Understanding complex perturbative effective potentials

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We examine the situation where the perturbatively calculated effective potential develops an imaginary part. We show that his imaginary part has a natural interpretation as a decay rate per unit volume of a well-defined state and that it agrees quantitatively with an independent calculation of this rate. We examine in some detail the nature of this decay process.

I. INTRODUCTION

The effective potential, $V_{\rm eff}$, has been found to be a useful tool for investigating spontaneous symmetry breaking and other properties of quantum field theories. However, in certain theories one encounters the difficulty that perturbative calculations suggest that $V_{\rm eff}$ becomes complex for some values of the fields, in contradiction with naive expectation and formal proofs. In this paper we show that this does not signal an utter breakdown of the calculational method, but is instead indicative of a physical instability, whose nature we investigate. It will be seen that the imaginary part of the perturbatively calculated $V_{\rm eff}$ has a natural interpretation as the decay rate per unit volume corresponding to this process and that it agrees quantitatively with an independent calculation of this rate.

Formally, $V_{\rm eff}$ is defined in terms of a Legendre transformation of $W(J)$, the generating functional of connected Green's functions. It can then be shown that $V_{\rm eff}(\phi_c)$ is the expectation value of the energy density in that state $|\Psi\rangle$ which minimizes $\langle\Psi|H|\Psi\rangle$ subject to the condition that $\langle\Psi|\phi(\mathbf{x})|\Psi\rangle=\phi_c$. From this it immediately follows that $V_{\rm eff}(\phi_c)$ must be real. It can also be shown that $V_{\rm eff}$ must be everywhere convex.[1,2]

In practice the effective potential must usually be calculated perturbatively. The results of the one-loop approximation to $V_{\rm eff}$ are well known.[3] For the simplest case, the theory of a single scalar field with Lagrangian

$$L=\int d^3x\left[\tfrac{1}{2}(\partial_\mu\phi)^2-V(\phi)\right], \tag{1.1}$$

the one-loop approximation is

$$V_{1\ \rm loop}=V(\phi)+\frac{1}{64\pi^2}\{[V''(\phi)]^2\ln V''(\phi)\}+P(\phi), \tag{1.2}$$

where $P(\phi)$ is a polynomial whose coefficients are fixed by the renormalization conditions.

If $V''(\phi)<0$ for some range of ϕ (as will be the case in any theory with tree-level spontaneous symmetry breaking), the logarithm in Eq. (1.2) will cause $V_{1\ \rm loop}$ to be complex; this is the difficulty we wish to resolve.[4] A not unrelated difficulty, which we will discuss first, is that in this same range of ϕ the perturbatively calculated effective potential is nonconvex, although $V_{\rm eff}$ should not be so. It is widely understood that the resolution of this latter difficulty lies in the recognition that the perturbative calculation, although not being everywhere a good approximation to $V_{\rm eff}$, is an approximation to a closely related and physically meaningful quantity. [It is possible to modify the standard perturbative calculation so as to approximate the true, everywhere convex, $V_{\rm eff}$ (Refs. 5–13).]

The essential features can be understood at the classical level, if we take the classical analog of $\langle\Psi|\phi(\mathbf{x})|\Psi\rangle$ to be the spatial average of ϕ, $\bar\phi$. A classical effective potential V_c could then be defined as the minimum value of the energy density among all states with a given value of $\bar\phi$. Now consider a theory with a potential $V(\phi)$, such as the one shown in Fig. 1, which has two equal minima at $\phi=\pm\sigma$. If $|\bar\phi|>\sigma$, the energy density is minimized by a static homogeneous configuration with $\phi(\mathbf{x})=\bar\phi$ everywhere, and V_c is identical to $V(\phi)$. On the other hand, when $|\bar\phi|<\sigma$, the energy is minimized by an inhomogeneous mixed state in which $\phi(\mathbf{x})=\sigma$ in a fraction $f=(\bar\phi+\sigma)/(2\sigma)$ of space and $\phi(\mathbf{x})=-\sigma$ elsewhere. In the infinite-volume limit the surface energy at the boundary between the two regions can be neglected, and the energy density is equal to $V(\sigma)=V(-\sigma)$. $V_c(\phi)$ is thus flat in the region between the two minima. [In the general case with $V''(\phi)<0$, V_c would be given in the corre-

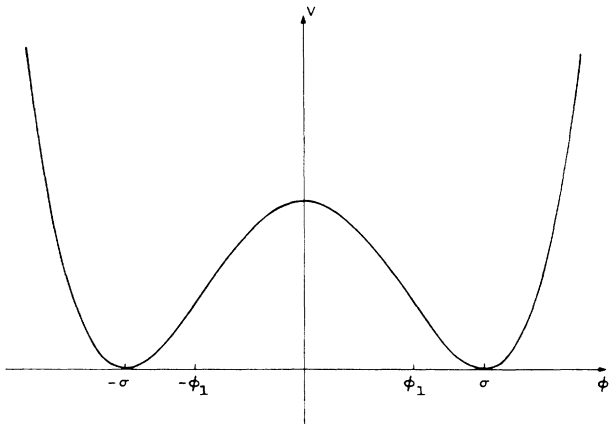

FIG. 1. A potential of the type considered here. $V(\phi)$ has two equal minima at $\phi=\pm\sigma$. $V''(\phi)<0$ in the region $-\phi_1<\phi<\phi_1$.

sponding region by a straight line which could be obtained by Maxwell's construction.] Of course, if we were interested only in homogeneous states and did not want to consider mixed phase states, V_c would not be relevant when $|\bar{\phi}|<\sigma$; in this region, as elsewhere, the minimum energy density for a homogeneous state would be given by $V(\phi)$.

