Relationship between supersymmetry and solvable potentials

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We investigate whether a general class of solvable potentials, the Natanzon potentials (those potentials whose solutions are hypergeometric functions), and their supersymmetric partner potentials are related by a discrete reparametrization invariance called "shape invariance" discovered by Gendenshtein. We present evidence that this is not the case in general. Instead we find that the Natanzon class of potentials is not the most general class of solvable potentials but instead belongs to a wider class of potentials generated by supersymmetry and factorization whose eigenfunctions are sums of hypergeometric functions. The series of Hamiltonians, together with the corresponding supersymmetric charges form the graded Lie algebra sl(1/1)⊗SU(2). We also present a strategy for solving, in a limited domain, the discrete reparametrization invariance equations connected with "shape invariance."

I. INTRODUCTION

The study of exactly solvable problems in nonrelativistic quantum mechanics has a long and cherished history.[1] By exactly solvable, one means those problems for which the Schrödinger equation can be transformed to hypergeometric (or confluent hypergeometric) form. The first unified venture in this direction was made by Schrödinger[2] who introduced the famous factorization method. This approach was subsequently generalized by Infeld and Hull and others.[3] More recently,[4,5] Schrödinger's factorization method was recognized to be a rediscovery of a technique attributable to Darboux.[6]

Fifteen years ago, Natanzon[7] attacked this problem from another angle. He obtained the most general form of the potential for which the Schrödinger equation reduces to hypergeometric form. A subclass of these solvable potentials were later rediscovered by one of the present authors (J.G.) who also obtained scattering functions for this subclass of potentials.[8] In recent years, a group at Yale has made a detailed study of these solvable potentials from a group-theory point of view and have shown that both the bound and scattering states of these solvable potentials are related to the unitary representations of certain groups.[9]

Starting in 1981 with the pioneering work of Witten,[10] it was recognized that supersymmetry (SUSY), a symmetry containing commuting and anticommuting operators and relating bosons to fermions, could be applied to quantum mechanics as a limiting case ($d=1$) of field theory. The subsequent development of supersymmetric quantum mechanics has followed two main paths. In one, the goal has been to clarify the relationship between supersymmetric quantum mechanics and various formulations of supersymmetric quantum field theory, and to use this simple model to gain understanding of supersymmetric theories in general.[11] The second line of investigation studies supersymmetric quantum mechanics as an interesting theory in its own right. Several authors[12,13] recognized that the supersymmetry transformation is closely related to the Schrödinger factorization method and, therefore, to the Darboux transformation. The formal theory of supersymmetric quantum mechanics has been developed by many authors,[12–15] while a number of interesting applications have been developed.[16]

A conceptual breakthrough in understanding the connection between solvable potentials and SUSY was made by Gendenshtein[17] who introduced a discrete reparametrization invariance called "shape invariance." Gendenshtein showed that whenever the "shape invariance" relationship was satisfied by the two Hamiltonians related by SUSY the spectra and wave functions could be determined by purely algebraic means.

The next conceptual advance came when several people realized that if one knew the ground-state wave function of a potential one could always factorize the Hamiltonian and cast it into supersymmetric form.[6,12–18] This led to the further realization that there was a sequence of related Hamiltonians generated by SUSY and factorization differing by having n bound states removed (where $n<N=$number of bound states of H_1) (Refs. 4 and 12).

In light of these developments we wanted to know whether the Natanzon[7] class of potentials (which depends on six parameters) and a restricted class[8] (Ginocchio class) of Natanzon potentials (which depends on two parameters) were shape invariant. If that were the case then we could have used this fact to algebraically solve for the energy eigenstates and also determine all the wave functions without solving the Schrödinger equation. The sequence of Hamiltonians generated by SUSY and factorization would all then belong to the same class and depend on the same six (or two) parameters of the Natanzon (restricted Natanzon) class. In fact, Gendenshtein[17] proposed in his paper that all soluble potentials would have the shape invariance property.

If this did not turn out to be the case (as we subsequently discovered) then we would discover a new class of solvable potentials whose wave functions would be more complicated then hypergeometric (or Gegenbauer in the restricted class) polynomials, but whose spectra would be related to those of the Natanzon class of potentials, being identical except for the absence of a certain number of bound states.

We were able to prove that the restricted class (Ginocchio class) of Natanzon potentials did not get mapped into itself under the transformation that produced the tower of Hamiltonians. Also, the new Hamiltonians had solutions that were sums of Gegenbauer polynomials. In the general case, the new Hamiltonians have eigenfunctions that are sums of hypergeometric functions which suggest that in the general case the new Hamiltonians are also not in the original Natanzon class. However, we do not have complete proof of this.

Our results do not preclude the possibility that these potentials may belong to a class of potentials having more than six parameters, and in that class, shape invariance might hold so that one might be able to solve these potentials algebraically.

We arrange our paper as follows. In Sec. II we review the relevant aspects of SUSY quantum mechanics. In Sec. III we investigate whether or not the restricted class of Natanzon potentials (Ginocchio class) are shape invariant and prove that they are not. In Sec. IV we investigate the shape invariance of the general class of Natanzon potentials and are able to prove that when three of the parameters are restricted they are not shape invariant. We also generate the first few Hamiltonians of the tower of Hamiltonians related by SUSY and factorization and the ground-state wave functions of these Hamiltonians. Since the ground-state wave function of these Hamiltonians are sums of hypergeometric functions it is unlikely that they can be written in terms of a single hypergeometric function of a related variable as required by shape invariance. In Sec. V we show that the generalized Hamiltonian generated by SUSY and factorization together with the related SUSY charges Q_i and $Q_i^\dagger$ $(i=1,2,\dots,n-1)$ can be put into a representation of the graded Lie algebra sl(1/1)⊗SU(2). In Sec. VI we give an ansatz for obtaining solutions to the shape invariance equations of Gendenshtein and recover all known solutions of these equations.

II. REVIEW OF SUSY QUANTUM MECHANICS

In the Schrödinger picture, SUSY quantum mechanics can be described by a pair of related bosonic Hamiltonians:

$$H_\pm\Psi=i\partial\Psi/\partial t=[-d^2/dx^2+V_\pm(x)]\Psi\ , \tag{2.1}$$

where $V_\pm(x)$ are given by

$$V_\pm(x)=W^2(x)\pm W'(x)\ . \tag{2.2}$$

Here, a prime denotes differentiation with respect to x and we have set $\hbar=2m=1$.

The Hamiltonians $H_\pm$ can be factorized as

$$H_+=AA^\dagger,\quad H_-=A^\dagger A\ , \tag{2.3}$$

where

$$A=d/dx+W,\quad A^\dagger=-d/dx+W\ . \tag{2.4}$$

One can show that the Hamiltonians $H_\pm$ are supersymmetric partners. To this end, let us consider two-component notation and introduce SUSY charges Q and $Q^\dagger$ by

$$Q=\begin{bmatrix}0&0\\A&0\end{bmatrix},\quad Q^\dagger=\begin{bmatrix}0&A^\dagger\\0&0\end{bmatrix}\ . \tag{2.5}$$

In this case the SUSY Hamiltonian $H_{\rm SUSY}$ has the form

$$H_{\rm SUSY}=\{Q,Q^\dagger\}=\begin{bmatrix}H_-&0\\0&H_+\end{bmatrix}\ . \tag{2.6}$$

Note further that

$$Q^2=(Q^\dagger)^2=[H_{\rm SUSY},Q]=[H_{\rm SUSY},Q^\dagger]=0\ . \tag{2.7}$$

The supersymmetric algebra (2.6) and (2.7) is called the sl(1/1) superalgebra.[19] From the above one can immediately derive the following facts about the eigenvalues and eigenfunctions of the two partner Hamiltonians $H_\pm$ (Ref. 10).

(i) The ground state of H_- has zero energy ($E_0^-=0$) provided the ground-state wave function $\Psi_0^-(x)$ given by ($A\Psi_0^-=0$)

$$\Psi_0^-(x)=N_0\exp\left[-\int^x W(x')dx'\right] \tag{2.8}$$

is square integrable. In this case the supersymmetry can be shown to be unbroken. We shall not discuss the case of broken SUSY in this paper.

