antichiral spin-$\frac{1}{2}$ field in the fermionic sector. The summary of this section is given in Table II.

TABLE II. Yang-Mills constraints where F_n is defined as $\mathrm{Tr}F^{2n}/(2\pi)^{2n}$.

D	k	YM constraints
6	2	$\epsilon F_2 = \frac{1}{4}\alpha(6\alpha - \epsilon F_1)$
10	3	$\epsilon F_3 = \frac{1}{48}(\epsilon F_2)(\epsilon F_1) - \frac{1}{14\,400}(\epsilon F_1)^3$
14	4	$\epsilon F_2 = -\frac{1}{196}(\epsilon F_1)^2$
		$\epsilon F_4 = -\frac{1}{36}(\epsilon F_3)(\epsilon F_1) + \frac{5}{7\,112\,448}(\epsilon F_1)^4$
18	5	$\epsilon F_2 = \frac{1}{100}(\epsilon F_1)^2$
		$\epsilon F_3 = \frac{1}{7200}(\epsilon F_1)^3$
		$\epsilon F_5 = \frac{1}{16}(\epsilon F_4)(\epsilon F_1) - \frac{7}{69\,120\,000}(\epsilon F_1)^5$
26	7	No YM matter allowed

B. Constraints expressed in terms of indices

Now, in this subsection we reduce the YM constraints to the equations for indices of various irreps. Thereby, we make it possible to solve constraints group-theoretically. For general discussion on indices, please refer to Ref. 9. We also discuss the condition that the spin connection can be embedded into the gauge group (holonomy compactification).

Using the results of Appendix B and Ref. 9, we can always reduce traces of an irrep, Λ, into traces of another irrep, which we denote $\Box$. This irrep, $\Box$, is usually taken to be a lowest-dimensional rep. Then, we define a pth order index by $Q_p^j = D_p(\Lambda_j)/D_p(\Box)$ for an irrep Λ_j. Thus, in principle, all the trace constraints can be written in terms of indices of irreps (see, e.g., Appendix B). Hereafter, we omit the suffix j for the summation symbol $\sum$. For the fourth-order and the sixth-order constraints, the results for a rep, $\Lambda = \sum \epsilon_j m_j \Lambda_j$, are as follows.

Fourth order:

$$(\epsilon F_2) = K(\epsilon F_1)^2$$

is equivalent to

$$\sum \epsilon_j m_j Q_4^j = 0 , \quad (2.16)$$
$$\sum \epsilon_j m_j X_4^j = K\left[\sum \epsilon_j m_j Q_2^j\right]^2 . \quad (2.17)$$

Sixth order ($Q_3 = 0$):

$$(\epsilon F_3) = \tfrac{1}{48}(\epsilon F_2)(\epsilon F_1) - \tfrac{1}{14\,400}(\epsilon F_1)^3 \quad (2.18)$$

is equivalent to

$$\sum \epsilon_j m_j Q_6^j = 0 , \quad (2.19)$$
$$\sum \epsilon_j m_j X_6^j = \tfrac{1}{48}\left[\sum \epsilon_j m_j Q_2^j\right]\left[\sum \epsilon_j m_j Q_4^j\right] , \quad (2.20)$$
$$\sum \epsilon_j m_j Y_6^j = \left[\sum \epsilon_j m_j Q_2^j\right] \times \left[\tfrac{1}{48}\sum \epsilon_j m_j X_4^j - \tfrac{1}{14\,400}\left[\sum \epsilon_j m_j Q_2^j\right]^2\right] , \quad (2.21)$$

where

$$X_4^j = A_4^j (Q_2^j)^2 - A_4(\Box) Q_4^j , \quad (2.22)$$
$$X_6^j = B_6^j Q_2^j Q_4^j - B_6(\Box) Q_6^j , \quad (2.23)$$
$$Y_6^j = A_6^j (Q_2^j)^3 - A_6(\Box) Q_6^j - A_4(\Box) X_6^j . \quad (2.24)$$

Note the fact that $X_4(\Box) = X_6(\Box) = Y_6(\Box) = 0$, since $Q_p(\Box) = 1$.

For exceptional groups, we have extra relations, $B_6^j = 0$ and $Q_4^j = 0$, and thus the conditions become weaker.

Fourth order:

$$\sum \epsilon_j m_j A_4^j (Q_2^j)^2 = K\left[\sum \epsilon_j m_j Q_2^j\right]^2 . \quad (2.25)$$

Sixth order:

$$\sum \epsilon_j m_j Q_6^j = 0 \ , \tag{2.26}$$

$$\sum \epsilon_j m_j [A_6^j (Q_2^j)^3 - A_6(\Box) Q_6^j] = \left[\sum \epsilon_j m_j Q_2^j\right] \left[\tfrac{1}{48} \sum \epsilon_j m_j A_4^j (Q_2^j)^2 - \tfrac{1}{14\,400} \left[\sum \epsilon_j m_j Q_2^j\right]^2\right] . \tag{2.27}$$

Eighth order:

$$\sum \epsilon_j m_j Q_8^j = 0 \ , \tag{2.28}$$

$$\sum \epsilon_j m_j C_8^j Q_2^j Q_6^j = -\tfrac{1}{36} \left[\sum \epsilon_j m_j Q_2^j\right] \left[\sum \epsilon_j m_j Q_6^j\right] , \tag{2.29}$$

$$\sum \epsilon_j m_j [A_8^j (Q_2^j)^4 - C_8^j Q_2^j Q_6^j A_6(\Box)]$$
$$= \left[\sum \epsilon_j m_j Q_2^j\right] \left[-\tfrac{1}{36} \sum \epsilon_j m_j [A_6^j (Q_2^j)^3 - A_6(\Box) Q_6^j] + \tfrac{5}{7\,112\,448} \left[\sum \epsilon_j m_j Q_2^j\right]^3\right] . \tag{2.30}$$

Now, we write the condition that the spin connection can be embedded into the gauge group. This condition is needed if one wants to use the holonomy group for dimensional reduction. Here, we use the Schellekens criterion:[4] The embedding is possible if $c = 1/p$ (p an integer), where c is given by the Green-Schwarz factor normalized as

$$c(\epsilon F_1) - \frac{\sum \epsilon_j m_j l_2(\Lambda_j, G)}{l_2(v, \mathrm{SO}(D))} R_1 \ ,$$

where D is the dimension of space-time, v is the vector rep, and l_2 denotes the second-order index normalized as follows:[10] For a vector rep, 1 for SU(N) and Sp(N), and 2 for SO(N), while for an adjoint rep, $2N$ for SU(N), $N+2$ for Sp(N) (N even), $2(N-2)$ for SO(N) ($N>5$) and 8, 18, 24, 36, 60 for G_2, F_4, E_6, E_7, E_8, respectively. This normalization determines the dimension of the parameter space of instantons. This parameter c is related to our α' by

$$c = -\alpha' \frac{\sum \epsilon_j m_j l_2(\Lambda_j, G)}{l_2(v, \mathrm{SO}(D))}$$
$$= -\frac{\alpha'}{2} \sum \epsilon_j m_j l_2(\Lambda_j, G) \ . \tag{2.31}$$

If this parameter c is equal to one, then the solution is called *regular*.

In the case of $D=6$, we can look for a regular solution, since the YM constraints are weaker than those of other dimensions. Then, for $D=6$ we choose α to be

$$\alpha = \alpha'(\epsilon F_1)$$

and

$$\alpha' = -\frac{2}{l_2(\Box) \left[\sum \epsilon_j m_j Q_2^j\right]} \ . \tag{2.32}$$

Then, the fourth-order constraints for a regular solution in $D=6$ are given by

$$\sum \epsilon_j m_j Q_4^j = 0 \ , \tag{2.33}$$

$$\sum \epsilon_j m_j l_2^j (2 A_4^j l_2^j - 1) = 12 \ . \tag{2.34}$$

It turns out that these two conditions are still weak enough to allow rather general solutions, as we will show later.

III. SOLUTIONS WITH A SINGLE IRREP

The trace identities derived in a previous section require *two* conditions to be satisfied: (1) vanishing indices and (2) matching coefficients. Vanishing indices means that if $\mathrm{Tr}F^{2k}$ is expressed in terms of traces of lower powers of F, then the dominant index Q_{2k} has to vanish.[9] Furthermore, if a particular power of traces is missing from the trace identity, then the index corresponding to that power has to vanish also. Indices of a simple Lie algebra depend only on an irrep (independent of the definition of F) and are calculable. Matching coefficients means that for an irrep all the coefficients of a trace identity are calculable and thus one must find a rep to match them. Both conditions are so strong that we sometimes end up having only a few solutions or none.

In this section, we discuss solutions with a single irrep in $D=6,10,14,18$. First, we limit ourselves to an adjoint rep. Then, we show the uniqueness of solutions in $D=14$ and 18 without restricting the rep to being adjoint. For $D=10$, the uniqueness of single-irrep solutions does not hold and we give two solutions with a nonadjoint irrep. Solutions with two and three irreps are discussed in Secs. IV and V.

A. Adjoint rep only

1. Vanishing indices

From Table II we find that the following indices should vanish for a particular dimension. (See Table

TABLE III. Vanishing indices for anomaly cancellation.

D	k	Vanishing indices
6	2	Q_4
10	3	Q_3, Q_6
14	4	Q_3, Q_4, Q_8
18	5	$Q_3, Q_4, Q_5, Q_6, Q_{10}$

TABLE IV. Indices for adjoint reps.

	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	Q_{10}
SU(N)	$2N$	0^*	$2N$	0^*	$2N$	0^*	$2N$	0^*	$2N$
SO(N)	$N-2$	0	$N-8$	0	$N-32$	0	$N-128$	0	$N-512$
Sp(N)	$N+2$	0	$N+8$	0	$N+32$	0	$N+128$	0	$N+512$
G_2	4	0	0	0	-26	0	0	0	0
F_4	3	0	0	0	-7	0	17	0	0
E_6	4	0	0	0^*	-6	0	18	0^*	0
E_7	3	0	0	0	-2	0	10	0	-2
E_8	1	0	0	0	0	0	1	0	0

III.) For an adjoint rep. we can easily calculate various indices as shown in Table IV (Ref. 9), where an asterisk indicates the fact that it may not be vanishing for other irreps. Thus, the possible groups are shown in Table V, where the bracket indicates the dimension of an adjoint rep.

2. *Matching coefficients*

Now, we see if the coefficients in the YM trace identities are matched. We discuss cases dimension by dimension.

(i) $D=6$ ($k=2$): We have to satisfy

$$\epsilon\,\mathrm{Tr}F^4=\tfrac{1}{4}\alpha(6\alpha-\epsilon\,\mathrm{Tr}F^2) \tag{3.1}$$

while for any adjoint rep with $Q_4=0$ we have the trace identity (see Appendix B)

$$\mathrm{Tr}F^4=\frac{5}{2(a+2)}(\mathrm{Tr}F^2)^2\ , \tag{3.2}$$

where a denotes the dimension of an adjoint rep. Let us *assume* that $\alpha=\alpha'\epsilon\,\mathrm{Tr}F^2$ (similar to cases of $k\geq 3$). Then, we have the equation for α':

$$6\alpha'^2-\alpha'-\frac{10\epsilon}{a+2}=0\ . \tag{3.3}$$

If we cannot find a real α' for a particular a, then that solution is not allowed. For $\epsilon=+1$, the discriminant for α' is always positive definite; therefore, for any a, there exist two real solutions for α'. Hence, all of the solutions found in Sec. III A 1 for $D=6$ are solutions. However, for $\epsilon=-1$, the discriminant is given by $1-240/(a+2)$, which requires $a>238$. Thus, only the E_8 case is the solution.

If we impose the Schellekens criterion[4] discussed in Sec. II B, all of the groups for $D=6$ listed in Table V have the $c=1$ (called *regular*) solutions for $\epsilon=+1$. The values of α' are

	G_2	F_4	E_6	E_7	E_8	SU(2)	SU(3)	SO(8)
α'	$-\frac{1}{4}$	$-\frac{1}{9}$	$-\frac{1}{12}$	$-\frac{1}{18}$	$-\frac{1}{30}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{6}$

The value $c=1$ is also found in the $D=10$ case with an adjoint of SO(32) or $(\mathbf{248},1)+(1,\mathbf{248})$ of $E_8\times E_8$. Thus, these solutions at $D=6$ should stand up as equally as the $D=10$ solutions. The only difference is the need for many gauge-singlet antichiral spin-$\frac{1}{2}$ fields. The number of antichiral spin-$\frac{1}{2}$ fields is given by $l=\epsilon n_G+245$. We believe that it is very difficult to obtain these solutions, using a dimensional reduction of a higher-dimensional theory. The reason is that an adjoint rep of all exceptional groups decomposes into not only an adjoint rep and singlets but also some nontrivial reps of an exceptional group. Just an adjoint rep plus singlets is hard to come by.