In the quantum theory the behavior of $V_{\rm eff}$ in the region between the two minima is quite similar. The main difference from the classical case is that the state which minimizes the energy density is not a spatially inhomogeneous mixture of two phases but rather a quantum superposition of two vacuum states. Just as in the classical theory, the effective potential is not particularly useful in this region if one is interested in "homogeneous" states in which the field is concentrated about a single value. (One might imagine obtaining such states by starting with one of the vacua and then gradually shifting the expectation value of ϕ by applying some external source.) For such states $V(\phi)$ is not quite the appropriate quantity either. Even when $|\phi|>\sigma$, the zero-point energies of the quantum fluctuations cause $V_{\rm eff}$ to differ from $V(\phi)$. To describe the homogeneous states when $|\phi|<\sigma$, one would want a modified effective potential which would take into account these zero-point fluctuations. This might be defined to be the expectation value of the energy density in that state $|\Psi\rangle$ which minimizes $\langle\Psi|H|\Psi\rangle$ subject to the condition that $\langle\Psi|\phi(\mathbf{x})|\Psi\rangle=\phi_c$ and subject to the further restriction that the wave functional for $|\Psi\rangle$ be concentrated on configurations with $\phi(\mathbf{x})\approx\phi_c$. (Note that this is more restrictive than simply requiring that the state be formed by applying local fields to a single vacuum state, rather than a linear combination of vacua.) To leading order, this would differ from $V(\phi_c)$ by the sum of the zero-point energies of the quantum fluctuations of the field about ϕ_c and would thus agree with the perturbative $V_{1\text{ loop}}$ in the region where the latter is real.

While this shows that the real part of the nonconvex perturbative effective potential can be physically meaningful, we must still account for the imaginary part. The key to this is to recognize that, because the restriction to configurations with $\phi(\mathbf{x})\approx\phi_c$ does not commute with the Hamiltonian, the states described above are unstable for $|\phi_c|<\sigma$. These unstable states will have decay rates which, in the usual fashion, can be interpreted as imaginary contributions to the energy.

This suggests the idea of using the state $|\Psi\rangle$ described above to define a function $\tilde{V}(\phi_c)$ whose real part is the expectation value of the energy density and whose imaginary part is half the decay rate per unit volume. Our claim is that it is just such a function which the perturbative calculation is approximating. Indeed, it seems likely that this function is in fact the analytic continuation of $V_{\rm eff}(\phi)$ to the region $|\phi|<\sigma$ (Ref. 14). (For other variations on the effective potential with similarities to our $\tilde{V}$, see Refs. 15–18.)

This definition is still somewhat vague, since we have not yet defined precisely what we mean by the requirement that the wave functional be concentrated on configurations with $\phi(\mathbf{x})\approx\phi_c$. We will address this question in Sec. II and show that there is a physically reasonable statement of this requirement which leads to a $\tilde{V}$ which agrees with the perturbative results. In Sec. III we use the results of the preceding section to analyze the nature of the instability. We contrast the case $V''(\phi_c)<0$, for which the instability is seen perturbatively, and the case $V''(\phi_c)>0$, but $|\phi_c|<\sigma$, in which the decay is by nonperturbative processes. We then make some concluding remarks. The evaluation of some integrals is discussed in the Appendix.

II. NORMAL-MODE ANALYSIS AND CALCULATION OF THE DECAY RATE

In Sec. I a modified effective potential $\tilde{V}(\phi_c)$ was defined in terms of the state which minimized $\langle\Psi|H|\Psi\rangle$ subject to the requirements that $\langle\Psi|\phi(\mathbf{x})|\Psi\rangle=\phi_c$ and that the wave functional be concentrated on configurations with $\phi(\mathbf{x})\approx\phi_c$. In this section we will make this definition more precise by developing a more detailed statement of this last requirement. Two opposing considerations must be taken into account. On the one hand, we want to approximate the classical homogeneous state, in which $\phi(\mathbf{x})$ is fixed precisely. On the other, the uncertainty principle tells us that the more precisely we fix the coordinates the greater will be the uncertainty in the conjugate momenta. Fixing the coordinates too narrowly will thus lead to a state with large values of these momenta. This will both raise the energy and increase the rate at which the wave function spreads. We will see that a reasonable compromise between these two leads to a $\tilde{V}$ which agrees with the perturbative calculation of the effective potential.