(ii) In the case where SUSY is unbroken one can show that, apart from the ground state of H_-, the partner Hamiltonians $H_\pm$ have identical bound-state spectra. In particular, they satisfy

$$E_{n+1}^-=E_n^+,\quad n=0,1,2,\dots\ . \tag{2.9}$$

(iii) The eigenfunctions of $H_\pm$ corresponding to the same eigenvalue are related by

$$A\Psi_{n+1}^-(x)=(E_n^+)^{1/2}\Psi_n^+(x)\ , \tag{2.10a}$$

$$A^\dagger\Psi_n^+(x)=(E_n^+)^{1/2}\Psi_{n+1}^-(x)\ . \tag{2.10b}$$

It may be noted here that given a Hamiltonian H_- the corresponding partner H_+ is in general not unique, but one has a class of Hamiltonians $H_+(\lambda)$ that could be partner Hamiltonians. This in turn has indicated[20] a deep relationship between SUSY quantum mechanics and the inverse scattering method of Gelfand and Levitan.[21]

The next conceptual advance was the realization that the superpotential $W(x)$ and, therefore, the factorization of the Hamiltonian could be generated from the ground-state solution of the Schrödinger equation.[6,18] This in turn led to the fact that there are a hierarchy of related Hamiltonians, H_1, H_2, $H_3,\dots,H_n$, having the same bound-state spectra except m states are missing and $0<m<N$, where $N+1$ is the number of bound states of H_1 (Ref. 12). To recapitulate this result we follow Sukumar.[12] If one has the Hamiltonian

$$H=-\nabla^2+V(x) \tag{2.11}$$

and we want to factorize the Hamiltonian as in (2.3) and (2.4) with

$$H = A^\dagger A + \epsilon \ , \tag{2.12}$$

then W needs to satisfy the equation

$$W_2 - W' = V - \epsilon \ . \tag{2.13}$$

One solution to this equation is

$$W = -\partial_x \Psi(x,\epsilon)/\Psi(x,\epsilon) \ , \tag{2.14}$$

where Ψ obeys the equation

$$H\Psi = \epsilon\Psi \tag{2.15}$$

and to preserve the positive semidefinitness of $A^\dagger A, \Psi$ must have no nodes so that $\Psi(x,\epsilon)$ is $\Psi_0(x)$ and $\epsilon = E_0$.

Therefore, once we know the ground-state energy $\epsilon_0^{(1)}$ and ground-state wave function $\Psi_0^{(1)}$ of H^1 we can determine W^1 from

$$W^1 = -d \ln(\Psi_0^{(1)})/dx \ . \tag{2.16}$$

We first generate the partner Hamiltonian

$$H_2 = A_1 A_1^\dagger + \epsilon_0^{(1)} \tag{2.17}$$

which by factorization and (2.9) can also be written as

$$H_2 = A_2^\dagger A_2 + \epsilon_1^{(1)} \ . \tag{2.18}$$

One obtains W_2 by the SUSY relation (2.10):

$$\Psi_n^{(2)} = (\epsilon_{n+1}^{(1)} - \epsilon_0^{(1)})^{-1/2} A_1 \Psi_{n+1}^{(1)} \ , \tag{2.19a}$$

and the factorization result (2.14) by

$$W_2 = -d \ln \Psi_0^{(2)}/dx \ . \tag{2.19b}$$

One then keeps iterating this method. For example, the SUSY partner Hamiltonian of H_2 is H_3 which can be written as

$$H_3 = A_2 A_2^\dagger + \epsilon_0^{(2)} = A_3^\dagger A_3 + \epsilon_2^{(1)} \ , \tag{2.20a}$$

where

$$\Psi_0^{(3)} \propto A_2 A_1 \Psi_2^{(1)} \tag{2.20b}$$

and

$$W_3 = -d \ln \Psi_0^{(3)}/dx \ . \tag{2.21}$$

So we see if we know exactly the spectra and eigenfunctions of a given Hamiltonian we can generate a sequence of Hamiltonians by this method. In Sec. V we will show that the degenerate states of this collection of Hamiltonians can be unified by having a super Hamiltonian which has an additional SU(2) algebra formed from the A_i. (A picture of the sequence of Hamiltonians and their degenerate spectra is given in Fig. 4 for the case of a Ginocchio class potential for H_1.)

In a remarkable paper, Gendenshtein[17] explored from a different perspective the relationship between SUSY, the hierarchy of Hamiltonians, and solvable potentials. Let us consider the pair potentials $V_\pm(x,a)$, where a is a set of parameters. Gendenshtein calls these potentials "shape invariant" if they satisfy the relationship

$$\begin{aligned} V_+(x,a) &\equiv W^2(x,a) + W'(x,a) \\ &= W^2(x,a_1) - W'(x,a_1) + C(a_1) \\ &\equiv V_-(x,a_1) + C(a_1) \ , \end{aligned} \tag{2.22}$$

where $a_1 = f(a)$ and $C(a_1)$ is a constant independent of x. Using $E_0^- = 0$ and Eq. (2.9) it then immediately follows that the complete bound-state spectrum of H_- is

$$E_n^- = \sum_{k=1}^{n} C(a_k) \ . \tag{2.23}$$

The remarkable thing is that the well-known solvable potentials, such as Coulomb, oscillator, Morse, Rosen-Morse, Eckart, Poschl-Teller, are all shape invariant [in the sense of Eq. (2.22)] and, hence, Eq. (2.23) immediately gives their energy eigenvalue spectra. In the same paper, Gendenshtein[17] then conjectured that shape invariance is not only sufficient but may even be necessary for a potential to be solvable.

In a recent paper,[22] Dutt, Khare, and Sukhatme, have further shown that using SUSY one can also obtain the bound-state energy eigenfunctions of H_- for shape-invariant potentials. In particular, by using the fact that the ground-state wave function of H_- is $\Psi_0^-(x,a)$ [as given by Eq. (2.8)] and using Eq. (2.10b) they show that the nth-state eigenfunction $\Psi_n^-(x,a)$ is given by

$$\Psi_n^-(x,a) = A^\dagger(x,a) A^\dagger(x,a_1) \cdots A^\dagger(x,a_{n-1}) \Psi_0^-(x,a_n) \ . \tag{2.24}$$

This is thus a generalization of the operator method of constructing Ψ_n's for the harmonic-oscillator problem for which case $a_n = a_{n-1} = \cdots = a_1 = a$. These authors have also shown[22] that the leading-order SUSY WKB approximation[23] also reproduces the exact spectra (2.23) for any shape-invariant potential.

We notice that (2.24) is exactly the same equation as that found for the general hierarchy of Hamiltonians (2.10) and (2.20), except that now one has that, from shape invariance,

$$A_n(x) = A_1(x,a_{n-1}) \tag{2.25}$$

and

$$\Psi_0^{(n)} = N_n \exp\left[-\int_x W_1(x,a_{n-1}) dx \right] \ . \tag{2.26}$$

From these results one realizes that one can always cast a solvable potential into supersymmetric form because we know explicitly the ground-state wave function and thus W. We can therefore ask whether a particular potential we are studying belongs to a class of shape-invariant potentials. If that is so we could have solved for the bound-state spectra and wave functions without solving the Schrödinger equation.

III. SOLVABLE POTENTIALS AND SHAPE INVARIANCE

In this section we wish to discuss the six-parameter solvable class of potentials of Natanzon[7] and the restricted class of Natanzon potentials having two parameters (Ginocchio class[8]) to see if these classes of potentials are shape invariant [that is, can we write V_2 in terms of a V_1 which is still in the class of Natanzon (or Ginocchio)].

A. Restricted class of Natanzon potentials

In this case, the Schrödinger equation in dimensionless units

$$r = bx \equiv (2mv_0/\hbar^2)^{1/2}x \tag{3.1}$$

is given by

$$[-d^2/dr^2 + V(r)]\Psi_n(r) = \epsilon_n \Psi_n(r) , \tag{3.2}$$

where

$$V(x) = v_0 V(r), \quad \epsilon_n = E_n/v_0 . \tag{3.3}$$

The potential function $V(r)$ is given here only in terms of the function $y(r)$ by[8]

$$V(r) = \{-\lambda^2 v(v+1) + \tfrac{1}{4}(1-\lambda^2)[5(1-\lambda^2)y^4 - (7-\lambda^2)y^2 + 2]\}(1-y^2) , \tag{3.4}$$

where v and λ are dimensionless parameters which measure the depth and the shape of the potential, respectively. Note that y is not explicitly known in terms of r. However, r is explicitly known in terms of y and is given by

$$r\lambda^2 = \{ \operatorname{arctanh}(y) - (1-\lambda^2)^{1/2} \times \operatorname{arctanh}[(1-\lambda^2)^{1/2}y]\}, \quad \lambda < 1 \tag{3.5a}$$

or equivalently

$$r\lambda^2 = \{ \operatorname{arctanh}(y) + (\lambda^2-1)^{1/2} \times \operatorname{arctanh}[(\lambda^2-1)^{1/2}y]\}, \quad \lambda > 1 . \tag{3.5b}$$

Thus, as $-\infty \le r \le \infty$, y ranges through the finite interval $-1 \le y \le 1$ and is symmetrical around $r=0$. From Eqs. (3.5a) or (3.5b) we find that

$$dy/dr = (1-y^2)[1+(\lambda^2-1)y^2] . \tag{3.6}$$

As shown in Ref. 8 the bound-state eigenfunctions for this case are given by

$$\Psi_n \propto (1-y^2)^{\mu_n/2}[g(y)]^{-(2\mu_n+1)/4} \times C_n^{(\mu_n+1/2)}(\lambda y/[g(y)]^{1/2}) ,$$

where

$$g(y) = 1 + (\lambda^2-1)y^2 \tag{3.7}$$

while the corresponding energy eigenvalues are given by

$$\epsilon_n = -\mu_n{}^2\lambda^4, \quad \mu_n > 0 , \tag{3.8a}$$

where

$$\mu_n\lambda^2 = [\lambda^2(v+1/2)^2 + (1-\lambda^2)(n+1/2)^2]^{1/2} - (n+1/2) . \tag{3.8b}$$

Here $C_n^{(\alpha)}(z)$ is a Gegenbauer polynomial when n is a non-negative integer ($n=0,1,2,\ldots,n_{\max}$; $n_{\max} < v$).