(ii) $D=10$ ($k=3$): With a single irrep we have to satisfy

$$\mathrm{Tr}F^6=\tfrac{1}{48}\mathrm{Tr}F^4\mathrm{Tr}F^2-\tfrac{1}{14\,400}(\mathrm{Tr}F^2)^3\ , \tag{3.4}$$

while for an adjoint rep of a group with $Q_3=Q_6=0$ but $Q_4\neq 0$, we have, in general[9] (see Appendix B),

$$\mathrm{Tr}F^6=\frac{21}{2(a+8)}\mathrm{Tr}F^4\mathrm{Tr}F^2-\frac{35}{2(a+4)(a+8)}(\mathrm{Tr}F^2)^3\ . \tag{3.5}$$

Thus, we *uniquely* select $a=496$, which is the dimension of an adjoint rep of SO(32). Consequently, we have $l=1$, i.e., just one antichiral spin-$\frac{1}{2}$ field is necessary. Note that this is a *regular* solution, since $c=\frac{1}{30}\times 2\times(32-2)/2=1$. For the case where $Q_4=0$ also, we can use Eqs. (3.4) and (B5) to find

$$\mathrm{Tr}F^6=\frac{748-a}{14\,400(a+2)}(\mathrm{Tr}F^2)^3\ , \tag{3.6}$$

while we have, on the other hand, using identities Eqs. (B5) and (B6) which hold for any adjoint rep for a group with $Q_3=Q_4=Q_6=0$,

$$\mathrm{Tr}F^6=\frac{35}{4(a+2)(a+4)}(\mathrm{Tr}F^2)^3\ . \tag{3.7}$$

These two equations have solutions, $a=248$ or $a=496$. Only **248** of E_8 satisfies the condition $Q_4=0$, in addition

TABLE V. Candidates for anomaly-free solutions.

D	k	Possible groups
6	2	G_2(**14**), F_4(**52**), E_6(**78**), E_7(**133**), E_8(**248**) SU(2)(**3**), SU(3)(**8**), SO(8)(**28**)
10	3	SO(32)(**496**), E_8(**248**), SU(2)(**3**), Sp(4)(**10**)
14	4	G_2(**14**), SU(2)(**3**)
18	5	E_8(**248**), SU(2)(**3**)

to $Q_3=Q_6=0$. Obviously, one needs $l=-247$ for the cancellation of the pure gravitational anomaly. Again, this solution is *regular,* since $c=\frac{1}{30}\times\frac{60}{2}=1$.

(iii) $D=14$ ($k=4$): One of the trace identities is given by

$$\mathrm{Tr}F^4=\tfrac{1}{196}(\mathrm{Tr}F^2)^2 \quad (Q_4=0) \ . \tag{3.8}$$

The coefficient $\frac{1}{196}$ has to be equal to $5/[2(a+2)]$ for an adjoint rep, i.e., $a=488$. None of the simple groups in Table V has this dimension. Thus, there are no solutions.

(iv) $D=18$ ($k=5$): The equation $5/[2(a+2)]=\frac{1}{100}$ leads to $a=248$, which is the dimension of an adjoint rep of E_8. The rest of trace identities are satisfied by this rep. This solution is regular and needs $l=1$.

B. Uniqueness of solutions with a single irrep

Now, we drop the condition that the rep is adjoint. Actually, for $D=14$ and 18 the trace identities are indeed very strong conditions. For $D=14$ and 18, we can show that no simple group has a single irrep solution, except for the adjoint rep of E_8 at $D=18$. For exceptional groups (except E_6 at $D=18$), we can use Appendix B to prove this with much algebra. E_6 is excluded, because E_6 can have nonvanishing Q_5 which will modify the trace identities. We cannot use the same method for classical groups either, since in general Q_4 is nonvanishing for them. Therefore, for these groups, we first try to find solutions for the fourth-order trace identity, Eq. (B1), which is applicable to any simple group.

Using the inequality (see Appendix B for definition of A_4)

$$\frac{1}{d}\le A_4\le\frac{3}{d} \quad (d=\text{dimension of an irrep}) \ ,$$

together with the fourth-order trace constraints in Eqs. (2.12) and (2.14), we can limit the dimension of an irrep to

$$\begin{aligned}&196\le d\le 588 \quad \text{for } D=14 \ ,\\ &100\le d\le 300 \quad \text{for } D=18 \ .\end{aligned} \tag{3.9}$$

The derivation of the lower bound for A_4 is rather involved and is not given here. Anyway, those groups with rank less than or equal to eight can be checked and ruled out by the table of McKay and Patera.[11] For SO(N) and Sp(N) with rank ≥ 9, only a few reps have dimensions within the range specified above: λ_1, λ_2, $2\lambda_1$, λ_3, and spinors. However, none of these gives $A_4=\frac{1}{196}$ or $\frac{1}{100}$. Thus, we turn to SU(N). However, for an irrep of SU(N) ($N\ge 4$), it is very difficult to satisfy $Q_4=0$ and we have checked that no single-irrep solution exists within the bounds of Eq. (3.9). For SU(2) and SU(3), we have $Q_4=0$ automatically and thus we must investigate A_4 case by case. After doing so, we were able to rule out these groups.

For $D=10$, the uniqueness no longer holds. In addition to the **496** of SO(32) and the **248** of E_8 (both adjoint and regular), we find the **495**(λ_2) of Sp(32) and the **495**(λ_4) of SO(12) satisfy the YM trace identities, although these two solutions are *neither* adjoint *nor* regular. It is interesting to observe that both of these solutions have the same dimension, 495, and require *no* YM singlet spin-$\frac{1}{2}$ field for the cancellation of gravitational anomalies.

IV. SOLUTIONS WITH TWO IRREPS

Allowing the rep to have two irreps, we loosen the constraints on the YM matter significantly. We are not required to have an irrep with vanishing indices, since the cancellation of an index can be achieved by the sum of two or more nonvanishing indices, using chirality and indices with negative values. In order to make things manageable, we put some restrictions on the possible reps used in the present search. We hope that in the future, we can relax the assumptions. Our assumptions are as follows.

Assumption 1. The rep is of the form

$$\Lambda=\epsilon_0\Lambda_0(\text{adjoint rep})+\epsilon m\Lambda_1(m \text{ copies of an irrep } \Lambda_1) \ ,$$

where ϵ_0 and ϵ denote the chirality of the reps. If Λ_1 is complex, then we allow the possibility

$$\Lambda=\epsilon_0\Lambda_0+\epsilon m\Lambda_1+\epsilon' m'\Lambda_1^* \ .$$

For SO($2n$), we allow two kinds of spinors.

Assumption 2. The YM group contains a $SU(2)_L$ weak group.

Assumption 3. The irrep Λ_1 contains the weak isospin components, $I_{3L}=0,\pm\frac{1}{2}$ only.

Using m copies of an irrep allows us to have the family structure. For a complex rep, we have $Q_p(\Lambda_1)=(-)^pQ_p(\Lambda_1^*)$. Thus, it sometimes happens that only the sum $(m+m')$ is fixed by the anomaly-free condition, as we will see later. The restriction on I_{3L} for Λ_1 imposes a not-too-drastic particle contents. All the possible reps satisfying Assumption 3 have been found in Ref. 12. The results are

SU($n+1$)	λ_j ($1\le j\le n$)
SO($2n+1$)	λ_1 or λ_n ($n\ge 3$)
Sp($2n$)	λ_j ($1\le j\le n$)
SO($2n$)	λ_1, λ_{n-1}, or λ_n ($n\ge 4$)
G_2	λ_2 (**7**)
F_4	λ_4 (**26**)
E_6	λ_1 or λ_5 (**27**)
E_7	λ_6 (**56**)
E_8	None

where λ_j ($j=1,2,\ldots,n$) denotes the fundamental weight of a Lie algebra of rank n. An immediate simple consequence of these three assumptions is that for SU($n+1$) ($n\ge 2$), $\Lambda_1=m(\lambda_j+\lambda_j^*)$ must be used for $D=10$, 14, and 18. This is because at these dimensions we must have $Q_3(\Lambda)=0$ in order to satisfy the trace identities, while by direct computation $Q_3(\lambda_j)\neq 0$ and $Q_3(\text{adjoint})=0$.

Another interesting consequence is that if an exceptional group is found to have a solution, then a lower-

TABLE VI. A branching rule for exceptional groups. The parentheses indicate the Lie algebra.

E_8:	**248**	$\rightarrow$	E_7:**133**+2 **56**+3 **1**	(a_1+e_7)
			E_6:**78**+3 **27**+3 **27***+8 **1**	(a_2+e_6)
			F_4:**52**+7 **26**+14 **1**	(f_4+g_2)
			G_2:**14**+26 **7**+52 **1**	(f_4+g_2)
E_7:	**56**	$\rightarrow$	E_6:**27**+**27***+2 **1**	
			F_4:2 **26**+4 **1**	(a_1+f_4)
			G_2:6 **7**+14 **1**	(c_3+g_2)
	133	$\rightarrow$	E_6:**78**+**27**+**27***+**1**	
			F_4:**52**+3 **26**+3 **1**	(a_1+f_4)
			G_2:**14**+14 **7**+21 **1**	(c_3+g_2)
E_6:	**27**	$\rightarrow$	F_4:**26**+**1**	
			G_2:3 **7**+6 **1**	(a_2+g_2)
	78	$\rightarrow$	F_4:**52**+**26**	
			G_2:**14**+8 **7**+8 **1**	(a_2+g_2)
F_4:	**26**	$\rightarrow$	G_2:3 **7**+5 **1**	(a_1+g_2)
	52	$\rightarrow$	G_2:**14**+5 **7**+3 **1**	

rank exceptional group also has a solution which satisfies all the assumptions. The reason is that all the reps listed above and adjoint reps of exceptional groups always have a branching into an exceptional group as a subgroup where they decompose into only an adjoint rep, plus many copies of the reps listed above, and singlets. See Table VI.

We examine the possibility of exceptional groups in Sec. IV A which are divided into cases with different dimensions. Then we discuss classical groups in Sec. IV B which are divided into many subsections, dealing with different dimensions and different classical groups. Note that hereafter we denote $F_j(\Lambda_0)=F_j^0$, $F_j(\Lambda_1)=F_j$, and $y_p=D_p(\Lambda_0)/D_p(\Lambda_1)$ for convenience.

A. Exceptional groups

For all exceptional groups, trace identities are simplified considerably as in Eqs. (2.25)–(2.30), due to the absence of a genuine fourth-order Casimir invariant.[9] We discuss each dimension separately.

1. $D=18$ for exceptional groups

For $D=18$, we know that an adjoint rep **248** of E_8 is the solution. Thus, using the argument above as well as Table VI, we know immediately that the following are also solutions:

$$\begin{aligned} &G_2:\ \mathbf{14}+26\,\mathbf{7}\ ,\\ &F_4:\ \mathbf{52}+7\,\mathbf{26}\ ,\\ &E_6:\ \mathbf{78}+3\,\mathbf{27}+3\,\mathbf{27}^*\ ,\\ &E_7:\ \mathbf{133}+2\,\mathbf{56}\ . \end{aligned} \tag{4.1}$$

We can show that these are unique. For $D=18$, we must satisfy $Q_3=Q_4=Q_5=Q_6=Q_{10}=0$. Thus, $\epsilon m=26,7,6,2$ for G_2,F_4,E_6,E_7, respectively, since $Q_6(\Lambda_1)=1$ and $Q_6(\Lambda_0)=-26,-7,-6,-2$ for G_2,F_4,E_6,E_7, respectively. For E_6, the combination $3\,\mathbf{27}+3\,\mathbf{27}^*$ is fixed uniquely by the condition $Q_5=0$ while maintaining $Q_6=0$. Note also $Q_{10}(\mathbf{56})=-2$ for E_7 and $Q_{10}=0$ for the remaining groups. Then, $Q_{10}(\Lambda)=0$ is automatically satisfied by the solutions of Eq. (4.1). Hence, we have shown the uniqueness of Eq. (4.1).