We consider a theory with a real scalar field and a Lagrangian given by Eq. (1.1). The potential $V(\phi)$ is assumed to have the form shown in Fig. 1. We work with a finite volume Ω, with the field satisfying periodic boundary conditions; the limit $\Omega\to\infty$ is understood to be taken at the end. The field ϕ may be decomposed as

$$\phi(\mathbf{x})=\phi_0+\tilde{\phi}(\mathbf{x})\ , \tag{2.1}$$

where

$$\phi_0=\frac{1}{\Omega}\int d^3x\ \phi(\mathbf{x})\ . \tag{2.2}$$

This implies that the integral of $\tilde{\phi}(\mathbf{x})$ over all space vanishes, so that when Eq. (2.1) is substituted into Eq. (1.1) there are no terms linear in $\tilde{\phi}(\mathbf{x})$, and one obtains

$$L=\Omega[\tfrac{1}{2}\dot{\phi}_0{}^2-V(\phi_0)]+\int d^3x[\tfrac{1}{2}(\partial_\mu\tilde{\phi})^2-\tfrac{1}{2}V''(\phi_0)\tilde{\phi}^2 -\tfrac{1}{6}V'''(\phi_0)\tilde{\phi}^3+\cdots]\ . \tag{2.3}$$

For translationally invariant states $\langle\Psi|\phi(\mathbf{x})|\Psi\rangle=\langle\Psi|\phi_0|\Psi\rangle$, so for any form of the effective potential the defining state will satisfy $\langle\Psi|\phi_0|\Psi\rangle=\phi_c$. For $\tilde{V}(\phi_c)$ there is a further requirement that the wave functional be concentrated on configurations with $\phi(\mathbf{x})$ near ϕ_c; this clearly should be interpreted to mean configurations with ϕ_0 near ϕ_c. In fact, Eq. (2.3) implies

that the momentum conjugate to ϕ_0 is $\Omega\dot{\phi}_0$, so that in the infinite-volume limit ϕ_0 can be treated as a classical variable; i.e., there are states of finite energy density in which ϕ_0 is fixed precisely. We therefore require that the wave functional be of the form $\Psi[\phi_0,\phi(\mathbf{x})] = \delta(\phi_0-\phi_c)\tilde{\Psi}[\tilde{\phi}(\mathbf{x})]$. (By contrast, the states which define the exact effective potential in its flat region between the two minima of $V(\phi)$ will have wave functionals of the form $\alpha_+\delta(\phi_0-\sigma)\tilde{\Psi}_+[\tilde{\phi}(\mathbf{x})]+\alpha_-\delta(\phi_0+\sigma)\tilde{\Psi}_-[\tilde{\phi}(\mathbf{x})]$.)

The remaining variables, which make up $\tilde{\phi}(\mathbf{x})$, cannot be fixed precisely in the quantum theory without leading to infinite energy. Instead, we can only require that $\tilde{\phi}(\mathbf{x})$ be, by some reasonable definition, small. At the least, such a definition should imply that, for sufficiently weak coupling, the terms in the Lagrangian which are of cubic or higher order in $\tilde{\phi}(\mathbf{x})$ can be treated as small perturbations. To lowest order, $\tilde{V}(\phi_c)$ is then determined by the quadratic Hamiltonian

$$H_{\text{quad}}=\Omega V(\phi_0)+\int d^3x\left[\tfrac{1}{2}\dot{\tilde{\phi}}^2+\tfrac{1}{2}(\nabla\tilde{\phi})^2+\tfrac{1}{2}V''(\phi_0)\tilde{\phi}^2\right] . \tag{2.4}$$

We now expand $\tilde{\phi}(\mathbf{x})$ as

$$\tilde{\phi}(\mathbf{x})=\frac{1}{\sqrt{2\Omega}}\sum_{\mathbf{k}}\left(\phi_{\mathbf{k},1}\sin\mathbf{k}\cdot\mathbf{x}+\phi_{\mathbf{k},2}\cos\mathbf{k}\cdot\mathbf{x}\right) . \tag{2.5}$$

[The sum is over all $\mathbf{k}$, but with the understanding that $\phi_{-\mathbf{k}j}\equiv(-1)^j\phi_{\mathbf{k}j}$.] The Hamiltonian then takes the form

$$H_{\text{quad}}=\Omega V(\phi_0)+\tfrac{1}{2}\sum_{\mathbf{k}}\sum_{j=1}^{2}\tfrac{1}{2}(\phi_{\mathbf{k}j}^2+\omega_{\mathbf{k}}^2\phi_{\mathbf{k}j}^2) , \tag{2.6}$$

where the first factor of $\frac{1}{2}$ cancels the double counting of modes from including both $\mathbf{k}$ and $-\mathbf{k}$ in the sum and

$$\omega_{\mathbf{k}}^2=\mathbf{k}^2+V''(\phi_c) . \tag{2.7}$$

When $V''(\phi_c)>0$, the $\omega_{\mathbf{k}}$ are all real and H_{quad} is a sum of harmonic-oscillator Hamiltonians. For the eigenstates of the harmonic oscillator, the lower the energy the more concentrated at the origin is the wave function. Consequently, many different formulations of our restriction on the wave functional will lead to the same minimum-energy state. Only if the field is required to be peaked more narrowly than in the oscillator ground state will a different result by obtained; such a requirement will give a rapidly spreading state of higher energy. Assuming that this is not the case, the desired state $|\Psi\rangle$ will be the ground state of H_{quad}. The lowest-order approximation to $\tilde{V}(\phi_c)$ will be its energy density