In this Ginocchio class of potentials, the number of bound states N is simply related to v:

$$\{v\} = N , \tag{3.9}$$

where $\{v\}$ means the largest integer smaller than v.

We next want to cast the potential (3.4) into supersymmetric form, i.e.,

$$V_-(r) \equiv V(r) - \epsilon_0 = W_1{}^2(r) - W_1'(r) , \tag{3.10}$$

where ϵ_0 is the ground-state energy as given by Eq. (3.8) and $W_1(r)$ is the superpotential. From Eq. (2.14) it follows that $W_1(r)$ can be computed from the ground-state wave function:

$$W_1(r) = -\Psi_0'(r)/\Psi_0(r) . \tag{3.11}$$

Since in our case the ground-state wave function is given by

$$\Psi_0(r) \propto (1-y^2)^{\mu_0/2}[g(y)]^{-(2\mu_0+1)/4} , \tag{3.12a}$$

hence,

$$W_1(r) = \tfrac{1}{2}(1-\lambda^2)y(y^2-1) + \mu_0 y\lambda^2 , \tag{3.12b}$$

where μ_0 is as given by Eq. (3.8b). As a check, one can immediately verify that this $W_1(r)$ indeed satisfies Eq. (3.10) when $V(r)$ and ϵ_0 are as given by Eqs. (3.4) and (3.8), respectively.

We would like to know whether this two-parameter family of potentials is shape invariant. If it is, then we can solve for the wave functions and energy spectra using (2.23) and (2.24). For shape invariance to be true, we need [since $y = y(\lambda, r)$]

$$W_1{}^2(y(\lambda,r),v,\lambda) + [dW_1(y(\lambda,r),v,\lambda)/dy]dy(\lambda,r)/dr = W_1{}^2(y(\lambda'r),v',\lambda') - [dW_1(y(\lambda',r),v,\lambda)/dy]dy(\lambda',r)/dr + C(v,\lambda) , \tag{3.13}$$

where $dy(\lambda,r)/dr$ is given by (3.6) and C is independent of r. It is quite difficult to solve (3.13) for $v'(v,\lambda)$, $\lambda'(v,\lambda)$, and $C(v,\lambda)$ because unless $\lambda=\lambda'$ we do not have an explicit way of determining $y(\lambda',r)$ in terms of $y(\lambda,r)$, since y is only an implicit function (unless $\lambda=1$) of λ, r as given by (3.5). To get around this difficulty we realize the shape invariance and the SUSY relationship (2.9) require that the energy levels of H_1 and H_2 be related:

$$E_{n+1}^{-}(\nu,\lambda)=E_n^{-}(\nu',\lambda')+f(\nu,\lambda)\ , \tag{3.14}$$

where f must be independent of the level number n. Also H_2 must have one fewer bound state than H_1 (for the same values of the parameters) so that if H_2 also belongs to the Ginocchio class of potentials one necessarily has, from (3.9),

$$\{\nu'\}=N-1\ . \tag{3.15}$$

We can use the condition (3.14) to show that, for this class of potentials, $\lambda'=\lambda$ is a necessary condition for (3.13) to be satisfied. Equation (3.14) leads to

$$\epsilon_{n+1}(\nu,\lambda)-\epsilon_1(\nu,\lambda)-\epsilon_n(\nu',\lambda')+\epsilon_0(\nu',\lambda')\equiv 0\ . \tag{3.16}$$

Using (3.8) leads to

$$(\lambda^2-2)(n+3/2)^2-(\lambda'^2-2)(n+1/2)^2+\epsilon_0(\nu',\lambda')-\epsilon_1(\nu,\lambda)=2(n+1/2)[\lambda'^2(\nu'+1/2)^2+(1-\lambda'^2)(n+1/2)^2]^{1/2} -2(n+3/2)[\lambda^2(\nu+1/2)^2+(1-\lambda^2)(n+3/2)^2]^{1/2}\ . \tag{3.17}$$

By squaring this equation, bringing all terms to the left-hand side except for the remaining square root, and then squaring again, this equation is reduced to an equation which is an eighth-order polynomial in n which is identically equal to zero. In order for this latter equation to be satisfied for all n, the coefficient of each order in the polynomial must be identically zero.

Satisfying these conditions results in

$$\lambda'=\lambda\ , \tag{3.18}$$

$$\lambda'=1\ , \tag{3.19}$$

and

$$\nu'=\nu-1\ . \tag{3.20}$$

Hence, the only solution to the shape-invariance condition is the Poschl-Teller potential which has already been shown to be shape invariant.[17]

When $\lambda=\lambda'$ we can directly use the Gendenshtein equation instead of (3.16) to rule out shape invariance. If shape invariance is true and if we do not change λ then $W_2(r)$ must be a cubic in y which is odd from (3.12):

$$V_+(r)\equiv W_1{}^2(r)+W_1'(r)=W_2{}^2(r)-W_2'(r)+\epsilon_1-\epsilon_0\ , \tag{3.21}$$

where

$$W_2(r)\equiv Ay^3+By \tag{3.22}$$

with A and B being constants to be determined from v and λ. In view of Eq. (2.8) it follows that this $W_2(r)$ will be an acceptable choice if and only if the corresponding ground-state wave function $\Psi_0^+(r)$ defined by

$$\Psi_0^+(r)\propto\exp\left[-\int^r W_2(r')dr'\right] =\exp\left[-\int^y W_2(y')(dr/dy')dy'\right] \tag{3.23}$$

is square integrable. In view of Eq. (3.5) this implies that Ψ_0^+ should vanish as $y\to\pm1$. Using Eqs. (3.6) and (3.22) in (3.23) and demanding the vanishing of Ψ_0^+ at $y=\pm1$ we obtain the constraint

$$A+B>0\ . \tag{3.24}$$

On the other hand, using Eqs. (3.12) and (3.22) in (3.21) gives us the following two solutions ($\lambda^2\neq1$).

Solution (i):

$$A=-\tfrac{1}{2}(1-\lambda^2),\quad B=-\mu_0+\tfrac{1}{2}(1-\lambda^2)\ . \tag{3.25}$$

Solution (ii):

$$A=\tfrac{7}{2}(1-\lambda^2)\ , \tag{3.26a}$$

$$B=-2-\mu_0/3-\tfrac{11}{6}(1-\lambda^2)\ , \tag{3.26b}$$

and

$$\tfrac{4}{9}[\tfrac{1}{2}(1-\lambda^2)-\mu_0]^2=(2-\lambda^2)^2-6(1-\lambda^2)\ . \tag{3.26c}$$

Straightforward algebra immediately shows that neither solution (3.25) nor (3.26) satisfies the square-integrability constraint (3.24) so that none of them is an acceptable choice.

There is yet an alternative way of proving that the class of potentials (3.4) is not shape invariant for $\lambda=\lambda'$. The point is that since the potentials (3.4) are exactly solvable we can directly compute $W_2(r)$. Using (2.10a), we obtain

$$\Psi_0^+(r)\propto[d/dr+W_1(r)]\Psi_1(r)\ , \tag{3.27}$$

and then using Eq. (2.14),

$$W_2(r)=-\Psi_0^{+\prime}(r)/\Psi_0^+(r)\ . \tag{3.28}$$

Using $\Psi_1(r)$ and $W_1(r)$ as given by Eqs. (3.7) and (3.12), respectively, we find that

$$\Psi_0^+(r)\propto(1-y^2)^{\mu_1/2}[g(y)]^{-(2\mu_1+3)/4}f(y)\ ,$$

where $g(y)$ is given by (3.7) and

$$f(y)=1+(\mu_0\lambda^2-\mu_1\lambda^2-1)y^2 \tag{3.29}$$

and, hence,

$$W_2(r)=\tfrac{3}{2}(1-\lambda^2)y(y^2-1)+\mu_1\lambda^2 y-[2g(y)(\mu_0\lambda^2-\mu_1\lambda^2-1)y(1-y^2)]/[f(y)] . \tag{3.30}$$

On comparing $W_1(r)$ and $W_2(r)$ as given by Eqs. (3.12) and (3.22), respectively, it is clear that the class of potentials (3.4) for fixed λ are not shape invariant except when $\lambda^2=1$. For $\lambda^2=1$, using (3.8b) we find that in (3.22) only the second term survives and in that case (3.4) reduces to the Poschl-Teller[1] potential which is a shape-invariant potential.