2. $D=14$ for exceptional groups

For $D=14$, we must have $Q_3=Q_4=Q_8=0$. The first two equations are automatically satisfied for exceptional groups. We can immediately fix the product $\epsilon_0\epsilon m$ for F_4, E_6, and E_7, since $Q_8(\Lambda_0)=17,18,10$ for F_4,E_6,E_7, respectively. The signs of ϵ_0 and ϵ can be determined by looking at the fourth-order trace identity: $\epsilon F_2=-\frac{1}{196}(\epsilon F_1)^2$ and Eq. (2.25). Interestingly, only $\epsilon_0=+1$ has integer solutions for all the groups and the results are

	G_2	F_4	E_6	E_7
ϵm	$(-11,-46)$	-17	-18	-10

Now, the G_2 case has still two possibilities but we confirm solutions for F_4, E_6, and E_7, now satisfying $Q_8=0$ also. The final check is done by the eighth-order trace identity:

$$\epsilon F_4=-\tfrac{1}{36}(\epsilon F_3)(\epsilon F_1)+\tfrac{5}{7\,112\,448}(\epsilon F_1)^4\ , \tag{4.2}$$

where we now have $\epsilon F_j=F_j(\Lambda_0)-mF_j(\Lambda_1)$. Using Eqs. (2.28)–(2.30), we obtain two equations for m, corresponding to the one for F_1F_3 and the other for $F_1{}^4$. All the coefficients are given in Table VII.

The resulting two equations for each exceptional group are

TABLE VII. Values of the coefficients in the trace identities for exceptional groups.

	y_2	y_6	A_4	A_4^0	A_6	A_6^0	A_8	A_8^0	C_8	C_8^0
G_2	4	-26	$\frac{1}{4}$	$\frac{5}{32}$	$\frac{5}{72}$	$\frac{35}{1152}$	$\frac{35}{1728}$	$\frac{1435}{221\,184}$	$\frac{2}{3}$	$\frac{20}{39}$
F_4	3	-7	$\frac{1}{12}$	$\frac{5}{108}$	$\frac{5}{576}$	$\frac{5}{1728}$	$\frac{25}{25\,056}$	$\frac{425}{2\,029\,536}$	$\frac{7}{24}$	$\frac{5}{24}$
E_6	4	-6	$\frac{1}{12}$	$\frac{1}{32}$	$\frac{79}{8856}$	$\frac{7}{5248}$	$\frac{2047}{1\,912\,896}$	$\frac{67}{1\,007\,616}$	$\frac{26}{81}$	$\frac{4}{27}$
E_7	3	-2	$\frac{1}{24}$	$\frac{1}{54}$	$\frac{181}{78\,912}$	$\frac{7}{14\,796}$	$\frac{37\,919}{263\,250\,432}$	$\frac{4739}{333\,176\,328}$	$\frac{31}{174}$	$\frac{8}{87}$

$$\begin{aligned}
&G_2: \ (m-46)(m^3+39m^2+1746m+178\,772)=0\ ,\\
&\qquad (m-46)(m-44)=0\ ,\\
&F_4: \ (m-17)(58m^3+290m^2+8062m+322\,527)=0\ ,\\
&\qquad (m-17)(m-\tfrac{21}{2})=0\ ,\\
&E_6: \ (m-18)(135m^3+270m^2+17\,820m+682\,708)=0\ ,\\
&\qquad (m-18)(m-\tfrac{76}{9})=0\ ,\\
&E_7: \ (m-10)(8062m^3-16\,124m^2+274\,108m+4\,366\,395)=0\ ,\\
&\qquad (m-10)(m-\tfrac{75}{29})=0\ .
\end{aligned} \tag{4.3}$$

It is amazing that there is just one integer solution for each exceptional group:

$$\begin{aligned}
&G_2: \ \mathbf{14}-46\,\mathbf{7}\ ,\\
&F_4: \ \mathbf{52}-17\,\mathbf{26}\ ,\\
&E_6: \ \mathbf{78}-m\,\mathbf{27}-m'\,\mathbf{27^*}\quad (m+m'=18)\ ,\\
&E_7: \ \mathbf{133}-10\,\mathbf{56}\ .
\end{aligned} \tag{4.4}$$

Only E_6 has the arbitrariness in fixing m and m': Only the sum, $m+m'$, is fixed. This arbitrariness follows from the fact that $Q_p(\mathbf{27})=(-)^p Q_p(\mathbf{27^*})$ ($p=2,5,6,8,9,12$) and the YM constraints here are only $Q_3=Q_4=Q_8=0$. The cases for G_2 and F_4 follow from the E_7 cases, using Table VI. Here, we have directly reproduced the Schellekens solution for E_7 which was derived from the dimensional reduction of the $D=18$ E_8 solution by compactifying four dimensions as K_3. In our approach, there is no special mystery attached to E_7. The subgroup approach from E_7 produces one of the E_6 solutions, $\mathbf{78}-9\,\mathbf{27}-9\,\mathbf{27^*}$. For all the solutions in Eq. (4.4), the spin connection can be embedded into the gauge group, since $c=1$ (i.e., regular). It is enough to check this for E_7: $l_2(\mathbf{133})=36$, $l_2(\mathbf{56})=12$, and $l_2(v,\mathrm{SO}(14))=2$.

3. $D=10$ for exceptional groups

For $D=10$, we know that all the solutions at $D=18$ are solutions at this dimension also. In particular, since we are limiting ourselves to a rep, consisting of an adjoint plus many copies of a lowest-dimensional irrep, we find only these solutions for exceptional groups. The only exception is that we can have $\mathbf{78}+m\,\mathbf{27}+m'\,\mathbf{27^*}$ ($m+m'=6$) for E_6, since at $D=10$ Q_5 can be nonvanishing, whereas at $D=18$, Q_5 must be vanishing in order to satisfy the trace identities.

4. $D=6$ for exceptional groups

For $D=6$, the situation completely changes because of the weak constraints. We assume that $\alpha=\alpha'(\epsilon F_1)$ and look for a regular solution ($c=1$) where the connection can be embedded into the YM field easily. That is, we look for solutions to Eq. (2.34). Amazingly, with the rep of the form: an adjoint plus ϵm copies of a lowest-dimensional irrep, we have $2l_2A_4-1=0$ and $l_2^0(2l_2^0A_4^0-1)=12$ for all the groups G_2,F_4,E_6,E_7. This can be seen from $(l_2,l_2^0,A_4,A_4^0)=(2,8,\frac{1}{4},\frac{5}{32})$, $(6,18,\frac{1}{12},\frac{5}{108})$, $(6,24,\frac{1}{12},\frac{1}{32})$, and $(12,36,\frac{1}{24},\frac{1}{54})$ for G_2, F_4, E_6, and E_7, respectively. Consequently, we must have $\epsilon_0=+1$ (the chirality of an adjoint is fixed). Furthermore, we can satisfy the trace condition for an *arbitrary* ϵm: Given ϵm, we get α' from Eq. (2.32). Then, Eq. (2.34) is always satisfied and the solution is *regular*. Consequently, the rep, consisting of an adjoint plus an arbitrary number and chirality of a lowest-dimensional irrep is a solution at $D=6$ for G_2, F_4, E_6, and E_7. This includes $\epsilon m=0$, which was discussed in Sec. III. Obviously, solutions which are converted from those at $D=14$ and 18 are also included in this general statement (except the E_8 case).

B. Classical groups

Using the local isomorphism of algebras, $A_1=B_1=C_1$, $A_3=D_3$, $B_2=C_2$, we limit ourselves to the following cases: SU(n) ($n\geq 2$), SO($2n$) ($n\geq 4$), SO($2n+1$) ($n\geq 3$), and Sp($2n$) ($n\geq 2$). Now, we have to deal with $Q_4\neq 0$ for an irrep. For SO($2n$) (n even), one must be careful, since in general they have two independent nth-order Casimir invariants and hence two nonzero indices: Q_n and Q_n' for SO($2n$). For a spinor, Q_n' is nonvanishing and $Q_n'(\lambda_n)=-Q_n'(\lambda_{n-1})$, whereas for a vector and an adjoint it is vanishing. The values of various coefficients are given in Table VIII. Now, we discuss SO(N), Sp(N), and SU(N).

1. $D=6$ for SO(N) and Sp(N)

Because of the weak constraints at $D=6$, we look for a *regular* solution, i.e., α' is given by Eq. (2.32). Then, we have, from Eqs. (2.33) and (2.34),

$$\epsilon m=-\epsilon_0 y_4\ , \tag{4.5}$$

$$\epsilon_0[l_2^0(2l_2^0A_4^0-1)-l_2y_4(2l_2A_4-1)]=12\ . \tag{4.6}$$

Because of the nonvanishing y_4, ϵm is fixed. If ϵ_0 given by the second equation were not ± 1, then we could not have a regular solution. *Remarkably,* we have $\epsilon_0=+1$

TABLE VIII. Various coefficients for SO($2n$), SO($2n+1$), and Sp($2n$), where $X_4=X_6=Y_6=0$ for a vector.

	SO($2n$)	SO($2n+1$)	Sp($2n$)
Q_2^0	$2n-2$	$2n-1$	$2n+2$
Q_4^0	$2n-8$	$2n-7$	$2n+8$
Q_6^0	$2n-32$	$2n-31$	$2n+32$
Q_2^s	2^{n-4}	2^{n-3}	
Q_4^s	-2^{n-5}	-2^{n-4}	
Q_6^s	2^{n-4}	2^{n-3}	
X_4^0	3	3	3
X_6^0	15	15	15
Y_6^0	0	0	0
X_4^s	$3\times 2^{n-7}$	$3\times 2^{n-6}$	
X_6^s	$-15\times 2^{n-8}$	$-15\times 2^{n-7}$	
Y_6^s	$15\times 2^{n-10}$	$15\times 2^{n-9}$	

for Λ_1=vector of SO(N) or Sp(N) (N even) or Λ_1=spinor of SO(N) (both N even and odd). Thus, if ϵm is found to be an integer, then that is a regular solution with α' given by Eq. (2.32).

For Λ_1=vector (i.e., λ_1) of SO(N) or Sp(N), we have $\epsilon m=-(N\pm 8)$ [+ for Sp(N) and − for SO(N)]. Thus, for an *arbitrary* N, the rep, consisting of an adjoint, $(N\pm 8)$(vectors), is a regular solution at $D=6$. Incidentally, the value of α' and the number of shadow matter fields are $\alpha'=-\frac{1}{6}$ and $l=245-N(N-15)/2$ for SO(N), and $\alpha'=\frac{1}{3}$ and $l=245-N(N+15)/2$ for Sp(N). The (absolute) fewest number of shadow matter fields is achieved at SO(31) (adjoint−23 vectors) or Sp(16) (adjoint−24 vectors) with $l=-3$.

For $\Lambda_1=\lambda_j$ $(1\le j\le n)$ of Sp($2n$), we have

TABLE IX. Solutions for Sp($2n$) with $2n\le 200$.

j	n	ϵm
2	2	3
	3	7
	5	−9
	6	−5
	8	−3
	12	−2
3	3	2
	5	2
	6	5
4	4	1

$$d(\lambda_j)=\frac{2(n+1-j)(2n+1)!}{j!(2n+2-j)!}\ , \tag{4.7}$$

$$Q_2(\lambda_j)=\frac{(n+1-j)(2n)!}{n(j-1)!(2n+1-j)!}\ , \tag{4.8}$$

$$Q_4(\lambda_j)=\frac{2(n+1-j)(2n-2)!}{(n-1)(j-1)!(2n+1-j)!}[2n^2+3n+4-3j(2n+2-j)]\ , \tag{4.9}$$

$$Q_6(\lambda_j)=\frac{8(n+1-j)(2n-5)!}{(j-1)!(2n+1-j)!}[4n^4+40n^3+95n^2+170n+96-15j(2n+2-j)(2n^2+5n+8)+30j^2(2n+2-j)^2]\ . \tag{4.10}$$

Using ϵm fixed by Eq. (4.5), we obtain

$$\epsilon m l_2^j(2l_2^jA_4^j-1)=\epsilon_0\frac{2(n-1)(n+4)(2n-1)}{2n^2+n+2}\ , \tag{4.11}$$

$$\epsilon l_2^0(2l_2^0A_4^0-1)=-\epsilon_0\frac{2(n+1)(2n^2-9n-8)}{2n^2+n+2}\ , \tag{4.12}$$

which leads to $12\epsilon_0=12$ in Eq. (2.34). Note that Eq. (4.11) is independent of j. This means that as long as $\epsilon m=-Q_4^0/Q_4^j$ is an integer, Λ=adjoint+$\epsilon m\lambda_j$ is a solution. Note that $Q_4^0=2(n+4)$ and Q_4^j is given in Eq. (4.9). As we know from the previous paragraph, $j=1$ is always a solution. Solutions with $j\ne 1$ up to Sp(200) are very rare. The results are listed in Table IX.