$$\begin{aligned}\frac{E_{\text{quad}}}{\Omega}&=V(\phi_c)+\frac{1}{\Omega}\sum_{\mathbf{k}}\tfrac{1}{2}\omega_{\mathbf{k}}\\&=V(\phi_c)+\frac{1}{(2\pi)^3}\int d^3k\,\tfrac{1}{2}[\mathbf{k}^2+V''(\phi_c)]^{1/2} ,\end{aligned} \tag{2.8}$$

where the second equality holds in the infinite-volume limit. (The integral on the right-hand side is actually divergent. The divergent part can be written as a polynomial in ϕ_c and is canceled by the counterterms of the theory.)

In fact, Eq. (2.8) is in agreement with the perturbative calculation. This can be most easily seen by differentiating with respect to ϕ_c the integral expressions for the two quantities; if the results are the same, the two potentials differ by at most a physically uninteresting constant. In fact, only the corrections to $V(\phi)$ need be compared; since ϕ_c enters these only through V'', it is simplest to take derivatives with respect to the latter quantity. The one-loop corrections contained in Eq. (1.2) are obtained from the integral

$$I(\phi_c)=\frac{-i}{4\pi}\frac{1}{(2\pi)^3}\int d^3k\,dk_0\{\ln[k_0^2-\mathbf{k}^2-V''(\phi_c)]\} . \tag{2.9}$$

Differentiating this gives

$$\begin{aligned}\frac{dI(\phi_c)}{dV''}&=\frac{i}{4\pi}\frac{1}{(2\pi)^3}\int d^3k\,dk_0\frac{1}{k_0^2-\mathbf{k}^2-V''(\phi_c)}\\&=\frac{1}{(2\pi)^3}\int d^3k\tfrac{1}{4}[\mathbf{k}^2+V''(\phi_c)]^{-1/2}\end{aligned} \tag{2.10}$$

which is the same as the derivative of the integral in Eq. (2.8).

Things become more interesting when $V''(\phi_c)<0$. The modes with $\mathbf{k}^2>|V''|$ remain harmonic oscillators and presumably contribute to $\tilde{V}$ as before. However, the long-wavelength modes, with $\mathbf{k}^2<|V''|$, now have imaginary frequencies. These can be considered upside-down harmonic oscillators with Hamiltonians of the form

$$\hat{H}=\tfrac{1}{2}\dot{q}^2-\tfrac{1}{2}\eta^2q^2 . \tag{2.11}$$

In contrast with the standard harmonic oscillator, there is no lower bound on the energy.[19] However, for defining $\tilde{V}$ we are interested only in states which are concentrated about small values of $\tilde{\phi}$, so we need only consider those states of the upside-down oscillator whose wave functions are concentrated about small values of q. When restricted to these states, $\hat{H}$ does have a lower bound; we will see that this bound is rather sensitive to the precise form of the constraint which we put on the state.

No matter what the exact form of the constraint, it cannot be maintained indefinitely. To be explicit, consider imposing the requirement

$$\langle\psi|q^2|\psi\rangle\le a\eta^{-1} . \tag{2.12}$$

If $a\lesssim 1$, uncertainty principle arguments show that the conjugate momentum must be large and that the wave function will rapidly spread; this is just as in the case of the normal oscillator. The wave function will also spread if $a\gtrsim 1$, but in this case the spreading reflects the fact that a classical particle would roll, with an exponentially growing speed, down such a potential. (A detailed analysis of the correspondence between the classical and quantum upside-down oscillators has been given by Guth and Pi.[20])

The state $|\psi\rangle$ which minimizes the energy subject to this constraint can be found by using Lagrange multipliers. To do this we first require that $\langle\psi|\psi\rangle=1$ and $\langle\psi|q^2|\psi\rangle=a\eta^{-1}$. Minimizing $\hat{H}$ subject to these two constraints gives

$$(\hat{H}+\tfrac{1}{2}\lambda_1\eta^2q^2)|\psi\rangle=\lambda_2\eta|\psi\rangle \ . \tag{2.13}$$

(Note that factors of η have been extracted in order to make the Lagrange multipliers λ_j dimensionless.) Solutions to this exist if $\lambda_1>1$; they are the states of the harmonic oscillator with frequency

$$\tilde{\omega}^2=(\lambda_1-1)\eta^2 \ . \tag{2.14}$$

For the state with quantum number n,

$$\lambda_2=\sqrt{\lambda_1-1}(n+\tfrac{1}{2}) \tag{2.15}$$

and

$$\langle\psi|q^2|\psi\rangle=(n+\tfrac{1}{2})\frac{1}{(\lambda_1-1)^{1/2}\eta} \ . \tag{2.16}$$

Setting the latter quantity equal to $a\eta^{-1}$ gives

$$\lambda_1=1+(n+\tfrac{1}{2})^2a^{-2} \tag{2.17}$$

and hence

$$\langle\psi|\hat{H}|\psi\rangle=(\lambda_2-\tfrac{1}{2}\lambda_1a)\eta=\tfrac{1}{2}\eta\left[\frac{(n+\frac{1}{2})^2}{a}-a\right] \ . \tag{2.18}$$