We have explicitly verified that Eq. (3.21) is satisfied by $W_1(r)$ and $W_2(r)$ as given by Eqs. (3.12) and (3.30), respectively, when use is made of Eq. (3.8). Thus, although W_2 is a ratio of polynomials $P_5(y)/Q_2(y)$, the quantity

$$W_2{}^2-(dW_2/dy)dy/dr \tag{3.31}$$

is a sixth-order polynomial in y. This is not so with the potential for H_3:

$$W_2{}^2+(dW_2/dy)dy/dr \tag{3.32}$$

which is the ratio of a tenth-order polynomial in y over a fourth-order polynomial in y: $P_{10}(y)/Q_4(y)$ and Q is not a factor of P. Thus for fixed λ we are discussing potentials which are quite different from those expressed by (3.4) which is a sixth-order polynomial in y. As we will see later the wave functions of the Hamiltonians generated by factorization and SUSY will also differ from those given by (3.7).

Before finishing the discussion about the potential (3.4) we would like to remark that the Ginocchio class of potentials is quite rich and can have multiple-well structure. It is a sixth-order polynomial in y and can have four real zeros apart from $y=\pm 1$ in the region $-1<y<1$ which maps into the whole real axis $-\infty<r<\infty$. In particular, it is easy to show that the potential (3.4) can have double-well structure when the quantity $\lambda^2\nu(\nu+1)$ is less than $\frac{1}{4}(\lambda^2-1)(\lambda^2-9)$. For a given ν this implies that the potential has a double-well structure when λ^2 is either sufficiently small or sufficiently large.

In conclusion we see from our above discussion that if we restrict our potentials to have the two parameters of the Ginocchio class that we cannot satisfy the shape-invariance condition. Instead we generate potentials which are more complicated functions of y (ratios of polynomials).

We could of course have asked if a more general class of potentials which are sixth-order polynomials in y (for fixed λ) can be shape invariant. This question is easily answered in the negative.

Suppose we generalize the potential of Eq. (3.4) by choosing

$$W_1(y)=Ay^3+By^2+Cy+D , \tag{3.33a}$$

so that we now have a five-parameter family of potentials, but we do not allow λ to change under reparametrization. So we allow in principle four parameters to change under reparametrization. The first thing we find is that in order for the ground state to be square integrable we obtain the conditions that

$$A+C>0, \quad B=-D , \tag{3.33b}$$

so that

$$W_1(y)=Ay^3+Cy+B(y^2-1) .$$

If we now try to rewrite

$$\begin{aligned} W^2(A,B,C)&+[dW(A,B,C)/dy]dy/dr \\ &=W^2(A',B',C')-[dW(A',B',C')/dy]dy/dr \\ &\quad +f(A,B,C) , \end{aligned} \tag{3.33c}$$

we find there are no solutions that satisfy the square-integrability condition on the wave function $A'+C'=0$. Thus, by enlarging the class of potentials in this trivial way to keep the potentials sixth order in y was not sufficient to obtain a shape-invariant class of potentials.

It would be nice to know if it is possible to enlarge the parameter space of this class of potentials to have only a finite number of parameters and still be shape invariant.

B. General Natanzon class of potentials

In this case, the Schrödinger equation in dimensionless form is again given by Eq. (3.2) [see also Eqs. (3.1) and (3.3)] but now the potential $V(r)$ is given in terms of the function $z(r)$ by[7]

$$\begin{aligned} V(r)=&[fz(z-1)+h_0(1-z)+h_1 z+1]/R \\ &+\{a+[a+(c_1-c_0)(2z-1)]/[z(z-1)] \\ &-5\Delta/4R\}z^2(1-z)^2/R^2 , \end{aligned} \tag{3.34}$$

where

$$R=az^2+(c_1-c_0-a)z+c_0 \tag{3.35}$$

and

$$\Delta=(a-c_0-c_1)^2-4c_0c_1 . \tag{3.36}$$

This potential is a function of six dimensionless parameters f, h_0, h_1, a, c_0, and c_1. The subclass of potentials considered in Sec. III A has only two parameters. The parameters are $c_0=0$, $c_1=1/\lambda^4$, $a=c_1-1/\lambda^2$, $h_0=-\frac{3}{4}$, $h_1=-1$, and $f=(\nu+1/2)^2-1$.

As shown by Natanzon[7] the transformation from r to z which transforms the Schrödinger equation to a hypergeometric form is such that $0\le z\le 1$ as $-\infty\le r\le\infty$ and, furthermore,

$$dz/dr=2z(1-z)/R^{1/2} . \tag{3.37}$$

This equation like (3.6) implicitly defines $z(r)$.

For simple cases of $R(z)$ one can explicitly solve for $z(r)$ and one obtains all of the known potentials which are explicitly given as functions of r. These potentials are also known to be shape invariant. When $R(z)$ is of the form $R=c_1 z$ one obtains the Poschl-Teller potential;[1] for $R=c_1 z^2$ one obtains the Manning-Rosen[1] potential; for $R=c_1$ one obtains the Rosen-Morse[1] potential; for

$R = a(1-z)^2$ one obtains the Eckart[1] potential. Thus, the Natanzon class of potentials includes all solvable potentials whose potentials are explicitly known and these potentials are known to be shape invariant.

In the general case, the energy eigenvalues are given by[7]

$$2n+1=(1-a\epsilon_n+f)^{1/2}-(1-c_0\epsilon_n+h_0)^{1/2} -(1-c_1\epsilon_n+h_1)^{1/2} \equiv \alpha_n-\beta_n-\delta_n \quad (3.38a)$$

while the corresponding eigenfunctions are

$$\Psi_n \propto R^{1/4}z^{\beta_n/2}(1-z)^{\delta_n/2} \times {}_2F_1(-n,\alpha_n-n;1+\beta_n;z) , \quad (3.38b)$$

where ${}_2F_1(\sigma_1,\sigma_2;\sigma_3;z)$ is a hypergeometric function.

Using (3.38a) we can proceed as we did in the restricted case to determine whether the supersymmetry and shape-invariance conditions (2.9) and (2.22) can be satisfied. This leads to the condition (3.14) with the two parameters ν,λ replaced by the six parameters of the Natanzon class. When applied to (3.38a), Eq. (3.14) leads to the expression

$$2=\alpha_{n+1}-\alpha'_n-\beta_{n+1}-\delta_{n+1}+\beta'_n+\delta'_n , \quad n=0,1,\ldots , \quad (3.39a)$$

where the functions defined in (3.38a) depend on the same energy, but different parameters, i.e.,

$$\alpha_{n+1}=(1-a\epsilon_{n+1}+f)^{1/2} , \quad \alpha'_n=(1-a'\epsilon_{n+1}+f')^{1/2} , \quad (3.39b)$$

etc.

The condition (3.39a) must be valid for all ϵ_{n+1} and this leads to a 32nd-order polynomial in ϵ_{n+1} which must vanish identically. Hence, the 33 coefficients of this polynomial must be zero from which we can determine the six primed parameters in terms of the original unprimed parameters. Since this is an overdetermined system, one expects that only a small subspace of the six-parameter space of unprimed variables will allow for a solution. If these conditions are not met in general then we can conclude that the Natanzon class of potentials are not, in general, shape invariant. Unfortunately, the algebra involved in solving the 33 simultaneous algebraic equations was too complex to attempt although straightforward.

By studying (3.39) we discovered that there are rather general solutions which are not connected with shape invariance. Thus, even obtaining a solution to (3.39) does not necessarily prove that the Natanzon class of potentials are shape invariant. For example, if we set $c_0=0$ in these equations, then β and β' are independent of n and are constant. Solutions to (3.39) are $a'=a$, $c'_1=c_1$, $f'=f$, $h'_1=h$, $\beta'=\beta+2$. However, this subclass is not shape invariant as we shall see below, unless in addition either $c_1=0$, $c_1=a$, or $a=0$.