For Λ_1=spinor of SO(N), we calculate

$$\epsilon m=\begin{cases}\dfrac{2(n-4)}{2^{n-5}} & \text{for SO}(2n)\ (n\ge 5)\ ,\\[2ex] \dfrac{2n-7}{2^{n-4}} & \text{for SO}(2n+1)\ .\end{cases}$$

Only a few n have integer ϵm's and the results are

SO(10): $\mathbf{45}+m\,\mathbf{16}+m'\mathbf{16}^*$ $(m+m'=2)$, $\alpha'=-\frac{1}{12}$, $l=322$,

SO(12): $\mathbf{66}+m\,\mathbf{32}+m'\mathbf{32}^*$ $(m+m'=2)$, $\alpha'=-\frac{1}{18}$, $l=375$,

SO(16): $\mathbf{120}+\mathbf{128}$ (or $\mathbf{128}^*$), $\alpha'=-\frac{1}{30}$, $l=493$,

SO(7): $\mathbf{21}-2\,\mathbf{8}$, $\alpha'=-\frac{1}{3}$, $l=250$,

SO(9): $\mathbf{36}+\mathbf{16}$, $\alpha'=-\frac{1}{9}$, $l=297$.

2. $D=6$ for SU(N)

For the nonadjoint rep, we use $\Lambda_1=\lambda_j$ $(1\le j\le N-1)$, which is a totally antisymmetric irrep. The indices are calculated to be[9]

$$Q_2(\lambda_j)=\frac{j(N-j)}{N(N-1)}d(\lambda_j)\ , \tag{4.13}$$

$$Q_4(\lambda_j)=\frac{N(N+1)-6j(N-j)}{(N-2)(N-3)}Q_2(\lambda_j)\ , \tag{4.14}$$

$$Q_6(\lambda_j)=\frac{N(N+1)(N^2+15N-4)-30N(N+3)j(N-j)+120j^2(N-j)^2}{(N-2)(N-3)(N-4)(N-5)}Q_2(\lambda_j)\ , \tag{4.15}$$

$$d(\lambda_j)=\begin{bmatrix}N\\ j\end{bmatrix}\ \text{(binomial coefficient)}\ . \tag{4.16}$$

For an adjoint rep, we have

$$Q_2^0=Q_4^0=2N\ . \tag{4.17}$$

First of all, we show that $\Lambda_1=\lambda_1$ does not work for $N\ge 4$: $\epsilon F_1=(\epsilon_0 Q_2^0+\epsilon m)F_1{}^1=0$, because $Q_p(\lambda_1)=1$ and Eq. (2.16). Meanwhile, $\sum \epsilon_j m_j X_4^j=\epsilon_0 X_4^0+\epsilon m X_4(\lambda_1)=6\epsilon_0$. Hence, we have $\epsilon F_2=6\epsilon_0(F_1{}^1)^2\propto(\epsilon F_1)^2$. Thus, $F_1{}^1=\mathrm{Tr}F^2(\lambda_1)=0$, which is impossible. Q.E.D. For $N<4$, i.e., SU(3) and SU(2), $Q_4=0$ automatically and thus they must be discussed separately. However, it is easy to show that the rep, an adjoint + arbitrary number and chirality of **3** and **3*** for SU(3) is a solution, because SU(3) is a maximal subgroup of G_2 and **7** of $G_2\to$**3**+**3***+**1** of SU(3) and **14** of $G_2\to$**8**+**3**+**3*** of SU(3). Note that it does not matter what kind of combinations of **3** and **3*** are used, since there exists no constraint on the third-order indices.

Now, we show that the rep, adjoint$+\epsilon m\lambda_j$ $(2\le j\le N-2)$, is a regular solution as long as $\epsilon m=-Q_4^0/Q_4^j=y_4$ is an integer. Note we must exclude the case where $Q_4^j=0$, i.e., $N(N+1)-6j(N-j)=0$ (for example, $N=8$ and $j=2$). We look for a regular solution, i.e., a solution to Eq. (4.6). Using Eqs. (4.13)–(4.16), we obtain

$$l_2^j y_4(2l_2^j A_4^j-1)=-\frac{2(N+1)(N-2)(N-3)}{N^2+1}\quad \text{(independent of } j)\ , \tag{4.18}$$

$$l_2^0(2l_2^0A_4^0-1)=-\frac{2N(N^2-10N+1)}{N^2+1}\ , \tag{4.19}$$

which reduce Eq. (4.6) to $12\epsilon_0=12$. The Green-Schwarz coefficient α' is given by

$$\alpha'=\frac{N(N+1)-6j(N-j)}{6N(j-1)(N-1-j)}$$

from which we can tell again that we should exclude $j=1$ and $j=N-1$. The condition that $\epsilon m=-y_4$ is an integer has a small number of solutions up to SU(100). The results are listed in Table X.

Note that only the sum of the numbers, λ_j and λ_j^*, is fixed. The case of $j=3$ and $N=9$ is guessed from the E_8 solution at $D=18$ $(\alpha'=-\frac{1}{30})$: **248** of $E_8\to$adjoint $+\lambda_3+\lambda_3^*$ of SU(9).

TABLE X. Solutions for SU(N) with $N\le 100$.

j	N	ϵm	α'
2	4	2	$-\frac{1}{6}$
	6	6	$-\frac{1}{18}$
	7	14	$-\frac{1}{42}$
	9	-18	$\frac{1}{54}$
	10	-10	$\frac{1}{30}$
	12	-6	$\frac{1}{18}$
	16	-4	$\frac{1}{12}$
	24	-3	$\frac{1}{9}$
3	6	2	$-\frac{1}{12}$
	9	2	$-\frac{1}{30}$
	12	8	$-\frac{1}{192}$
	13	-26	$\frac{1}{702}$
	18	-1	$\frac{1}{42}$
4	8	1	$-\frac{1}{18}$
	16	1	$-\frac{1}{198}$

3. $D=10$ for SO(N) and Sp(N)

For $\Lambda_1=$vector, we have the sixth-order constraint:

$$\epsilon_0(N\pm 32)+\epsilon m=0\ , \tag{4.20}$$

$$15\epsilon_0=\tfrac{1}{40}[\epsilon_0(N\pm 8)+\epsilon m][\epsilon_0(N\pm 2)+\epsilon m]\ , \tag{4.21}$$

$$0=[\epsilon_0(N\pm 2)+\epsilon m]\{\tfrac{1}{16}\epsilon_0-\tfrac{1}{14\,400}[\epsilon_0(N\pm 2)+\epsilon m]^2\}\ , \tag{4.22}$$

where + [−] for Sp(N) [SO(N)]. The solution is $\epsilon_0=+1$ and $\epsilon m=-(N\pm 32)$ [+ for Sp(N) and − for SO(N)]. Consequently, for an arbitrary N, the rep, adjoint$-(N\pm 32)$ vectors [− for Sp(N) and + for SO(N)], is a solution. This solution is regular for SO(N) $(c=(\frac{1}{30})[2(N-2)-2(N-32)]/2=1)$, while it is not for Sp(N) $(c=(\frac{1}{30})[(N+2)-(N+32)]/2=-\frac{1}{2})$. It is easy to see why it is a solution for SO(N) $(N<32)$, since adjoint of SO(32)$\to$adjoint$+(32-N)$ vectors of SO(N) for

$N<32$. However, the solution is valid for $N>32$ also. The number of shadow matter fields is given by $l=-495-N(N\mp 63)/2$ [$-$ for SO(N) and $+$ for Sp(N)]. Consequently, its absolute value is minimized at $N=31$ and 32 for SO(N), while for Sp(N) it is always greater than 495. The case for $N=32$ corresponds to the Green-Schwarz solution with $l=1$.

For $\Lambda_1=\lambda_j$ $(2\le j\le n)$ of Sp($2n$), we must have $\epsilon m=-\epsilon_0 Q_6^0/Q_6^j$ to be an integer. Except for $j=1$, only a few integer solutions exist up to Sp(200): $j=2$ and $n=8,12,14,15,17,18,20,24,32,48$, or $j=3$ and $n=30$. Thus, we investigate only $j=2$ and 3 for an arbitrary n. For $j=2$, we have

$$X_4{}^2=X_4^0=3,\quad X_6{}^2=X_6^0=15,\quad Y_6{}^2=Y_6^0=0\ ,$$
$$\epsilon m=-\epsilon_0\frac{n+16}{n-16}\quad (n\neq 16)\ , \tag{4.23}$$

which lead to

$$n^2+8\epsilon_0(n-16)=0$$

for both Eqs. (2.20) and (2.21). Hence, the only solution is $(\epsilon_0=+1, n=8)$ **136**(adjoint)$+3$ **119**(λ_2) of Sp(16). Unfortunately, this is the subgroup solution of **496** of SO(32), because an adjoint of SO($4n$) goes into an adjoint$+3\lambda_2+3$ singlets of Sp($2n$). For $j=3$, we have

$$X_4{}^3=6(n-2),\quad X_6{}^3=30(n-5),\quad Y_6{}^3=15\ ,$$

which lead to no solution.

For $\Lambda_1=$spinor of SO($2n$) $(n\ge 4)$, we have three equations. Amazingly, the case where $\epsilon_0=+1$ and $\epsilon m=-2(n-16)/2^{n-4}$ satisfies all three equations, while the case where $\epsilon_0=-1$ does not. Consequently, if ϵm is an integer for a particular n, then it is a solution. The exceptions are $n=4$ and 6. This comes about because of the nonvanishing Q_4' for $n=4$ and Q_6' for $n=6$. Thus, for $n=4$ and 6, we use two kinds of spinors, λ_n and λ_{n-1}, to cancel these nonvanishing indices. However, for $n=6$, $\epsilon m=5$ and thus it is impossible to do so. Anyway, we have found solutions:

SO(8): **28**$+12$ **8**$+12$ **8′** ,

SO(10): **45**$+m$ **16**$+m'$**16*** $(m+m'=11)$,

SO(16): **120**$+$ **128** (or **128***) .

The SO(16) solution is easy to get, because of the general result that $D=18$ solutions are $D=10$ solutions also and the fact that SO(16) is a maximal subgroup of E_8 and **248**$\to$**120**$+$**128**. The SO(10) solution cannot be obtained from SO(16), since **120**$\to$**45**$+6$ **10**$+15$ **1** and **128**$\to 4$ **16**$+4$ **16***. Both solutions are regular, since $c=[(2\times 5-2)+11\times 2^{(5-4)}]/30=1$ for SO(10) and $c=[(2\times 8-2)+2^{(8-4)}]/30=1$ for SO(16). The number of shadow matter fields is given by $l=-274$ for SO(10), and -247 for SO(16).

For $\Lambda_1=$spinor of SO($2n+1$) $(n\ge 3)$, we can show that the case where $\epsilon_0=+1$ and $\epsilon m=-(2n-1)/2^{n-3}$ satisfies all three equations, while the case where $\epsilon_0=-1$ does not. The only integer solution is $\epsilon m=25$ for $n=3$. Thus, the rep

SO(7): **21**$+25$ **8**

is a regular solution, since $c=[(6-1)+25\times 2^{(3-3)}]/30=1$. The number of shadow matter fields is given by $l=-274$. By the way, the SO(7) solution is not a subgroup solution of the SO(10) or SO(16) either.

4. D = 10 for SU(N)

As we mentioned earlier, we must use the rep, consisting of an adjoint$+m(\lambda_j+\lambda_j^*)$, for dimensions higher than 6 in order to have $Q_3=0$. At $D=10$, we have to satisfy Eqs. (2.19)–(2.21). The first equation, $\epsilon_0 Q_6^0+2\epsilon m Q_6^j=0$, has only a few integer ϵm solutions, except $j=1$ where ϵm is always an integer. Up to SU(100), only $j=2$ and 3 have integer solutions. Thus, we discuss only $j=1$, 2, and 3 for an arbitrary N.

First of all, we can show that $j=1$ cannot be used. The reason is that $Q_p^1=1$ and $Q_2^0=Q_4^0=Q_6^0=2N$, while $X_6^1=0$ and $X_6^0=30(N^2+1)/(N^2+7)$. Hence, Eq. (2.20) cannot be satisfied.

For $j=2$, $N=16$ is the only solution of Eq. (2.20). However, this is the subgroup solution of SO(32): **496** of SO(32)$\to$**255**$+$**120**$+$**120***$+1$ of SU(16). For $j=3$, no solution exists for Eq. (2.20).