To get the minimum energy, we set $n=0$. So far, we have required $\langle\psi|q^2|\psi\rangle=a\eta^{-1}$, although we only want to impose the inequality $\langle\psi|q^2|\psi\rangle\le a\eta^{-1}$. However, since the right-hand side of Eq. (2.18) is a monotonically decreasing function of a, the energy is minimized when the equality is saturated. Thus, the desired state is the one with

$$\langle\psi|\hat{H}|\psi\rangle=\frac{\eta}{4}\left[\frac{1}{2a}-2a\right] \tag{2.19}$$

and

$$\psi(q)=\langle q|\psi\rangle=\left[\frac{\eta}{2a\pi}\right]^{1/4}e^{-\eta q^2/4a} \ . \tag{2.20}$$

As has been noted, these states will not be stationary for any value of a. Their "decay rate" can be extracted from the time development of a state whose wave function is given by Eq. (2.20) at $t=0$. The time development of such states was found in Ref. 20. Adapting the results of that paper to our notation we have

$$\psi(q,t)=\left[\frac{\eta\sin2\phi}{2\pi}\right]^{1/4}\frac{1}{[\cos(\phi-i\eta t)]^{1/2}}\exp\left[-\frac{\eta q^2}{2}\tan(\phi-i\eta t)\right] \ , \tag{2.21}$$

where

$$\phi=\arctan\left[\frac{1}{2a}\right] \ . \tag{2.22}$$

The overlap of the state at time t with the initial state is given by

$$\hat{P}(t)=|\langle\psi(0)|\psi(t)\rangle|^2=\left[1+\frac{\sinh^2(\eta t)}{\sin^2(2\phi)}\right]^{-1/2} \ . \tag{2.23}$$

At large t this behaves as

$$\hat{P}(t)\sim2\sin2\phi e^{-\eta t} \ . \tag{2.24}$$

Extracting a decay rate from the exponent gives $\hat{\Gamma}=\eta$, no matter what the choice for a.

Another measure of the spreading of the state is the growth in time of $\langle\psi|q^2|\psi\rangle$. Using Eq. (2.21), one obtains

$$\langle\psi(t)|q^2|\psi(t)\rangle=\frac{1}{2\omega_{\mathbf{k}}}\left[\frac{\cos2\phi+\cosh2\eta t}{\sin2\phi}\right]=\langle\psi(0)|q^2|\psi(0)\rangle+\frac{\sinh^2\eta t}{\omega_{\mathbf{k}}\sin2\phi} \ . \tag{2.25}$$

Thus, $\langle\psi|q^2|\psi\rangle$ grows exponentially with time at large t.

Returning to field theory, we can now sum the energies in each of the modes to obtain the real part of $\tilde{V}$. There is a contribution of $\frac{1}{2}\omega_{\mathbf{k}}$ from each of the modes with real frequencies; referring to Eq. (2.10), we see that the sum of these exactly reproduces the real part of the one-loop effective potential. The contribution from the imaginary-frequency modes depends on the precise form of the constraint. If Eq. (2.12) is imposed, then Eq. (2.19) leads to

$$\mathrm{Re}\tilde{V}(\phi_c)=\mathrm{Re}V_{1\ \mathrm{loop}}(\phi_c)+\frac{1}{(2\pi)^3}\int d^3k\frac{\eta_{\mathbf{k}}}{4}\left[\frac{1}{2a_{\mathbf{k}}}-2a_{\mathbf{k}}\right]\times\theta(|V''(\phi_c)|-\mathbf{k}^2) \ , \tag{2.26}$$

where $\eta_{\mathbf{k}}=|\omega_{\mathbf{k}}|$ and the two modes with wave number $\mathbf{k}$ are assumed to satisfy a constraint of the form of Eq. (2.12) with $a=a_{\mathbf{k}}$.

Agreement between the real parts of $\tilde{V}$ and $V_{1\ \mathrm{loop}}$ is obtained by choosing the constraint corresponding to the choice $a=\frac{1}{2}$ (i.e., $\phi=\pi/4$) for each of the upside-down oscillator modes. Equations (2.24) and (2.25) show that at all times this choice both maximizes the overlap with

the initial state and minimizes the width (relative to the initial width) of the wave function, so one might (in retrospect) even argue that this was the most natural choice of constraint.

The wave functional $\tilde{\Psi}[\tilde{\phi}(\mathbf{x}),t]$ is the product of the wave functions for each of the modes. Since the modes with real frequencies are in stationary states, the overlap of the wave functional at time t with that at $t=0$ is determined by the corresponding overlaps for the imaginary-frequency modes. At large t this will fall as $e^{-\Gamma t}$, where the decay rate Γ is the sum of the decay rates of the individual modes; i.e., the sum of the $\eta_{\mathbf{k}}$ over all the unstable modes. The imaginary part of $\tilde{V}$ is $\frac{1}{2}\Gamma/\Omega$; in the infinite-volume limit this becomes

$$\begin{aligned}\mathrm{Im}\tilde{V}(\phi_c)&=\frac{1}{(2\pi)^3}\int d^3k\tfrac{1}{2}[\,|V''(\phi_c)|-\mathbf{k}^2]^{1/2}\\&\quad\times\theta(|V''(\phi_c)|-\mathbf{k}^2)\\&=\frac{1}{64\pi}[V''(\phi_c)]^2\ .\end{aligned}\tag{2.27}$$

Comparing with Eq. (1.2), we see that this is indeed equal to the imaginary part of $V_{1\text{ loop}}$. Note that this result is independent of the choice of the $a_{\mathbf{k}}$.