Using (2.14) we find that the superpotential $W_1(r)$ is given by

$$W_1(r)=[\delta_0 z-(1+\beta_0)(1-z)]/R^{1/2} +[(c_1-c_0-a)z+2c_0](1-z)/2R^{3/2} . \quad (3.40)$$

It is not difficult to verify that this $W_1(r)$ indeed satisfies Eq. (3.10) when $V(r)$ and ϵ_0 are given by Eqs. (3.34) and (3.38), respectively.

We have not been able to implement (because of the algebraic complexity) the general strategy to show that the full Natanzon class of potentials are not shape invariant. However, if under shape invariance (2.22) we do not allow for the parameters a, c_0, and c_1, which determine the transformation from z to r as given by (3.35) and (3.37), to vary then we do not have to deal with the change in z which would require solving implicit relations (relating r and z). In that case we can proceed as in the Ginocchio class when λ is held fixed. Note that keeping a, c_0, and c_1 fixed leads to a different class of potentials than the Ginocchio class.

In order to prove that this second type of restricted class of exactly solvable potentials is in general not shape invariant we shall use the same strategy as in Sec. III A and compute $\Psi_0^+(r)$ and then $W_2(r)$ by using formulas (3.27) and (3.28). Using $\Psi_1(r)$ and $W_1(r)$ as given by Eqs. (3.38b) and (3.40) we find

$$\Psi_0^+(r) \propto R^{-1/4}z^{\beta_1/2}(1-z)^{\delta_1/2}(Az^2+Bz+C)/(1+\beta_1) , \quad (3.41)$$

where

$$A=(\delta_1-\delta_0+\beta_1-\beta_0+2)(\alpha_1-1) ,$$
$$B=(\beta_1+1)(\delta_0-\delta_1)+(\alpha_1-\beta_1)(\beta_0-\beta_1)+2(1-\alpha_1) , \quad (3.42)$$
$$C=(\beta_1-\beta_0)(1+\beta_1) .$$

If we now calculate W_2 using (3.41) we find that W_2 has the form

$$W_2=P_5(z)/[(Az^2+Bz+C)R^{3/2}] , \quad (3.43)$$

where $P_5(z)$ is a fifth-order polynomial in z. Comparing $W_1(r)$ and $W_2(r)$ as given by Eqs. (3.40) and (3.43), it is immediately clear that the class of potentials (3.34) with a, c_1, and c_0 fixed even though solvable are not in general shape invariant.

Only in the cases $R=c_1$, $c_1 z$, $c_1 z^2$, $az(1-z)$, $c_0(1-z)$, $a(1-z)^2$ do we have shape invariance, but these choices lead to the known solvable potentials mentioned earlier.

Using (3.42) we find that when we calculate V_3 from

$$W_2{}^2(r)+(dW_2/dz)(dz/dr) ,$$

V_3 has a much more complicated dependence on z than the Natanzon class potential V_1, in that there are now extra ratios of polynomials in z multiplying the basic structure of (3.34). Since, in principle, this could lead to more zeros of the potential on the real axis, it is hard to believe that one stays in the same class of potentials (unless all of the extra real zeros in z occur for z greater than 1). Also, as we shall see later, the wave functions of the new Hamiltonians are sums of hypergeometric functions of

$z(a,c_0,c_1,r)$. Unless there are sum rules relating sums of hypergeometric functions of $z(a,c_0,c_1,r)$ in terms of a single hypergeometric function of $z(a',c_0'c_1',r)$, we leave the general class of Natanzon potentials when we generate new Hamiltonians by factorization and SUSY. Unfortunately, in the general case, Eq. (3.39) which needs to be solved to obtain the shape-invariance conditions is quite complicated. We were not, therefore, able to obtain a complete proof of the lack of shape invariance of the general class of Natanzon potentials as we were able to do in the Ginocchio class, where ϵ_n was explicitly given by (3.8).

C. Confluent hypergeometric case

For completeness we shall now discuss the class of potentials for which the Schrödinger equation reduces to confluent hypergeometric form.[7] Again the Schrödinger equation in dimensionless form is given by Eqs. (3.1)–(3.3), but now the potential $V(r)$, given in terms of the function $\rho(r)$, has the form

$$V(r)=(g_2\rho^2+g_1\rho+h_0+1)/R_1 - (\sigma_2+\sigma_1/\rho+\tfrac{5}{4}\Delta_1/R_1)\rho^2/R_1^{\,2} , \tag{3.44}$$

where

$$R_1=\sigma_2\rho^2+\sigma_1\rho+c_0 \tag{3.45a}$$

and

$$\Delta_1=\sigma_1^{\,2}-4\sigma_2c_0 . \tag{3.45b}$$

Here g_2, g_1, σ_2, σ_1, c_0, and h_0 are six dimensionless parameters and the mapping from r to ρ is characterized by

$$d\rho/dr=2\rho/R_1^{\,1/2} . \tag{3.46}$$

As shown in Ref. 7, the energy eigenvalues for this case are given by

$$\begin{aligned}2n+1 &= -(g_1-\sigma_1\epsilon_n)/[2(g_2-\sigma_2\epsilon_n)^{1/2}] \\ &\quad -(h_0-c_0\epsilon_n+1)^{1/2} \\ &\equiv -(g_1-\sigma_1\epsilon_n)/(2\alpha_n)-\beta_n ,\end{aligned} \tag{3.47}$$

while the corresponding eigenfunctions are

$$\Psi_n \propto R_1^{\,1/4}\rho^{\beta_n/2}\exp(-\alpha_n\rho/2)\,{}_1F_1(-n;1+\beta_n;\alpha_n\rho) , \tag{3.48}$$

where ${}_1F_1(\alpha;\beta;\rho)$ is a confluent hypergeometric function.

Using Eqs. (2.14) and (3.46)–(3.48) the superpotential $W_1(r)$ turns out to be

$$W_1(r)=[\alpha_0\rho-(1+\beta_0)]/(R_1)^{1/2}+(\sigma_1\rho+2c_0)/(2R_1^{\,3/2}) . \tag{3.49}$$

As expected, this $W_1(r)$ satisfies Eq. (3.10) with $V(r)$ and ϵ_0 as given by Eqs. (3.44) and (3.47), respectively. By exactly following the same procedure as before we are now able to show that this class of exactly solvable potentials for the restricted case c_0,σ_1,σ_2 fixed is in general not shape invariant. In particular, we have

$$\Psi_0^{+}(r)\propto R_1^{\,-1/4}\rho^{\beta_1/2}\exp(-\alpha_1\rho/2)(A\rho^2-B\rho+C) \tag{3.50}$$

and, hence,

$$\begin{aligned}W_2(r)= &[\alpha_1\rho-(\beta-1)]/(R_1)^{1/2}-(\sigma_1\rho+2c_0)/(2R_1^{\,3/2}) \\ &-(2\rho/R^{\,1/2})(2A\rho-B)/(A\rho^2-B\rho+C) ,\end{aligned} \tag{3.51}$$

where

$$\begin{aligned}A&=(\alpha_1-\alpha_0)\alpha_1 , \\ B&=(2\alpha_1-\alpha_0)(1+\beta_1)-\alpha(1+\beta_0) , \\ C&=(\beta_1-\beta_0)(1+\beta_1)-2\alpha_1 .\end{aligned}$$

On comparing the $W_1(r)$ and $W_2(r)$ as given by Eqs. (3.49) and (3.51) we conclude, as before, that the restricted class of potentials with c_0,σ_1,σ_2 fixed is not, in general, shape invariant. The exceptions are where $R_1=c_0$ (Morse potential); $R_1=\sigma_2\rho^2$ (three-dimensional Coulomb); and $R_1=\sigma_1\rho$ (three-dimensional oscillator).

Again, when we calculate H_3 we find that the ρ dependence of V_3 is more complicated than in (3.44), in that there are extra ratios of polynomials $P_4(\rho)/Q_4(\rho)$ (the subscript refers to the degree of the polynomial) multiplying the general structure of $V_1(\rho)$ as given by (3.44). When we look at the eigenfunctions of the Hamiltonians H_2, H_3, etc., they consist of sums of confluent hypergeometric functions of $\rho(r,c_0,\sigma_1,\sigma_2)$. Unless these sums of confluent hypergeometric functions can be related to a single hypergeometric function of $\rho(r,c_0',\sigma_1',\sigma_2')$, these new potentials cannot be in the full Natanzon class (six parameters). Again it is not precluded that there is a yet larger class of potentials with more parameters in which one can have shape invariance and an algebraic solution to the spectra.