5. D = 14 and 18 for SO(N) and Sp(N)

First, we assume that $\Lambda_1=$vector. Then, the fourth-order constraints are

$$\epsilon_0(N\pm 8)+\epsilon m=0\ , \tag{4.24}$$

$$3\epsilon_0=K[\epsilon_0(N\pm 2)+\epsilon m]^2\ , \tag{4.25}$$

which leads to

$$K=\tfrac{1}{12}\epsilon_0\quad (\epsilon_0=\pm 1)\ . \tag{4.26}$$

Therefore, $\Lambda_1=$vector *cannot* be a solution at $D=14$ and 18, since $K=-\frac{1}{196}$ $(\frac{1}{100})$ for $D=14$ $(D=18)$. For $\Lambda_1=$spinor, the fourth-order constraint becomes, for SO($2n$),

$$\epsilon_0\left[-\frac{2(n-4)}{2^{n-5}}\right]+\epsilon m=0\ , \tag{4.27}$$

$$\epsilon_0\frac{384(n-2)}{2^{2n}}=K\left[\epsilon_0\frac{2(n-1)}{2^{n-4}}+\epsilon m\right]\ . \tag{4.28}$$

For $D=18$, K must be $\frac{1}{100}$, which yields $n=8$ $(\epsilon_0=+1, \epsilon m=+1)$, while for $D=14$, $K=-\frac{1}{196}$ $(\epsilon_0=-1)$, which yields no integer solutions for n. It is easy to see why the $n=8$ case has the solution **248** of $E_8\to$**120**(adjoint)$+$**128**(spinor) of SO(16) where SO(16) is a maximal subgroup of E_8. Thus, this solution is not particularly interesting.

For SO($2n+1$), we have

$$\epsilon_0\left[-\frac{2n-7}{2^{n-4}}\right]+\epsilon m=0\ , \tag{4.29}$$

$$\epsilon_0\frac{48(2n-3)}{2^{2n}}=K\left[\epsilon_0\frac{2n-1}{2^{n-3}}+\epsilon m\right]^2 , \quad (4.30)$$

which does not have any integer solution for $K=\frac{1}{100}$ or $-\frac{1}{196}$.

For $\Lambda_1=\lambda_j$ $(2\leq j\leq n)$ of Sp($2n$), we have looked for a solution up to Sp(200) and none is found.

6. $D=14$ and 18 for SU(N)

We must satisfy Eqs. (2.16) and (2.17) with $K=-\frac{1}{196}$ for $D=14$ and $K=\frac{1}{100}$ for $D=18$. Up to SU(100), only $j=1$, 2, 3, and 4 have solutions for Eq. (2.16). Again, we can show that $j=1$ for any N does not work. Using the relation

$$X_4^j=\frac{3[j(N-j)-(N-1)]}{(N-2)(N-3)}Q_2^j \quad (4.31)$$

we find no solution, satisfying Eqs. (2.16) and (2.17) for $j=2,3,4$ for any N.

V. SOLUTIONS WITH THREE IRREPS

Here, we consider *only* SO(N) with Λ=adjoint +vectors + spinors. Unfortunately, because $Q_4\neq 0$, we cannot use the results for eighth- and tenth-order trace identities given in Appendix B. However, we can use the fourth- or sixth-order identities to look for solutions and then, using results given in Appendix C, we explicitly verify that the higher-order identities are satisfied. In the future, we may be able to develop higher-order trace identities for those groups with $Q_4\neq 0$ and directly show the results presented here.

The coefficients needed for the calculations are listed in Table VIII. Because SO($2n$) is a maximal subgroup of SO($2n+1$), we hereafter discuss SO($2n+1$) in detail. For SO($2n$), one must watch out for the existence of two independent indices of a nth order in the case of n even.

For $D=10$ and 18, we have to satisfy the sixth-order identity, which turns out to be a weak condition for our choice of Λ_1: $\epsilon_0=+1$ and $Q_6=0$. Thus, the rep consisting of an adjoint $+\epsilon m$ vectors $+\epsilon' m'$ spinors is a solution at $D=10$, provided that $\epsilon m+2^{n-3}\epsilon' m'=31-2n$ for SO($2n+1$). We can show that this solution is regular.

For $D=18$, we have to satisfy two more constraints: fourth and tenth. Fortunately, the fourth-order trace identity fixes the number of spinors and vectors for $\epsilon_0=\pm 1$. The sign of ϵ_0 is fixed to be $+1$ by the sixth-order identity:

$$\epsilon m=15-2n \ , \quad \epsilon' m'=2^{7-n} \quad \text{for SO}(2n+1) \text{ at } D=18 \ .$$

However, this solution is not interesting in the sense that this is obtainable from the E_8 solution **248** of $E_8\to$ **120**+**128** of SO(16)→**105**+**15**+**128** of SO(15).

For $D=14$, the fourth-order identity completely fixes the number of spinors and vectors:

$$\epsilon m=-(2n+9) \ , \quad \epsilon' m'=-2^{8-n} \ , \quad \epsilon_0=+1 \ .$$

We have to check that this solution actually satisfies the eighth-order identity. This can be shown, using the results given in Appendix C. Thus, the rep, consisting of an adjoint, 25 vectors, spinor of SO(17) is a solution at $D=14$.

For $D=6$, we again look for a regular solution. It turns out that the fourth-order constraint is so weak that only $\epsilon_0=+1$ and $Q_4=0$. Thus, the rep of the form, an adjoint $+\epsilon m$ vectors$+\epsilon' m'$ spinors with $0=(2n-7)+\epsilon m-2^{n-4}\epsilon' m'$, is a solution.

Similarly, for SO($2n$) we can repeat the above calculations as long as we are careful to cancel Q_n' for n even. The results are given in the Introduction.

VI. NONSUPERGRAVITY THEORY

If a theory does not contain a gravitino, then the total anomaly at $D=4k-2$ is related to the $4k$-form given by

$$I_k=\left[\sum\epsilon n_G-l\right]\hat{A}_k+\sum_{m=0}^{k-1}\hat{A}_m\frac{(-)^{k-m}}{[2(k-m)]!}\epsilon F_{k-m} \ . \quad (6.1)$$

We require this to factorize as

$$(R_1+\alpha)(r_{k-1}+r_{k-2}+\cdots+r_1+r_0) \ , \quad (6.2)$$

where r_m is the term containing m curvature two-forms. Thus, we must have

$$R_1r_{k-1}=\left[\sum\epsilon n_G-l\right]\hat{A}_k \ , \quad (6.3)$$

$$R_1r_{k-2}+\alpha r_{k-1}=-\tfrac{1}{2}\epsilon F_1\hat{A}_{k-1} \ ,$$

$$\vdots$$

$$R_1r_0+\alpha r_1=\frac{(-)^{k-1}}{[2(k-1)]!}\epsilon F_{k-1}\hat{A}_1 \ , \quad (6.4)$$

$$\alpha r_0=\frac{(-)^k}{(2k)!}\epsilon F_k \ , \quad (6.5)$$

whose solution is

$$\sum\epsilon n_G-l=0 \ , \quad (6.6)$$

$$\epsilon F_1=\epsilon F_2=\cdots=\epsilon F_{k-2}=0 \ , \quad (6.7)$$

$$\epsilon F_k=-\frac{k(2k-1)}{24}\alpha(\epsilon F_{k-1}) \ . \quad (6.8)$$

Note that α is not fixed and thus one can choose any value for it as long as it contains a trace of square of a gauge field two-form F. The situation is similar to the $D=6$ case with a $N=1$ supergravity. Because of the constraints, Eq. (6.7), the higher the dimension, the more difficult it is to find a solution. A simple consequence of Eqs. (6.6)–(6.8) is that *for $k\geq 3$ (i.e., $D\geq 10$) there exists no solution with a single irrep of a simple group,* since F_1 is positive definite for a single irrep of a simple group. However, for $D=10$ we can find a solution with two irreps as follows.

For $D=10$, the YM constraints are

$$\epsilon F_1=0 \ , \quad \epsilon F_3=-\tfrac{5}{8}\alpha(\epsilon F_2) \ ,$$

which are equivalent to

$$\sum\epsilon_jm_jQ_2^j=0 \ , \quad (6.9)$$

$$\sum \epsilon_j m_j Q_6^j = 0 \ , \tag{6.10}$$

$$\sum \epsilon_j m_j X_6^j F_1(\square) = -\tfrac{5}{8}\alpha \left[\sum \epsilon_j m_j Q_4^j\right] \ , \tag{6.11}$$

$$\sum \epsilon_j m_j Y_6^j F_1(\square) = -\tfrac{5}{8}\alpha \left[\sum \epsilon_j m_j X_4^j\right] \ . \tag{6.12}$$

Eliminating α from the last two equations, we obtain

$$\left[\sum \epsilon_j m_j Y_6^j\right]\left[\sum \epsilon_j m_j Q_4^j\right] = \left[\sum \epsilon_j m_j X_6^j\right]\left[\sum \epsilon_j m_j X_4^j\right] \ . \tag{6.13}$$

The rep

$$\Lambda = m \text{ spinors} - m 2^{[(N-7)/2]} \text{ vectors of SO}(N) \tag{6.14}$$

with $\alpha = -F_1(\square)$ is a solution where m is an arbitrary positive or negative integer. Note that for even N, any combinations of two kinds of spinors are acceptable, except for $N=8$ and 12 where the numbers of the two different kinds of spinors have to be equal. The reason why this rep is a solution is as follows: for spinors of SO(N), it happens that $Q_2^s = Q_6^s$ so that no conflict arises between Eqs. (6.9) an (6.10). Furthermore, for a vector of SO(N), $X_4 = X_6 = Y_6 = 0$. In addition, Eq. (6.13) is trivially satisfied for any N. This can be verified directly from results given in Appendix C. For SO(8), we do not need any singlets:

$$\Lambda = m[\mathbf{8}(\text{spinor}) + \mathbf{8}'(\text{spinor}')] - 2m\,\mathbf{8}(\text{vector}) \tag{6.15}$$

for any positive or negative n. A more complicated solution is derived from the heterotic string theory by Alvarez-Gaume, Ginsparg, Moore, and Vafa and by Dixon and Harvey.[5]

VII. FURTHER CONSTRAINTS ON ANOMALY-FREE THEORIES

Having found many anomaly-free solutions, we investigate ways of further restricting them. In Sec. VII A we discuss the constraint that anomaly-free solutions yield a *chiral* four-dimensional theory. We generalize the argument by Candelas, Horowitz, Strominger, and Witten.[13] In Sec. VII B, we discuss the constraint for the absence of global anomalies.

A. Chiral four-dimensional theory

We assume that a compactified space has no isometries. The justification for this assumption is that the survival of supersymmetry at four dimension requires that the compact space be Ricci flat.[13] The isometries of a Ricci-flat space are at most U(1)'s (Ref. 14). It is not necessary to investigate the anomaly-free condition at four dimension, since there exists a general proof that the anomaly-free theory remains anomaly-free at lower dimensions, provided that the compact space has no isometries.[15] This is another reason for requiring no isometries for the compact space.

We have a gauge group G at $D=4k-2$. Upon compactification, this group is broken into $G_0 \times G'$. The reason is as follows: We must have

$$\int_K dH = \int_K (\text{Tr}R^2 + \alpha' \text{Tr}F^2) = 0 \ , \tag{7.1}$$

where K denotes the compact space and H is the field strength of the B field which appears in the $N=1$ supergravity theory. Because R acquires a vacuum expectation value (VEV) on K, R_0, we must give F the VEV on K, F_0. The group G_0 is the broken part of G. The unbroken part of G is G', which is the gauge symmetry at four dimensions when no isometry exists for the compact space. We assume that *both* G_0 *and* G' *are simple groups*. The original rep Λ of G decomposes into the sum of irreps of (G_0, G'): $\Lambda \to \sum_i (\Lambda_i^0, \Lambda_i')$.