III. NATURE OF THE DECAY PROCESS

It has been shown that the imaginary part of the one-loop approximation to the effective potential has a natural interpretation as half the decay rate perturbative unit volume of a particular quantum state. In this section we examine in more detail the nature of the decay process.

Before doing so, let us consider briefly the reality properties of $\tilde{V}(\phi)$ in that part of the region between the minima of $V(\phi)$ where $V''(\phi)>0$; for the potential of Fig. 1, this is the region $\phi_1<|\phi|<\sigma$. It was argued in Sec. I that $\tilde{V}(\phi)$ should have an imaginary part for $|\phi|\le\sigma$, since the state which defines it is not a stationary state. However, the perturbative $V_{1\text{ loop}}$ of Eq. (1.2) is real when $V''(\phi)\ge 0$. The reason for this is that once ϕ_0 has been fixed the static homogeneous configuration $\phi(\mathbf{x})=\phi_c$ is classically stable; i.e., any small fluctuation which varies $\tilde{\phi}(\mathbf{x})$ but leaves ϕ_0 unchanged necessarily increases the potential energy. [It is important to remember that the potential energy includes the contribution of the gradient terms in the Hamiltonian as well as those of $V(\phi)$.] There are configurations with the same value of ϕ_0 and lower potential energy, but these are separated from the homogeneous configuration by a potential-energy barrier. The field can tunnel through this barrier to form "bubbles" where $\tilde{\phi}(\mathbf{x})\approx\pm\sigma$, but since this is a nonperturbative process it does not result in an imaginary part for the perturbative effective potential. The tunneling rate can be calculated by methods of the sort used to study the decay of a metastable vacuum,[21] and will give $\tilde{V}$ an exponentially small imaginary part proportional to e^{-S}, where S is the action of the appropriate "bounce" solution.

We now return to the perturbative instability which occurs when $V''(\phi)<0$. One's first thought may be to attribute this to a uniform "rolling" of the field down the potential $V(\phi)$, but this cannot be so, since our understanding of $V_{1\text{ loop}}$, and indeed the definition of $\tilde{V}$, are based on holding the spatial average of the field fixed. The instability is due to the growth of the long-wavelength components of $\tilde{\phi}(\mathbf{x})$ for which $\omega_{\mathbf{k}}^2=\mathbf{k}^2+V''<0$. We should therefore expect it to be manifested as a breakup of the original homogeneous configuration into many uncorrelated domains. Within a domain the field would be correlated, and $\tilde{\phi}(\mathbf{x})$, the deviation from ϕ_c, would be increasing with time; these deviations would vary randomly from positive to negative from one domain to the next. These domains would be expected to have size at least as great as $v^{-1}\equiv|V''|^{-1/2}$, the minimum wavelength for an unstable mode.

This intuitive, somewhat classical, picture can be given a more precise quantum-mechanical meaning by analyzing two quantities. The first is the correlation function

$$F(r,t)=\langle\Psi(t)|\tilde{\phi}(\mathbf{x})\tilde{\phi}(\mathbf{x}+\mathbf{r})|\Psi(t)\rangle\ .\tag{3.1}$$

For $r\equiv|\mathbf{r}|\lesssim v^{-1}$ the two fields are very likely to be in the same domain and so should be correlated in sign; $F(r,t)$ then measures the increase in their magnitude. Since the individual unstable modes are growing exponentially [see Eq. (2.25)], we expect the overall growth of the field to be also exponential, with the rate set by the characteristic scale v, so that $F(r,t)\sim e^{2vt}$. On the other hand, the sign of the fields will be uncorrelated for large r and so $F(r,t)$ should not display this growth when $r\gg v^{-1}$.

The second quantity,

$$G(l,t)=\langle\Psi(t)|\tilde{\phi}_l^{\,2}(\mathbf{x})|\Psi(t)\rangle\tag{3.2}$$

involves the smeared field

$$\tilde{\phi}_l(\mathbf{x})=\frac{1}{(2\pi)^{3/2}l^3}\int d^3y\,\tilde{\phi}(\mathbf{y})e^{-(\mathbf{x}-\mathbf{y})^2/2l^2}\tag{3.3}$$

which averages $\tilde{\phi}(\mathbf{x})$ over a region of volume $\sim l^3$. When $l\lesssim v^{-1}$, $\tilde{\phi}(\mathbf{x})$ will be correlated over the entire smearing region and so we expect $G(l,t)\sim e^{2vt}$. In contrast, when l is much greater than the domain size ξ, the smearing averages $\tilde{\phi}(\mathbf{x})$ over $N\sim(1/\xi)^3$ domains. These will tend to cancel, with $G(l,t)$ being a measure of the fluctuations. We thus expect $G(l,t)\sim(\xi/l)^3e^{2vt}$.