IV. NEW CLASS OF SOLVABLE POTENTIALS

In this section we shall generate a series of new solvable Hamiltonians from the Natanzon class of potentials as given by Eq. (3.34). The central idea of generating a new solvable Hamiltonian from a given solvable one is reviewed in (2.11)–(2.21). Knowing Ψ_0^- and hence W_1, one can use Eqs. (2.19a) and (2.19b) to generate W_2 and hence a new solvable potential V_2 defined by

$$V_2(r)=W_2^{\,2}(r)-W_2'(r)+\epsilon_1 \tag{4.1}$$

which has an identical spectrum to that of $V(r)$ as given by (3.34) except that the ground state of $V(r)$ is missing and its eigenfunctions are related to those given by Eq. (3.38a) by relation (2.10a). In particular, on using $W_2(r)$ as given by Eq. (3.43), the new Hamiltonian takes the form

$$H_2=-d^2/dr^2+V_2(r) , \tag{4.2}$$

where

$$V_2(r)=\epsilon_0+[(\beta_0+\delta_0)(\beta_0+\delta_0+2)z(z-1)+(\beta_0{}^2-1)(1-z)+(\delta_0{}^2-1)z+1]/R$$
$$+\{a-[c_1(3\beta_0+\delta_0)+c_0(\beta_0+3\delta_0)+a(1-\beta_0-\delta_0)]/[z(z-1)]$$
$$-(2z-1)[(c_1-c_0)(\beta_0+\delta_0+1)+a(\beta_0-\delta_0)]/[z(z-1)]+7\Delta/(4R)\}z^2(1-z)^2/R^2\ . \quad (4.3)$$

Although this expression is as similar in form as a function of z to that of Natanzon's as given by Eq. (3.34), it is quite different in its dependence on the parameters. In particular, the term in the curly brackets must depend only on the parameters involved in the transformation from z to r (i.e., a,c_1,c_0) in order to be in the Natanzon class.

The eigenfunctions of H_2 are given by [see Eq. (2.10a)]

$$\Psi_{n-1}^{(2)}\propto[d/dr+W_1(r)]\Psi_n(r)$$
$$\propto R^{-1/4}z^{\beta_n{}'/2}(1-z)^{\delta_n{}'/2}\{(1-z)[\beta_n-\beta_0-z(\delta_n-\delta_0)]_2F_1(-n,\alpha_n-n;1+\beta_n;z)$$
$$-2z(1-z)n(n+1+\beta_n+\delta_n)_1F_2(-n+1;\alpha_n-n+1;\beta_n+2;z)\} \quad (4.4)$$

and are linear combinations of hypergeometric functions while the eigenfunctions of the Natanzon potential (3.34) are proportional to a single hypergeometric function. As we generate further Hamiltonians with n bound states deleted from the original bound-state spectrum, we find that the eigenfunctions are sums of $(n+1)$ hypergeometric functions and when $n>1$, the structure of the Hamiltonian as a function of z is changed from the Natanzon form (see below).

To generate the next Hamiltonian in the sequence of related Hamiltonians, we start from the second excited state wave function Ψ_2 as given by Eq. (3.38b) and by using Eq. (2.20b) we can obtain the ground-state wave function $\Psi_0^{(3)}$ of the third Hamiltonian H_3 and, hence, W_3 and finally H_3 itself, i.e.,

$$\Psi_0^{(3)}\propto(d/dr+W_2)(d/dr+W_1)\Psi_2(r) \quad (4.5)$$

and, hence,

$$W_3(r)=-d\{\ln[\Psi_0^{(3)}(r)]\}/dr\ , \quad (4.6)$$

so that

$$H_3=-d^2/dr^2+W_3{}^2(r)-W_3'(r)+\epsilon_2$$
$$\equiv-d^2/dr^2+V_3(r)\ . \quad (4.7)$$

The energy eigenvalue spectrum of H_3 is identical to that of the original Natanzon potential (3.34) except that two of its lowest states ϵ_0 and ϵ_1 are missing in H_3. The eigenfunctions of H_3 are given by an analogous relation to that of Eq. (4.4) and, in general, they are linear combination of three hypergeometric functions. By explicit construction one can demonstrate that $V_3(r)$ defined from (4.7) does not belong to the Natanzon class for fixed c_0,c_1,a. However, one can see this more simply by using (2.19), i.e.,

$$V_3(r)=W_3{}^2(r)-W_3'(r)+\epsilon_2=W_2{}^2(r)+W_2'(r)+\epsilon_1\ . \quad (4.8)$$

On using $W_2(r)$ as given by Eq. (3.43) we immediately see that $V_3(r)$ will have an extra piece of the form (P_{10} is a tenth-order polynomial in z)

$$P_{10}(z)/[R^3(Az^2+Bz+C)^2]\ . \quad (4.9)$$

When we compare V_3 with V_1 we find the original structure multiplied by the quantity $P_4(z)/(Az^2+Bz+C)^2$. It is interesting that W^2 and $(dW/dz)(dz/dr)$ are of the form as V_3. However, when we subtract the two terms we obtain the structure of V_2 which is similar to V_1.

Proceeding in the same way, we can start from the Natanzon potential (3.34) and generate a series of new Hamiltonians $H_4,H_5,\ldots,H_n$ which will have identical spectrum to that of (3.34) except that the first $3,4,\ldots,n-1$ levels of the original spectrum as given by Eq. (3.38) will be missing. However, the corresponding potentials will have extra factors of the type

$$(Az^2+Bz+C)^2(A_2z^2+B_2z+C_2)^2\cdots$$
$$\times(A_{n-2}z^2+B_{n-2}z+C_{n-2})^2 \quad (4.10)$$

in their denominators which are absent in (3.34). Furthermore, their eigenfunctions will, in general, be linear combinations of $4,5,\ldots,n$ hypergeometric functions. The number of such potentials will be equal to the total number of bound states N of the original Hamiltonian H_1.

To compare these potentials we look at the restricted class of potentials given in Sec. III A. Here there are two parameters: ν which determines the depth (the number of bound states $N=\{\nu\}$) and λ which determines the shape. For $\nu=7$ (thus, there are seven different related potentials) one has a single-well structure for $V(r)=V_1(r)+\epsilon_0$ as given by (3.4) when $\lambda=7$, and a double-well structure for $V(r)=V_1(r)+\epsilon_0$ when $\lambda=\frac{1}{10}$. To get a feel for what these families of potentials look like, we will plot the first three potentials and the first three eigenvalues (renormalizing V_1 to have ground-state energy zero). That is, we have

$$V_1(r)=W_1{}^2(r)-W_1'(r)\ , \quad (4.11a)$$

$$V_2(r)=W_1{}^2(r)+W_1'(r)\ , \quad (4.11b)$$

$$V_3(r)=W_2{}^2(r)+W_2'(r)+\epsilon_1-\epsilon_0\ , \quad (4.11c)$$

where W_1 is given by (3.12), W_2 is given by (3.30), and ϵ_n is given by (3.8). In general, it is easier to plot V vs y since then the whole domain of V can be shown. It is not that difficult to get a uniform approximation for the function $y(r)$ which is the inverse of Eq. (3.5). To accomplish that one makes a Taylor-series expansion of (3.5) and reverts the Taylor series. This gives a Taylor series of $y(r)$ of the form

$$y(r)=r\sum a_n r^{2n} \ . \tag{4.12}$$

We know, however, that as $r\rightarrow\infty$, $y\rightarrow 1$, so that a uniform approximation based on the Taylor series (4.12) is obtained first by squaring the right-hand side of Eq. (4.12), considering the sequence of larger and larger diagonal Padé approximants (which have a finite limit as $r\rightarrow\infty$) and second by taking the square root of the terms in this sequence. We find that by keeping 20 terms in the Taylor series and using the (10,10) Padé approximant we obtained an excellent approximation to $y(r)$ for all $|y|\leq 0.96$.

In Fig. 1(a) we plot the case $\nu=7$, $\lambda=7$ for $-1\leq y\leq 1$ which corresponds to the whole real axis $-\infty\leq r\leq\infty$. [From (3.6) we note the change of variables is monotonic.] We see that the potentials are higher than one another and we see how the ground states of the higher potentials are excited states of the lower potentials. In Fig. 1(b) we replot these potentials versus r for $|r|\leq 0.22$ which corresponds to $|y|\leq 0.88$. In terms of r we see these potentials resemble square wells.

For the case of $\lambda=\frac{1}{10}$, $\nu=7$, the energy spectra is more hydrogenlike and progresses as $1/n^2$. Here the sequence of potentials shows rather interesting changes in shape. In Fig. 2(a) we plot V_1 and V_2 from $-1\leq y\leq 1$. We see that V_1 is double welled, but, although the ground state is in the two-well region, because the barrier is very thin [see Fig. 2(b)] there is no almost-degenerate partner. The next two states are for V_2 connected by tunneling through two barriers since V_2 has triple-well structure but has no states purely in the lowest well. The energy splitting here follows a hydrogenic formula rather than exponential splitting. In Fig. 2(b) we replot these two potentials in terms of r. The range $|r|\leq 10$ corresponds to $|y|\leq 0.96$. From this picture we see that the barrier of V_1 is quite thin and we also see that it is only at quite a long range that the excited states are bound.