In order to have a chiral theory at four dimension, two conditions for fermions of the rep Λ_i' must be satisfied: (1) $n_{1/2}(\Lambda_i') \neq 0$; (2) if fermions of the complex conjugate rep $\Lambda_i'^*$ exist, then $n_{1/2}(\Lambda_i') \neq n_{1/2}(\Lambda_i'^*)$, where $n_{1/2}(\Lambda_i') = n_{1/2}^L(\Lambda_i') = n_{1/2}^R(\Lambda_i')$. The first condition is obvious, while the second condition comes about, because at *four* dimension a right-handed spinor with the rep Λ_i' is the same as a left-handed spinor with the rep $\Lambda_i'^*$. [If our space-time were $(4q-2)$ dimension, then the second condition would not be needed.] The second condition requires that G' *must have complex reps.*

In order to investigate the first condition, we use the fact that the number of chiral spin-$\frac{1}{2}$ zero modes of the rep Λ_i' at four dimensions is given by the index on the compact space for the rep Λ_i^0 (Ref. 13). Omitting the subscript i for simplicity, we have

$$n_{1/2}(\Lambda') = c\int_K \hat{A}(R_0)\text{Ch}(F_0) = c\int_K \left[\sum_{l=0}^{m-1} \hat{A}_l(R_0) \frac{i^{2(m-l)-1}\text{Tr}F_0^{\,2(m-l)-1}}{(2\pi)^{2(m-l)-1}}\right]$$
$$= c\int_K \left[\frac{i^{2m-1}\text{Tr}F_0^{\,2m-1}}{(2\pi)^{2m-1}} + \hat{A}_1(R_0)\frac{i^{2m-3}\text{Tr}F_0^{\,2m-3}}{(2\pi)^{2m-3}} + \cdots + \hat{A}_{m-1}(R_0)\frac{i\,\text{Tr}F_0}{2\pi}\right] \ , \tag{7.2}$$

where the dimension of the compact space is $D'=4m-2$, since the space-time at $D=4k-2$ is compactified into $D=4+(4m-2)$ with $m=k-1$. The coefficient c depends on whether or not the original chiral fermions are Majorana. Consequently, in order to have $n_{1/2}(\Lambda') \neq 0$, we must have that *at least one of the traces of odd powers of* F_0 *is nonvanishing.* That is, for theories at a suitably high dimension, G_0 must be one of the groups which have complex reps. If the rep is real or pseudoreal, then we have $F_0 = -SF_0^t S^{-1}$ for some matrix S, which leads to $\text{Tr}F_0^{\text{odd}} = 0$. One gets stronger constraints in lower dimensions. For theories at $D=6$, the index for a compact two-dimensional space is in proportion only to $\mathrm{Tr}F_0$. Thus, G_0 cannot be a simple group and must be U(1). For theories at $D=10$, the index for a compact six-dimensional space is in proportion to only $\mathrm{Tr}F_0{}^3$ if G_0 is a simple group. Hence, G_0 must be $\mathrm{SU}(N)$ $(N\geq 3)$ [which includes SO(6), because of the local isomorphism of SU(4) and SO(6)].

Hence, the first condition for a chiral four-dimensional theory at a suitably high dimension requires that both G' and G_0 must be one of the following groups which have complex reps: $\mathrm{SU}(N)$ $(N\geq 3)$, $\mathrm{SO}(4n-2)$ $(n\geq 3)$, or E_6. Because both Λ' and Λ^0 are complex, the second condition for a chiral theory at four dimensions is automatically satisfied: Using Eq. (7.2), we have

$$n_{1/2}(\Lambda')=\mathrm{index}_K(\Lambda^0)=-\mathrm{index}_K(\Lambda^{0*})=-n_{1/2}(\Lambda'^*)\ ,$$

since $F_0(\Lambda^{0*})=-SF_0(\Lambda^0)^tS^{-1}$ for some matrix S.

Consequently, we have shown that as long as the original rep Λ of G is decomposed into complex reps of both G' and G_0, we will get a chiral four-dimensional theory.

Now, we look for a simple group G that satisfies the requirements. Since the smallest rank which gives complex reps is two, the rank of G must be greater than or equal to four. If we assume that $G_0\times G'$ is the *maximal subgroup* of G and assume that both G' and G_0 are simple groups, then the original simple group G must be one of the following:

$$\begin{aligned}&\mathrm{SO}(4p)\rightarrow\mathrm{SO}(4q-2)\times\mathrm{SO}(4(p-q)+2)\\ &\qquad\qquad\qquad (p\geq 3, q\geq 2, p-q\geq 1)\ ,\\ &\mathrm{F}_4\rightarrow\mathrm{SU}(3)\times\mathrm{SU}(3)\ ,\\ &\mathrm{E}_7\rightarrow\mathrm{SU}(3)\times\mathrm{SU}(6)\ ,\\ &\mathrm{E}_8\rightarrow\mathrm{SU}(3)\times\mathrm{E}_6 \text{ or } \mathrm{SU}(5)\times\mathrm{SU}(5)\ .\end{aligned}\tag{7.3}$$

Note that having just an adjoint rep or vector reps for SO(4p) does not yield complex reps at the broken level. One must have a spinor rep (or, in general, a tensor-spinor rep) for SO(4p). On the other hand, adjoint reps for F_4, E_7, and E_8 have complex reps at the broken level:

$$\begin{aligned}&\mathbf{52}\text{ of }\mathrm{F}_4\rightarrow(6,3^*)+(6^*,3)+(1,8)+(8,1)\text{ of }(\mathrm{SU}(3),\mathrm{SU}(3))\ ,\\ &\mathbf{133}\text{ of }\mathrm{E}_7\rightarrow(3^*,15)+(3,15^*)+(1,35)+(8,1)\text{ of }(\mathrm{SU}(3),\mathrm{SU}(6))\ ,\\ &\mathbf{248}\text{ of }\mathrm{E}_8\rightarrow(3,27)+(3^*,27^*)+(1,78)+(8,1)\text{ of }(\mathrm{SU}(3),\mathrm{E}_6)\\ &\qquad\rightarrow(5,10^*)+(5^*,10)+(10^*,5)+(10,5^*)+(24,1)+(1,24)\text{ of }(\mathrm{SU}(5),\mathrm{SU}(5))\ .\end{aligned}$$

Note that although in the decomposition the same number of complex reps and complex-conjugate reps appear, the net chirality at four dimensions can be nonvanishing, because of the presence of the VEV of R_0 and F_0.

We have found that although many anomaly-free theories exist at various $4k-2$ dimensions, only a small subset of these theories have the possibility of providing a chiral four-dimensional theory, *provided* that both G' and G_0 are simple groups (not semisimple) and $G'\times G_0$ is the maximal subgroup of G.

If we relax the condition on G' and G_0, then the possibilities explode: If G' or G_0 is semisimple, then E_6 is also valid: $\mathrm{E}_6\rightarrow\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{SU}(3)$. If a nonmaximal subgroup is allowed, then all the groups with suitably high ranks are valid. For example, $\mathrm{Sp}(2n)$(rank n)$\rightarrow\mathrm{SU}(n)$(rank $n-1$)$\rightarrow\mathrm{SU}(p)\times\mathrm{SU}(n-p)$(total rank $n-2$). We do not know at this point what is the physical reason for requiring maximal subgroups.

B. Safe groups for the absence of global gauge anomalies

So far, we have considered the absence of local anomalies. In this subsection, we discuss the constraints for the absence of global gauge anomalies. Because we are concerned with gauge configurations which approach constant at infinity, we discuss Euclidian theories formulated on a D-dimensional sphere. Global gauge transformation is defined as gauge transformations which cannot be reached continuously from the identity. Therefore, the *possible* presence of global anomalies is signaled by the nontrivial homotopy group $\Pi_D(G)$ where G is the gauge group. Thus, we term those groups which have trivial homotopy groups, i.e., $\Pi_D(G)=0$, as *safe groups.* (However, the absence of the homotopy group does not guarantee the absence of global gauge anomalies for any manifold, as discussed by Witten.[16]) Note that both SO(32) and E_8 at $D=10$ are safe groups. By using safe groups for physical theories, we can limit the choices for possible groups. Here, we list safe groups for each $D=4k-2$ space. Note that the compact connected Lie group G are only the following: $\mathrm{SO}(n)$ $(n\geq 2)$, $\mathrm{Spin}(n)$ $(n\geq 3)$, $\mathrm{U}(n)$ $(n\geq 1)$, $\mathrm{SU}(n)$ $(n\geq 2)$, $\mathrm{Sp}(2n)$ $(n\geq 1)$, G_2, F_4, E_6, E_7, and E_8.

For classical groups, Bott's periodicity theorem makes it easier to draw conclusions on homotopy groups for suitably high-rank classical groups. For theories at $D=4k-2$, Bott's periodicity theorem states that[17]

$$\begin{aligned}&\Pi_D(\mathrm{SO}(n))=\Pi_D(\mathrm{Spin}(n))=0\quad\text{for } n\geq D+2\ ,\\ &\Pi_D(\mathrm{SU}(n))=\Pi_D(\mathrm{U}(n))=0\quad\text{for } n\geq\frac{D+1}{2}\ ,\\ &\Pi_D(\mathrm{Sp}(2n))=0\text{ for } n\geq\frac{D-1}{4}\ .\end{aligned}\tag{7.4}$$

Therefore, classical groups with suitably high ranks are always *safe.* Interestingly, those groups with smaller ranks than those specified above usually have nonvanishing homotopy groups, except SO(n) with $n=5,6,7$ at $D=6$ (Ref. 18). The list for each dimension is

Dimension	Safe groups
6	SO(n),Spin(n) ($n\geq 7$)
	SU(n),U(n) ($n\geq 4$)
	Sp($2n$) ($n\geq 2$)
10	SO(n),Spin(n) ($n\geq 12$)
	SU(n),U(n) ($n\geq 6$)
	Sp($2n$) ($n\geq 3$)
14	SO(n),Spin(n) ($n\geq 16$)
	SU(n),U(n) ($n\geq 8$)
	Sp($2n$) ($n\geq 4$)
18	SO(n),Spin(n) ($n\geq 20$)
	SU(n),U(n) ($n\geq 10$)
	Sp($2n$) ($n\geq 5$)

For exceptional groups, we have to investigate homotopy groups dimension by dimension. The result is[18]

Dimension	Safe groups
6	F_4,E_6,E_7,E_8
10	G_2,F_4,E_6,E_7,E_8
14	E_6,E_7,E_8
18	None

Consequently, E_8 group at $D=18$ may have the global anomaly and thus needs a careful analysis.

VIII. THEORIES WITH MORE THAN ONE GRAVITINO

Using the same fermionic content as before, the $4k$-form for theories with p gravitinos is given by

$$I_k=n_p\hat{A}_k+\sum_{m=0}^{k-1}\hat{A}_m\left[pG_{k-m}+\frac{(-)^{k-m}\sum\epsilon F_{k-m}}{[2(k-m)]!}\right],$$
$$n_p=\sum\epsilon n_G+p(4k-3)-l\ . \tag{8.1}$$

Thus, the dominant pure gravitational anomaly cancellation occurs when

$$n_p+(-)^k p\frac{4k}{B_k}=0\ . \tag{8.2}$$

Because n_p has to be an integer, one can find a smallest possible p for each k. The result is given in Table XI for k up to 15. For $k\geq 16$, $p>2.35\times 10^8$, which can be obtained from the inequality given in Sec. II. Therefore, as long as we limit ourselves to theories with the number of gravitinos less than 691, we do not have to consider theories at $D=22$ or $D>26$ among theories at $D=4k-2$. For those theories at $D=2$, 6, 10, 14, 18, and 26, the $4k$-form becomes

$$\frac{I_k}{p}=(-)^{k+1}\frac{4k}{B_k}\hat{A}_k+\sum_{m=0}^{k-1}\hat{A}_m\left[G_{k-m}+\frac{(-)^{k-m}}{[2(k-m)]!}\sum\epsilon\left[\frac{F_{k-m}}{p}\right]\right],$$

which is the same as Eq. (2.1) with Eq. (2.2) except the fact that F_j is replaced by F_j/p. Therefore, the YM trace constraints for p gravitinos are obtained by just replacing F_j with F_j/p. Solutions for the YM constraints are difficult to find, except those trivial solutions which are just p copies of one-gravitino solutions.

Nontrivial multigravitino solutions exist if we expand the fermionic content, by adding tensor spinors, which have both tensor indices and spinor indices, or multispinors, which are products of spinors. The well-known

TABLE XI. The smallest number of gravitinos for $D=4k-2$.

D	k	B_k	$\frac{4k}{B_k}$	Smallest p
2	1	$\frac{1}{6}$	24	1
6	2	$\frac{1}{30}$	240	1
10	3	$\frac{1}{42}$	504	1
14	4	$\frac{1}{30}$	480	1
18	5	$\frac{5}{66}$	264	1
22	6	$\frac{691}{2730}$	$\frac{65\,520}{691}$	691
26	7	$\frac{7}{6}$	24	1
30	8	$\frac{3617}{510}$	$\frac{16\,320}{3617}$	3 617
34	9	$\frac{43\,867}{798}$	$\frac{28\,728}{43\,867}$	43 867
38	10	$\frac{174\,611}{330}$	$\frac{13\,200}{174\,611}$	174 611
42	11	$\frac{854\,513}{138}$	$\frac{552}{77\,683}$	77 683
46	12	$\frac{236\,364\,091}{2730}$	$\frac{131\,040}{236\,364\,091}$	236 364 091
50	13	$\frac{8\,553\,103}{6}$	$\frac{24}{657\,931}$	657 931
54	14	$\frac{23\,749\,461\,029}{870}$	$\frac{6960}{3\,392\,780\,147}$	3 392 780 147
58	15	$\frac{8\,615\,841\,276\,005}{14\,322}$	$\frac{171\,864}{1\,723\,168\,255\,201}$	1 723 168 255 201

type-IIB theory contains a self-dual totally antisymmetric tensor, which is equivalent to the symmetric product of two spinors of the same chirality in the anomaly counting. Thierry-Mieg,[3] Schellekens,[4] and Schellekens and Warner[8] have found solutions which have additional matter fields mentioned above.