$F(r,t)$ can be calculated by substituting the expansion of Eq. (2.5) into Eq. (3.1). Because $|\Psi\rangle$ is a product of harmonic-oscillator ground states for each of the real-frequency modes and states of the form of Eq. (2.21) for the imaginary-frequency modes, $\langle\Psi|\phi_{\mathbf{k}j}\phi_{\mathbf{k}'j'}|\Psi\rangle$ vanishes unless $j=j'$ and either $\mathbf{k}=\mathbf{k}'$ or $\mathbf{k}=-\mathbf{k}'$, in which case it is given by

$$\begin{aligned}\langle\Psi|\phi_{\mathbf{k}j}^{\,2}|\Psi\rangle&=(-1)^j\langle\Psi|\phi_{\mathbf{k}j}\phi_{-\mathbf{k}j}|\Psi\rangle\\&=g(k)\\&=\begin{cases}\dfrac{1}{2\omega_{\mathbf{k}}}, & \omega_{\mathbf{k}}^2\ge 0\ ,\\[2ex]\dfrac{1}{2|\omega_{\mathbf{k}}|}\cosh 2|\omega_{\mathbf{k}}|t, & \omega_{\mathbf{k}}^2\le 0\ ,\end{cases}\end{aligned}\tag{3.4}$$

where the result for the unstable modes is obtained from Eq. (2.25) with $\phi=\arctan(1/2a)=\pi/4$. This gives

$$F(r,t)=\frac{1}{\Omega}\sum_k g(k)\cos\mathbf{k}\cdot\mathbf{r} \tag{3.5}$$

which in the infinite-volume limit becomes

$$F(r,t)=\frac{1}{(2\pi)^3}\int d^3k\, g(k)\cos\mathbf{k}\cdot\mathbf{r}\ . \tag{3.6}$$

It is convenient to write this as

$$F(r,t)=F_1(r)+F_2(r,t)\ , \tag{3.7}$$

where

$$F_1=-\frac{v}{8\pi r}N_1(vr) \tag{3.8}$$

is the contribution from the real-frequency modes with $k\geq v$ and

$$F_2(r,t)=\begin{cases}\dfrac{v}{8\pi(4t^2-r^2)^{1/2}}I_1(v(4t^2-r^2)^{1/2}), & r<2t\ ,\\ \dfrac{v}{8\pi(r^2-4t^2)^{1/2}}J_1(v(r^2-4t^2)^{1/2}), & r>2t\end{cases} \tag{3.9}$$

is due to the unstable modes with $k<v$. (The evaluation of these integrals is described in the Appendix.)

At $t=0$, F displays the typical behavior of field-theory correlation functions, including the usual short-distance divergence. At large time ($t\gtrsim v^{-1}$), the asymptotic behavior of the Bessel functions leads to

$$F(r,t)\approx\begin{cases}\left[\dfrac{v}{1024\pi^3t^3}\right]^{1/2}e^{2vt}, & r\ll 2t\ ,\\ \left[\dfrac{v}{16\pi^3r^3}\right]^{1/2}\sin rv, & r\gg 2t\ .\end{cases} \tag{3.10}$$

As expected, $F(r,t)$ grows exponentially with time at small distances, but shows no growth for large values of r.

Similarly, Eqs. (2.5), (3.2), and (3.3) lead to

$$G(l,t)=\frac{1}{(2\pi)^3}\int d^3k\, g(k)e^{-k^2l^2}\ . \tag{3.11}$$

It is again convenient to write this as the sum of two terms: one from the modes with $k\geq v$ and one from those with $k<v$. The former is

$$G_1(l)=\frac{v^2}{16\pi^2}e^{-v^2l^2/2}[K_0(\tfrac{1}{2}v^2l^2)+K_1(\tfrac{1}{2}v^2l^2)]\ . \tag{3.12}$$

Although the latter integral cannot be evaluated explicitly for all values of t, expressions valid at large and small times can be obtained (see the Appendix). At $t=0$ we have

$$G_2(l,0)=\frac{v^2}{16\pi}e^{-v^2l^2/2}[I_0(\tfrac{1}{2}v^2l^2)-I_1(\tfrac{1}{2}v^2l^2)]\ . \tag{3.13}$$

For $t\gg v^{-1}$,

$$G_2(l,t)\approx\frac{v^2}{32\pi^{3/2}}e^{2vt}(l^2v^2+tv)^{-3/2}\ . \tag{3.14}$$

This has the exponential growth predicted above. As expected, the dependence on the smearing size is weak for small values of l, but at large values $G(l,t)$ is proportional to l^{-3}.

To sum up, we have shown that the perturbatively calculated effective potential can be a physically meaningful quantity even in those situations where it becomes complex. It describes a well-defined, although unstable, spatially homogeneous quantum state. On times scales which are short relative to the decay rate this state will remain homogeneous, and $\tilde{V}(\phi)$ can be reliably used in studying its evolution.