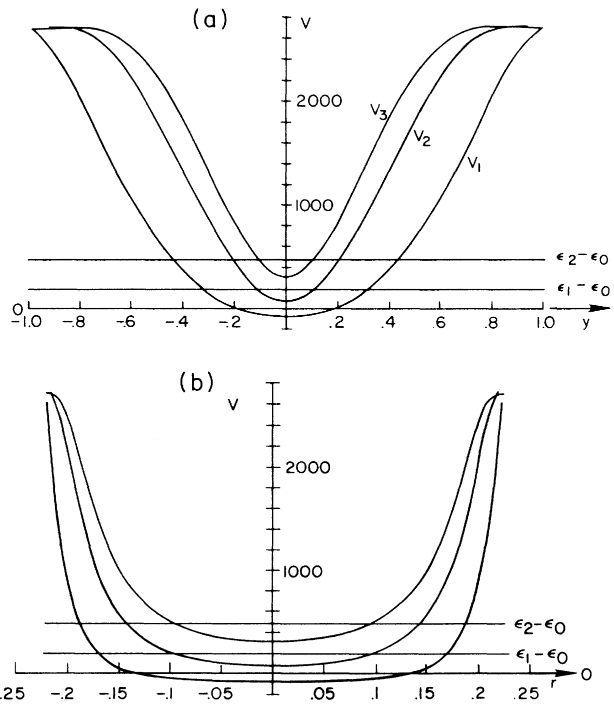

FIG. 1. V_1, V_2, V_3 given by Eq. (4.11) and the eigenvalues 0, $\epsilon_1-\epsilon_0$, $\epsilon_3-\epsilon_0$ for $\nu=7$, $\lambda=7$. In (a) we plot vs y. In (b) we plot vs r. The variables r and y are connected by (3.5).

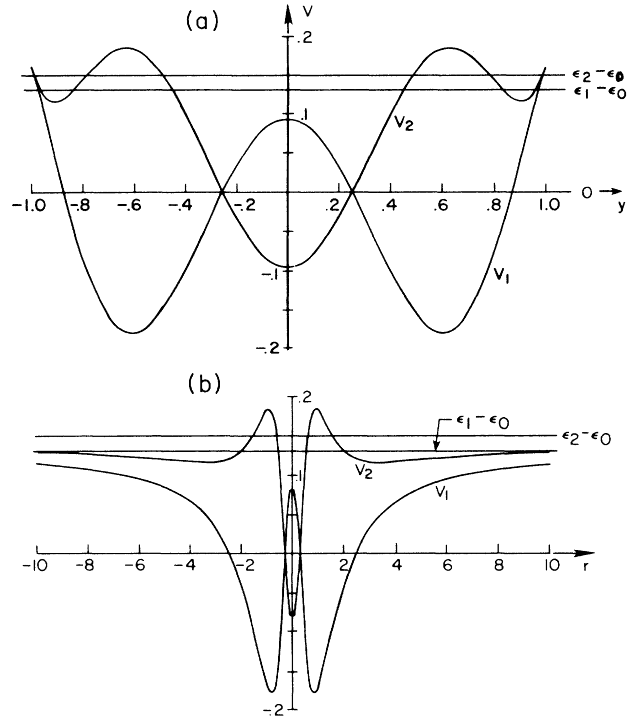

FIG. 2. V_1, V_2, and the eigenvalues 0, $\epsilon_1-\epsilon_0$, $\epsilon_2-\epsilon_0$ for $\nu=7$, $\lambda=\frac{1}{10}$. In (a) we plot vs y. In (b) we plot vs r.

In Fig. 3(a) we plot V_1, V_2, and V_3 in the range $-0.9\leq y\leq 0.9$. We see that V_3 returns to simple double-well structure, but again there is no exponential splitting between the states. The sequence of potentials $V_1, V_2, \ldots, V_7$ all have a finite asymptote at $y=\pm 1$ which corresponds to $r=\pm\infty$.

In Fig. 3(b) we replot V_1, V_2, V_3 vs r for $|r|\leq 10$ which corresponds to $|y|\leq 0.96$.

We have presented evidence that the Natanzon class of potentials (3.34) forms only a small subclass among the huge number of solvable potentials for which the energy eigenfunctions are proportional to a linear combination of hypergeometric functions. Clearly a similar analysis exists for the confluent hypergeometric case and we can generate a huge class of solvable new potentials which do not belong to the Natanzon class (3.44) and for which the energy eigenfunctions are a linear combination of confluent hypergeometric functions.

V. SYMMETRY GROUPS

We know from factorization and SUSY that if we have a Hamiltonian H_1 for which Ψ_0 and ϵ_0 are known, we can

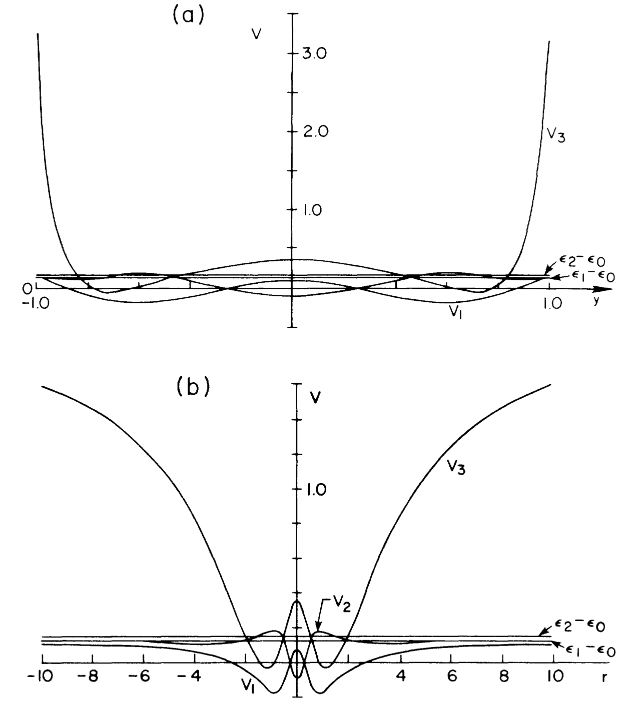

FIG. 3. V_1, V_2, V_3 for $\nu=7$, $\lambda=\frac{1}{10}$. In (a) we plot vs y. In (b) we plot vs r.

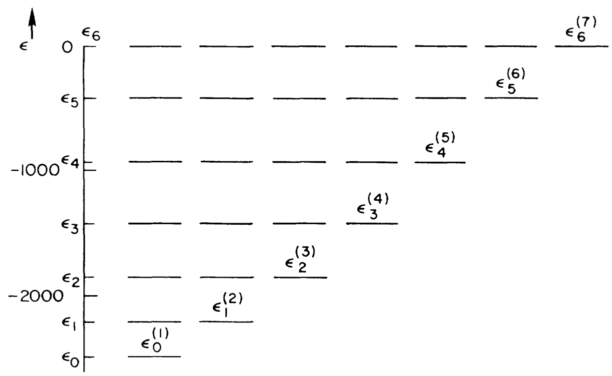

FIG. 4. Spectra of the related Hamiltonians $H_1, H_2, \ldots, H_7$ for the sequence of potentials related to the potential of (3.4). For specificity we chose $\nu=7$, $\lambda=7$ so that the first three potentials are shown in Fig. 1.

generate an entire class of Hamiltonians H_i, $i=2,3,\ldots,N$, where N is the total number of bound levels of the original Hamiltonian H_1, which have the same energy spectrum except that the first $1,2,\ldots,(i-1)$ levels are missing for H_i (Ref. 12). That is

$$H_i\Psi_n^{(i)}=\epsilon_n^{(i)}\Psi_n^{(i)}, \quad n=i-1,i,i+1,\ldots,N-1\ , \tag{5.1a}$$

where $\Psi_n^{(i)}$ and $\epsilon_n^{(i)}$ are the eigenfunctions and eigenenergies of H_i. This degeneracy implies that

$$\epsilon_n=\epsilon_n^{(1)}=\epsilon_n^{(2)}=\cdots=\epsilon_n^{(n+1)} \tag{5.1b}$$

and is illustrated in Fig. 4. We may ask if there is a symmetry group which determines this degeneracy. Previously, a twofold degeneracy has been explained in terms of a superalgebra[10] as discussed in the Introduction.