Note added. After completing our paper, we have noticed two papers on the search of anomaly-free field theories: A. N. Schellekens and N. P. Warner, Phys. Lett. B **181**, 339 (1986); Nucl. Phys. **B287**, 317 (1987). They used the modular invariance for the search and found an infinite class of anomaly cancellable theories, most of which seem to contain different matter fields from those we have used in our paper. Unfortunately, our unfamiliarity of their method makes it difficult for us to directly compare our results with theirs.

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APPENDIX A: GENERATING FUNCTIONS FOR ANOMALIES

In this appendix the generating function for the Dirac genus is shown to be given in terms of Bernoulli numbers. Actually, we develop general formulas for the generating functions of so-called multiplicative sequences. Hereafter we use the notation that, if no upper limit is indicated over the summation symbol, then the sum goes to infinity.

Note that the Pontryagin genus P, the Dirac genus $\hat{A}$, and the Hirzebruch genus L satisfy the two constraints: (1) $K(0)=1$; (2) $K(M\times N)=K(M)\cdot K(N)$ for the Cartesian product $M\times N$, where K stands for P, $\hat{A}$, or L and the parentheses denote the space they act on. Therefore, one can expect that generating functions for these polynomials K_j of $\mathrm{Tr}R^{2j}$,

$$K(z)=\sum_{j=0} K_j(R_1,\ldots,R_j)z^j \quad \left[R_j=\frac{\mathrm{Tr}R^{2j}}{(2\pi)^{2j}}\right]$$

should be of the form exp( ). In fact, the results are

$$P(z)=\sum_{j=0} p_j z^j=\exp\left[-\sum_{k=1}\frac{R_k}{2k}z^k\right], \tag{A1}$$

$$\hat{A}(z)=\sum_{j=0}\hat{A}_j z^j=\exp\left[\sum_{k=1}\frac{B_kR_k}{4k(2k)!}z^k\right], \tag{A2}$$

$$L(z)=\sum_{j=0} L_j z^j$$
$$=\exp\left[-\sum_{k=1}\frac{2^{2k}(2^{2k-1}-1)B_kR_k}{2k(2k)!}z^k\right], \tag{A3}$$

where the exponential is defined by the usual expansion and $R_k=\mathrm{Tr}R^{2k}/(2\pi)^{2k}$. The sequence $\{B_k\}$ are Bernoulli numbers,[19] e.g., $B_1=\frac{1}{6}$, $B_2=\frac{1}{30}$, $B_3=\frac{1}{42}$, $B_4=\frac{1}{30}$, $B_5=\frac{5}{66}$, $B_6=\frac{691}{2730}$, $B_7=\frac{7}{6}$, $B_8=\frac{3617}{510}$, ... Note that these generating functions are of the form

$$K(z)=\sum_{j=0}K_jz^j=\exp\left[\sum_{k=1}C_kR_kz^k\right] \tag{A4}$$

and the summation inside the large parentheses starts from $k=1$ (not $k=0$). We show later that this should be the case. This form makes the calculation of the coefficients of these polynomials easier, because of the following proposition which can be proved easily.[8]

Proposition 1. For a formal series

$$\sum K_jz^j=\exp\left[\sum_{k=1}C_kR_kz^k\right]$$
$$=\sum_{n=0}\frac{1}{n!}\left[\sum_{m=1}C_mR_mz^m\right]^n$$

the coefficient of $R_{a_1}^{b_1}R_{a_2}^{b_2}\cdots R_{a_r}^{b_r}$ in K_k ($k=\sum a_mb_m$ and $1\le a_1<a_2<\cdots<a_r\le k$) is given by

$$\frac{1}{H(1)}\prod_{n=1}^{r}\begin{bmatrix}H(n)\\ b_n\end{bmatrix}(C_{a_n})^{b_n}, \tag{A5}$$

where

$$H(n)=\sum_{m=n}^{r}b_m .$$

The parentheses after $\prod$ denote the binomial coefficient. In particular, the coefficient of R_k ($\propto \mathrm{Tr}R^{2k}$) in K_k is given by C_k. The coefficients of $(R_1)^k$ in K_k are given by $(C_1)^k/(k!)$.

Consequently, the calculation of the coefficients in K_k is reduced to getting the partition of the number k into $\{a_m,b_m\}$, which satisfy the constraints indicated in the proposition. For example, let us derive K_3. Because the partitions of 3 are 3, 2 + 1, 1 + 1 + 1, we have terms in K_3 such as $R_3, R_2R_1, R_1{}^3$. Using Proposition 1, we obtain

$$K_3=C_3R_3+C_2C_1R_2R_1+\frac{1}{3!}C_1{}^3R_1{}^3 .$$

In order to get $p_3,\hat{A}_3,L_3$, we take $C_k=-1/2k$, $B_k/[4k(2k)!]$, $-2^{2k}(2^{2k-1}-1)B_k/[2k(2k)!]$, respectively. The results are in complete agreement with Alvarez-Gaume and Ginsparg.[1] Note that for $\hat{A}_k$ they factorized $(4\pi)^{2k}$ out, instead of $(2\pi)^{2k}$. An easy way of getting the partitions is to use a Young tableaux of k boxes.[20] The generating function for the number of ways of partitioning a number k, $p(k)$, is given by[20]

$$\sum_{k=0}p(k)x^k=\prod_{n=1}(1-x^n)^{-1} \quad \text{(Euler)} . \tag{A6}$$

Now, we derive the results, Eqs. (A1)–(A3). In order to be as self-contained as much as possible, we review the terminology and some facts used to derive the main result, following Hirzebruch.[21] The sequence $\{K_j\}$ is called an m sequence (or multiplicative sequence) if every identity of the form

$$\sum_{j=0}p_jz^j=\left[\sum_{m=0}p'_mz^m\right]\left[\sum_{n=0}p''_nz^n\right] \tag{A7}$$

implies an identity

$$\sum_{j=0} K_j(p_1,\ldots,p_j)z^j = \left[\sum_{m=0} K_m(p'_1,\ldots,p'_m)z^m\right] \times \left[\sum_{n=0} K_n(p''_1,\ldots,p''_n)z^n\right] , \quad \text{(A8)}$$

where the summation goes to infinity and $p_0=p'_0=p''_0=K_0=1$. For our purpose, p_j is the Pontryagin class and K_j is the polynomial of p_j for a real manifold. Using the abbreviated notation

$$K\left[\sum_{j=0} p_j z^j\right] = \sum_{j=0} K_j(p_1,\ldots,p_j)z^j , \quad \text{(A9)}$$

we introduce the characteristic power series $Q(z)$ of the m sequence $\{K_j\}$ by

$$Q(z)=K(1+z)=\sum_{j=0} b_j z^j , \quad \text{(A10)}$$

where $b_0=1$ and $b_j=K_j(1,0,0,\ldots,0)$. In our case the $Q(z)$ are given as

$$Q(z)=1+z \quad \text{for } P ,$$
$$Q(z)=\frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)} \quad \text{for } \hat{A} , \quad \text{(A11)}$$
$$Q(z)=\frac{\sqrt{z}}{\tanh\sqrt{z}} \quad \text{for } L .$$

The following is the key lemma for our formulas due to Hirzebruch.[21]

Lemma. The m sequence $\{K_j\}$ is completely determined by its characteristic power series $Q(z)$ as

$$\sum_{j=0}^{m} K_j(p_1,\ldots,p_j)z^j + \sum_{j=m+1}^{\infty} K_j(p_1,\ldots,p_m,0,0,\ldots,0)z^j = \prod_{j=1}^{m} Q(\beta_j z) , \quad \text{(A12)}$$

where β_j are defined as the factorization

$$\sum_{j=0}^{m} p_j z^j = \prod_{j=1}^{m} (1+\beta_j z) . \quad \text{(A13)}$$

The proof is easy, using Eqs. (A9) and (A10). Any polynomial K_j with $j \le m$ is determined as a symmetric polynomial in the β_i and hence as a polynomial in the p_i. By introducing a formal factorization for $Q(z)$ as

$$\sum_{j=0} b_j z^j = \prod_{j=1} (1+\beta'_j z) \quad \text{(A14)}$$

we can write the m sequence formally as

$$\sum_{m=0} K_m(p_1,\ldots,p_m)z^m = \prod_{j=1}\left[\sum_{k=0} p_k \beta'^k_j z^k\right] , \quad \text{(A15)}$$

where we used Eq. (A13). Since the β'_j are expressed in terms of the b_j, K_k is completely fixed. Atiyah and Hirzebruch proved that for $\hat{A}$ and L polynomials, K_k can be written as a polynomial of Pontryagin classes with coprime integer coefficients divided by $\prod q^{[2k/(q-1)]}$ (a product over all primes q with $3 \le q \le 2k+1$) (times 2 to some power for $\hat{A}$) (Ref. 21). For mathematicians, it is important to express these polynomials in terms of Pontryagin classes. However, a great simplification occurs when we express p_k in terms of traces of a curvature two-forms. The situation is similar for the case of the Chern character Ch(F) expressed by traces or Chern classes $\{c_j\}$:

$$\begin{aligned}\mathrm{Ch}(F) &= \mathrm{Tr}\left[\exp\left[\frac{iF}{2\pi}\right]\right] \\ &= \dim(r) + \frac{i}{2\pi}\mathrm{Tr}F + \frac{i^2}{2(2\pi)^2}\mathrm{Tr}F^2 + \cdots \\ &= \dim(r) + c_1 + \frac{c_1{}^2 - 2c_2}{2} + \frac{c_1{}^3 - 3c_1c_2 + 3c_3}{3!} \\ &\quad + \cdots . \end{aligned} \quad \text{(A16)}$$

In order to derive the formulas, we find the expression for p_k in terms of $\mathrm{Tr}R^{2m}$ where R is the curvature two-form. Define the x_j's as two-form skew eigenvalues of the curvature two-form $R_{jk}/(2\pi)$, since $R_{jk}=-R_{kj}$. Then we have

$$\mathrm{Tr}R^{2k} = (-)^k 2(2\pi)^{2k}\sum (x_j{}^2)^k , \quad \text{(A17)}$$

where the summation is over j. Meanwhile, the Pontryagin classes p_j are given by elementary symmetric polynomials of $x_j{}^2$:

$$\begin{aligned} &p_1 = \sum_j x_j{}^2 , \quad p_2 = \sum_{i<j} x_i{}^2 x_j{}^2 , \\ &p_3 = \sum_{i<j<k} x_i{}^2 x_j{}^2 x_k{}^2, \ldots \end{aligned} \quad \text{(A18)}$$

which can be generated by

$$P(z) = \sum_{k=1} p_k z^k = \prod_{j=1} (1+x_j{}^2 z) . \quad \text{(A19)}$$

Using the formal Taylor expansion

$$\ln(1+x_j{}^2 z) = \sum_{n=1} (-)^{n-1}\frac{x_j{}^{2n}}{n} z^n , \quad \text{(A20)}$$

we obtain

$$\begin{aligned}\ln P(z) &= \ln\prod_{j=1}(1+x_j{}^2 z) \\ &= \sum_{j=1}\sum_{n=1}(-)^{n-1}\frac{x_j{}^{2n}}{n} z^n \\ &= \sum_{n=1}\frac{(-)^{n-1}}{n}\left[\sum_{j=1} x_j{}^{2n}\right] z^n . \end{aligned} \quad \text{(A21)}$$

Using Eq. (A17), we therefore have

$$P(z) = \sum_{k=0} p_k z^k = \exp\left[-\sum_{n=1}\frac{R_n}{2n} z^n\right] , \quad \text{(A22)}$$

where $R_n = \mathrm{Tr}R^{2n}/(2\pi)^{2n}$. Note that the series in the parentheses starts from $n=1$, not $n=0$. This fact makes our formulas useful. The inverse problem to express R_k in terms of p_j's has the solution called the Waring formula.[22] Another way is to use Newton's formula:

$$2kp_k=-R_k-\sum_{m=1}^{k-1}R_m p_{k-m} \ . \quad \text{(A23)}$$

By this formula, one obtains p_k (R_k) in terms of R_j (p_j), sequentially from $k=1$. Since $P(z)=\sum p_k z^k$, Eq. (A15) can be rewritten as

$$\sum_{j=0} K_j z^j=\prod_{j=1} P(\beta_j' z)$$
$$=\exp\left[-\tfrac{1}{2}\sum_{k=1}\left[\sum_{j=1}\beta_j'^k\right]\frac{R_k}{k}z^k\right]$$
$$=\exp\left[-\tfrac{1}{2}\sum_{k=1}\frac{S_k R_k}{k}z^k\right] ,$$

where $S_k=\sum\beta_j'^k$ and $R_k=\mathrm{Tr}R^{2k}/(2\pi)^{2k}$. Note again that the series in the parentheses starts from $k=1$, not $k=0$. The sequence $\{S_j\}$ can be derived by $Q(z)$ as

$$Q(z)\frac{d}{dz}\left[\frac{z}{Q(z)}\right]=\sum_{j=0}(-)^j S_j z^j \ . \quad \text{(A24)}$$

This is called Cauchy's formula.[21] This can be proven as follows: Using $Q(z)=\prod(1+\beta_j' z)$, we obtain

$$Q(z)\frac{d}{dz}\left[\frac{z}{Q(z)}\right]=1-z\frac{d}{dz}\ln Q(z)$$
$$=1-\sum\frac{\beta_j' z}{1+\beta_j' z}$$
$$=\sum(-)^k\sum(\beta_j')^k z^k=\sum(-)^k S_k z^k \ .$$

Consequently, we have proven our main result.