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APPENDIX

In this appendix we describe the evaluation of the integrals in Eqs. (3.6) and (3.11). We begin with the expression for $F(r,t)$, Eq. (3.6). After doing the angular integrations, this becomes

$$F(r,t)=\frac{1}{2\pi^2r}\int dk\, kg(k)\sin kr=\frac{-1}{2\pi^2r}\frac{d}{dr}\left[\int dk\, g(k)\cos kr\right]\ , \tag{A1}$$

where $g(k)$ is given by Eq. (3.4). The contribution from the real-frequency modes is

$$F_1(r)=\frac{-1}{4\pi^2r}\frac{d}{dr}\left[\int_v^\infty dk\frac{\cos kr}{(k^2-v^2)^{1/2}}\right]=\frac{-1}{4\pi^2r}\frac{d}{dr}\left[\int_0^\infty dx\cos(rv\cosh x)\right]\ , \tag{A2}$$

where the second equality is obtained by using the change of variables $k=v\cosh x$. The second integral is simply $-\frac{1}{2}\pi N_0(vr)$, which leads immediately to Eq. (3.8).

The contribution from the unstable modes is

$$F_2(r,t)=\frac{-1}{4\pi^2r}\frac{d}{dr}\left[\int_0^v dk\frac{\cos kr}{(v^2-k^2)^{1/2}}\cosh[2t(v^2-k^2)^{1/2}]\right]\ . \tag{A3}$$

By making the substitution $k=v\sin\theta$ we obtain

$$F_2(r,t)=\frac{-1}{4\pi^2 r}\frac{d}{dr}\left[\int_0^{\pi/2}d\theta\cos(rv\sin\theta)\cosh(2tv\cos\theta)\right]=\frac{-1}{16\pi^2 r}\frac{d}{dr}\left[\int_0^{2\pi}d\theta\, e^{irv\sin\theta-2tv\cos\theta}\right]. \tag{A4}$$

If $r>2t$ we define $\beta=\text{arcsinh}[2t/(r^2-4t^2)^{1/2}]$. Equation (A4) can then be rewritten as

$$F_2(r,t)=\frac{-1}{16\pi^2 r}\frac{d}{dr}\left[\int_0^{2\pi}d\theta\, e^{iv(r^2-4t^2)^{1/2}\sin(\theta+i\beta)}\right]=\frac{-1}{16\pi^2 r}\frac{d}{dr}\left[\int_C d\theta\, e^{iv(r^2-4t^2)^{1/2}\sin\theta}\right], \tag{A5}$$

where the contour C runs parallel to the real axis from $i\beta$ to $2\pi+i\beta$. By using the periodicity of the integrand and the fact that it is an entire function, it is easy to show that the contour can be replaced by any parallel contour. In particular, it can be replaced by a contour along the real axis from 0 to 2π, in which case the integral reduces to a standard representation for the Bessel function $J_0[(r^2-4t^2)^{1/2}]$. This leads to the expression given in Eq. (3.9). The case $r<2t$ can be done by similar methods, or else simply by analytic continuation.

$G(l,t)$ is given by Eq. (3.11). After performing the angular integration, one obtains the expression

$$G_1(l)=\frac{1}{4\pi^2}\int_v^\infty dk\frac{k^2}{(k^2-v^2)^{1/2}}e^{-k^2l^2} \tag{A6}$$

for the contribution of the real-frequency modes. By making the change of variables $k=v\cosh(y/2)$ this can be rewritten as

$$G_1(l)=\frac{v^2}{16\pi^2}e^{-v^2l^2/2}\int_0^\infty dy(1+\cosh y)e^{-(v^2l^2\cosh y)/2}. \tag{A7}$$

Recognizing the integral representations for the Bessel functions K_0 and K_1, we obtain Eq. (3.12).

The contribution from the unstable modes is

$$G_2(l,t)=\frac{1}{4\pi^2}\int_0^v dk\frac{k^2}{(v^2-k^2)^{1/2}}e^{-k^2l^2}\cosh[2t(v^2-k^2)^{1/2}]. \tag{A8}$$

The substitution $k=v\sin(\phi/2)$ converts this to

$$G_2(l,t)=\frac{v^2}{16\pi^2}e^{-v^2l^2/2}\int_0^\pi d\phi(1-\cos\phi)e^{(v^2l^2\cos\phi)/2}\cosh[2vt\cos(\phi/2)]. \tag{A9}$$

For $t=0$ the integral is a sum of two Bessel functions and we obtain Eq. (3.13). Although the integral cannot be evaluated explicitly when $t\neq0$, an approximate expression valid when t is large can be obtained. To do this, we note that for $tv\gg1$ the integral is dominated by the region $\phi\approx0$. We therefore make little error if we replace the upper limit of the integration by ∞ and expand the integrand about $\phi=0$. This gives

$$G_2(l,t)\approx\frac{v^2}{64\pi^2}e^{2vt}\int_0^\infty d\phi\,\phi^2 e^{-(v^2l^2+vt)\phi^2/4} \tag{A10}$$

which can be evaluated exactly to give Eq. (3.14).

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