Proposition 2. Using the S_k defined above, any m sequence can be expressed formally as

$$\sum K_k z^k=\exp\left[-\sum_{k=1}\frac{S_k R_k}{2k}z^k\right] , \quad \text{(A25)}$$

where $R_{2k}=\mathrm{Tr}R^{2k}/(2\pi)^{2k}$.

For the Pontryagin polynomial, we take $Q(z)=1+z$. Thus, we obtain

$$Q(z)\frac{d}{dz}\left[\frac{z}{Q(z)}\right]=\frac{1}{1+z}=1+\sum_{k=1}(-)^k z^k$$

which leads to

$$S_k=1 \ \text{ or } C_k=-\frac{1}{2k} \quad (k\geq 1) , \quad \text{(A26)}$$

where C_k is defined in Eq. (A4). Therefore, we obtain Eq. (A1).

For the Dirac genus $\hat{A}$, we take $Q(z)=\sqrt{z}/[2\sinh(\sqrt{z}/2)]$. Thus, we have

$$Q(z)\frac{d}{dz}\left[\frac{z}{Q(z)}\right]=\frac{1}{2}\left[\frac{\sqrt{z}/2}{\tanh(\sqrt{z}/2)}+1\right]$$
$$=1+\sum_{k=1}(-)^{k-1}\frac{B_k}{2(2k)!}z^k ,$$

where we used the definition of Bernoulli numbers given by

$$\frac{x/2}{\tanh(x/2)}=\frac{x}{e^x-1}+\frac{x}{2}=1+\sum_{k=1}\frac{(-)^{k-1}B_k}{(2k)!}x^{2k} \ .$$

This is the reason why Bernoulli numbers appear. Thus, we have

$$S_k=-\frac{B_k}{2(2k)!} \quad (k\geq 1) , \quad S_0=1 ,$$

or (A27)

$$C_k=\frac{B_k}{4k(2k)!} \quad (k\geq 1) \ .$$

The explicit coefficients C_k for the Dirac genus are given by

$$C_1{}^{-1}=2^4\times 3 , \quad C_2{}^{-1}=2^7\times 3^2\times 5 ,$$
$$C_3{}^{-1}=2^7\times 3^4\times 5\times 7 , \quad C_4{}^{-1}=2^{12}\times 3^3\times 5^2\times 7 ,$$
$$C_5{}^{-1}=2^{11}\times 3^5\times 5^2\times 7\times 11 ,$$
$$C_6{}^{-1}=(691)^{-1}2^{14}\times 3^7\times 5^3\times 7^2\times 11\times 13 ,$$
$$C_7{}^{-1}=2^{14}\times 3^6\times 5^2\times 7^2\times 11\times 13 ,$$
$$C_8{}^{-1}=(3617)^{-1}2^{21}\times 3^7\times 5^4\times 7^2\times 11\times 13\times 17 \ .$$

These will allow us to calculate anomaly coefficients up to $D=30$.

For the L polynomial, we obtain

$$S_k=\frac{2^{2k}(2^{2k-1}-1)B_k}{(2k)!} ,$$
$$S_0=1 , \quad C_k=-\frac{S_k}{2k} \quad (k\geq 1) \quad \text{(A28)}$$

since $Q(z)=\sqrt{z}/\tanh\sqrt{z}$ leads to

$$Q(z)\frac{d}{dz}\left[\frac{z}{Q(z)}\right]=\frac{1}{2}\left[\frac{2\sqrt{z}}{\sinh(2\sqrt{z})}+1\right] .$$

The reason why we get this form for S_k becomes clear when we notice that

$$\frac{2x}{\sinh(2x)}=2\frac{2x}{e^{2x}-1}-\frac{4x}{e^{4x}-1} \ .$$

Delbourgo and Matsuki do not appear to have noticed this relation.[8]

APPENDIX B: EVEN-ORDER TRACE IDENTITIES

Here, we give formulas for even-order trace identities. For the fourth- and sixth-order results, see Ref. 9 for derivation. For the eighth- and tenth-order results, this paper is the first place they appear, but we refer to the future paper for derivation. We use the following notations: a is the dimension of an adjoint rep Λ_0, d the dimension of an irrep Λ, $y_0=a/d$, $y_p=D_p(\Lambda_0)/D_p(\Lambda)=Q_p(\Lambda_0)/Q_p(\Lambda)$, D_p=trace of the pth-order Casimir invariant for Λ, f_p=quantity independent of an irrep, and X=generic representation matrix of Λ. Note that the index Q_p is the normalized D_p (Ref. 9).

First, we list the order p for which D_p is in general nonvanishing:

Simple group	Order p
SU(N)	2,3,4, . . . , N
SO($2n+1$)	2,4,6, . . . , $2n$
Sp($2n$)	2,4,6, . . . , $2n$
SO($2n$)	2,4,6, . . . , $(2n-2)$, and n
G_2	2,6
F_4	2,6,8,12
E_6	2,5,6,8,9,12
E_7	2,6,8,10,12,14,18
E_8	2,8,12,14,18,20,24,30

Now, we give general formulas for trace identities as follows.

Fourth order: We have

$$\mathrm{Tr}X^4=f_4D_4+A_4(\mathrm{Tr}X^2)^2 , \tag{B1}$$

where

$$A_4=\frac{3}{a+2}(y_0-\tfrac{1}{6}y_2) .$$

Sixth order: For those groups with $D_3=0$, then we have

$$\mathrm{Tr}X^6=f_6D_6+B_6\mathrm{Tr}X^2[\mathrm{Tr}X^4-A_4(\mathrm{Tr}X^2)^2]$$

where

$$B_6=\frac{15}{a+8}(y_0-\tfrac{1}{3}y_2+\tfrac{1}{30}y_4) ,$$

$$A_6=\frac{15}{(a+2)(a+4)}(y_0{}^2-\tfrac{1}{2}y_0y_2+\tfrac{1}{12}y_2{}^2) .$$

Eighth order: For those groups with $D_3=D_4=0$, we have

$$\mathrm{Tr}X^8=f_8D_8+C_8\mathrm{Tr}X^2[\mathrm{Tr}X^6-A_6(\mathrm{Tr}X^2)^3] + A_8(\mathrm{Tr}X^2)^4 , \tag{B3}$$

where

$$C_8=\frac{28}{a+12}(y_0-\tfrac{1}{2}y_2-\tfrac{1}{42}y_6) ,$$

$$A_8=\frac{105}{(a+2)(a+4)(a+6)}(y_0{}^3-y_0{}^2y_2+\tfrac{1}{3}y_0y_2{}^2-\tfrac{1}{18}y_2{}^3) + \frac{35}{(a+2)^2(a+6)}y_2{}^2(\tfrac{1}{4}y_0-\tfrac{1}{24}y_2) .$$

Tenth order: For those groups with $D_3=D_4=D_5=0$, the tenth-order trace identity is given by

$$\mathrm{Tr}X^{10}=f_{10}D_{10}+E_{10}\mathrm{Tr}X^2\{\mathrm{Tr}X^8-C_8\mathrm{Tr}X^2[\mathrm{Tr}X^6-A_6(\mathrm{Tr}X^2)^3]-A_8(\mathrm{Tr}X^2)^4\} +C_{10}[\mathrm{Tr}X^6-A_6(\mathrm{Tr}X^2)^3](\mathrm{Tr}X^2)^2+A_{10}(\mathrm{Tr}X^2)^5 , \tag{B4}$$

where

$$E_{10}=\frac{5}{(a+16)}(9y_0-6y_2+\tfrac{3}{10}y_8) ,$$

$$C_{10}=\frac{5}{2(a+2)(a+12)(a+14)} \times\{42(a+2)(6y_0-7y_2)y_0+\tfrac{21}{2}(9a+28)y_2{}^2 +[(9a+28)y_2-12(a+7)y_0]y_6\} ,$$

$$A_{10}=\frac{35}{(a+2)^2(a+4)(a+6)(a+8)} \times[9(a+2)y_0{}^3(3y_0-5y_2)+\tfrac{45}{4}(3a+8)y_0{}^2y_2{}^2 +\tfrac{5}{16}(7a+24)y_2{}^3(y_2-6y_0)] .$$

For an adjoint rep, trace identities become a function of only the dimension of the adjoint rep

For $D_4=0$,

$$\mathrm{Tr}X^4=\frac{5}{2(a+2)}(\mathrm{Tr}X^2)^2 . \tag{B5}$$

For $D_3=D_6=0$,

$$\mathrm{Tr}X^6=\frac{21}{2(a+8)}\mathrm{Tr}X^4\mathrm{Tr}X^2 -\frac{35}{2(a+4)(a+8)}(\mathrm{Tr}X^2)^3 . \tag{B6}$$

For $D_3=D_4=D_8=0$,

$$\mathrm{Tr}X^8=\frac{40}{3(a+12)}\mathrm{Tr}X^2\mathrm{Tr}X^6 -\frac{175(11a+12)}{24(a+2)^2(a+6)(a+12)}(\mathrm{Tr}X^2)^4 . \tag{B7}$$

For $D_3=D_4=D_5=D_6=D_{10}=0$,

$$\mathrm{Tr}X^{10}=\frac{33}{2(a+16)}\mathrm{Tr}X^2\mathrm{Tr}X^8 -\frac{385(9a+8)}{8(a+2)^2(a+4)(a+8)(a+16)}(\mathrm{Tr}X^2)^5 . \tag{B8}$$

APPENDIX C: SO(N) TRACE IDENTITIES

Here, we express the traces of adjoint and spinor irreps in terms of traces of a vector irrep. We use the abbreviation that $X_k^j = \mathrm{Tr}X^{2k}$ ($j=0$ for an adjoint rep and s for a spinor rep) and $x_k = \mathrm{Tr}X^{2k}$ (for a vector rep).

For an adjoint rep, we calculate for SO(N) (upper sign) and Sp(N) (lower sign):

$$X_1^0 = (N \mp 2)x_1 ,$$
$$X_2^0 = (N \mp 8)x_2 + 3x_1^2 ,$$
$$X_3^0 = (N \mp 32)x_3 + 15x_1x_2 ,$$
$$X_4^0 = (N \mp 128)x_4 + 28x_1x_3 + 35x_2^2 ,$$
$$X_5^0 = (N \mp 512)x_5 + 45x_1x_4 + 210x_2x_3 ,$$

For a spinor of SO(N) ($N=2n+1$) or a sum of two spinors, λ_n and λ_{n-1}, of SO($N-1$) ($N=2n+1$), we find

$$X_1^s = 2^{n-3}x_1 ,$$
$$X_2^s = -2^{n-4}x_2 + 3\times 2^{n-6}x_1^2 ,$$
$$X_3^s = 2^{n-3}x_3 - 15\times 2^{n-7}x_1x_2 + 15\times 2^{n-9}x_1^3 ,$$
$$X_4^s = -17\times 2^{n-5}x_4 + 7\times 2^{n-4}x_1x_3 + 35\times 2^{n-12}(-24x_1^2x_2 + 3x_1^4 + 16x_2^2) ,$$
$$X_5^s = 31\times 2^{n-3}x_5 - 765\times 2^{n-8}x_1x_4 - 105\times 2^{n-6}x_2x_3 + 315\times 2^{n-8}x_1^2x_3 + 1575\times 2^{n-11}x_1x_2^2 - 1575\times 2^{n-12}x_1^3x_2 + 945\times 2^{n-15}x_1^5 .$$

Note that the coefficient of x_p for X_p^s ($j=0$ or s) is precisely Q_{2p}^j ($j=0$ or s).

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