Instead of the two-dimensional matrix considered in (4.5) and (4.6) we introduce n-dimensional matrices. For convenience we use the notation E_{ij} to denote the $n\times n$ matrix with a 1 in the ith row and jth column and zeros elsewhere, i.e.,

$$(E_{ij})_{kl}=\delta_{i,k}\delta_{j,l}\ . \tag{5.2}$$

We introduce the $n-1$ charges

$$Q_i^{\dagger}=A_i^{\dagger}E_{i,i+1}, \quad i=1,2,\ldots,n-1\ , \tag{5.3a}$$

$$Q_i=A_iE_{i+1,i}\ , \tag{5.3b}$$

where

$$A_i^{\dagger}=-d/dx+W_i \tag{5.3c}$$

which is a natural generalization of the charges in (4.5). By construction, the supersymmetry implies

$$A_i^{\dagger}A_i+\epsilon_{i-1}=A_{i-1}A_{i-1}^{\dagger}+\epsilon_{i-2}\ . \tag{5.4}$$

Using these relations we find that these charges satisfy

$$Q_iQ_j^{\dagger}=\delta_{i,j}(H-\epsilon_{i-1})E_{i,i}\ . \tag{5.5}$$

The Hamiltonian in the matrix formalism is given by

$$H=\sum_{i=1}^{n}(A_i^{\dagger}A_i+\epsilon_{i-1})E_{ii}=\sum_{i=1}^{n}H_iE_{ii}\ . \tag{5.6}$$

With (5.4) we can prove that the charges commute with H:

$$[Q_i,H]=[Q_i^{\dagger},H]=0\ . \tag{5.7}$$

Consequently any product of charges also commute with H. The eigenfunctions of H are then given in terms of column vectors:

$$\chi_n^{(1)}=(\Psi_n^{(1)},0,\ldots)\ , \tag{5.8a}$$

$$\chi_n^{(2)}=(0,\Psi_n^{(2)},\ldots),\ \text{etc.} \tag{5.8b}$$

Then from (5.1) and (5.6),

$$H\chi_n^{(i)}=\epsilon_n\chi_n^{(i)}\ , \tag{5.9}$$

which expresses the degeneracy of H.

We can show that there are at least two symmetry groups of H for which the eigenfunctions (5.9) for $i=1,2,\ldots,n$ are in one n-dimensional representation of H for all n. One of these groups is a supergroup and the other a normal group.

Because Q_i commutes with H we can define a charge

$$Q=\sum_{i\ \mathrm{odd}}[H/(H-\epsilon_{i-1})]^{1/2}Q_i\ . \tag{5.10}$$

Using (5.5) we can show that

$$\{Q,Q^{\dagger}\}=H\ , \tag{5.11a}$$

$$\{Q^{\dagger},Q^{\dagger}\}=\{Q,Q\}=[Q,H]=[Q^{\dagger},H]=0\ . \tag{5.11b}$$

That is, this charge satisfies the same subalgebra sl(1/1) as the two-dimensional version described in the Introduction. However, this n-dimensional representation is reducible under this superalgebra. From (5.3) and (5.10) and the fact that

$$A_i \Psi_{i-1}^{(i)} = 0 \ , \tag{5.12}$$

we find

$$Q\chi_n^{(i)} = \chi_n^{(i+1)}, \quad i \text{ odd} \ , \tag{5.13a}$$

$$Q\chi_n^{(n)} = 0, \quad n \text{ odd} \ , \tag{5.13b}$$

$$Q^\dagger \chi_n^{(i+1)} = \chi_n^{(i)}, \quad i \text{ odd} \ , \tag{5.13c}$$

$$Q\chi_n^{(i)} = 0, \quad i \text{ even} \ , \tag{5.13d}$$

$$Q^\dagger \chi_n^{(i+1)} = 0, \quad i \text{ even} \ . \tag{5.13e}$$

From these relations we see that the representation is block diagonal with $(\chi_n^{(i)}, \chi_n^{(i+1)})$, i odd $< n$ forming doublets, and $\chi_n^{(n)}$ forming a singlet for n odd.

However, we can enlarge the algebra and get all the states in one irreducible representation if we introduce the boson generators

$$L_+ = \frac{1}{2} \sum_{i \text{ odd}}^{n-3} [(i+1)(\underline{n}-i-1)]^{1/2} \{ [(H-\epsilon_i)(H-\epsilon_{i-1})]^{-1/2} Q_i^\dagger Q_{i+1} + [(H-\epsilon_{i+1})(H-\epsilon_i)]^{-1/2} Q_{i+1}^\dagger Q_{i+2}^\dagger \} \ , \tag{5.14a}$$

$$L_- = L_+^\dagger \ , \tag{5.14b}$$

$$L_0 = \frac{1}{2} \sum_{i \text{ odd}}^{n-1} (\underline{n}/2 - i)(E_{i,i} + E_{i+1,i+1}) \ , \tag{5.14c}$$

where $\underline{n} = n$ for n even and $\underline{n} = n-1$ for n odd.

This algebra is an SU(2) algebra. The n-dimensional representation can be labeled by the "angular momentum" l associated with this group. The allowed values of l are

$$l = \frac{1}{2}(\underline{n}/2 - 1) \ , \tag{5.15a}$$

and for n odd only, also

$$l = 0, \quad n \text{ odd} \ . \tag{5.15b}$$

For each state in the representation, the eigenvalue of L_0 is m_i given by

$$m_i = \frac{1}{2}(\underline{n}/2 - i), \quad i \text{ odd} \ , \tag{5.16a}$$

$$m_i = \frac{1}{2}(\underline{n}/2 - i + 1), \quad i \text{ even} \ , \tag{5.16b}$$

$$m_n = 0, \quad n \text{ odd} \ . \tag{5.16c}$$

The generators are ladder operators which step between the sl(1/1) two-dimensional representations:

$$L_+ \chi_n^{(i)} = [(l - m_i)(l + m_i + 1)]^{1/2} \chi_n^{(i-2)} \ , \tag{5.17a}$$

$$L_- \chi_n^{(i)} = [(l + m_i)(l - m_i + 1)]^{1/2} \chi_n^{(i+2)} \ . \tag{5.17b}$$

These ladder operators combined with those of (5.13) mean that we can reach any state in the representation except for n odd in which case $\chi_n^{(n)}$ is separate. Furthermore,

$$[L_\pm, Q^\dagger] = [L_\pm, Q] = [L_\pm, H] = 0 \ . \tag{5.18}$$

Hence, the group

$$\text{sl}(1/1) \otimes \text{SU}(2) \tag{5.19}$$

is a symmetry group for H which has all the eigenstates in one irreducible representation for n even. For n odd the representation is reducible, such that all eigenstates $i = 1, \ldots, n-1$ are in one irreducible representation and the last state with $i = n$ is scalar.

However, there is an alternative symmetry group which has all the eigenstates in one irreducible representation for each n. Consider the generators

$$J_+ = \sum_{i=1}^{n-1} [i(n-i)/(H - \epsilon_{i-1})]^{1/2} Q_i^\dagger \ , \tag{5.20a}$$

$$J_- = J_+^\dagger \ , \tag{5.20b}$$

$$J_0 = \frac{1}{2} \sum_{i=1}^{n-1} (n + 1 - 2i) E_{ii} \ . \tag{5.20c}$$

These operators form an SU(2) algebra which has all the states, $i = 1, \ldots, n$, in one irreducible representation with the "angular momentum" j given by

$$j = (n-1)/2 \tag{5.21a}$$

and the state labeled μ_i is given by

$$\mu_i = (n + 1 - 2i)/2 \ . \tag{5.21b}$$

The ladder operators are given by

$$J_+ \Psi_n^{(i+1)} = [i(n-1)]^{1/2} \Psi_n^{(i)} \ . \tag{5.22}$$

Furthermore,

$$[J_\pm, H] = [J_0, H] = 0 \ . \tag{5.23}$$

Hence, an alternative symmetry group is

$$\text{U}(1) \otimes \text{SU}(2) \tag{5.24}$$

with U(1) being generated by the Hamiltonian H.

The analysis presented above holds in general for the hierarchy of Hamiltonians generated by supersymmetry and factorization and is not restricted to the Natanzon class of potentials.

VI. CLASSIFICATION OF SHAPE-INVARIANT POTENTIALS

In this section we wish to raise the interesting question of the classification of various solutions to the shape-invariance condition (2.22). This is clearly a very important problem because once such a classification is available, one could hopefully discover new shape-invariant potentials which are solvable by purely algebraic methods. Of course this is very difficult and we have not been able to answer this question in its full generality. However, by making use of some physical constraints on the observed Schrödinger spectrum we are able to classify a fairly large class of solutions to the condition (2.22).

To begin with, we notice that the eigenvalue spectrum of the Schrödinger equation is always such that the nth eigenvalue E_n at large n obeys the constraint[24]

$$1/n^2 \le E_n \le n^2 \,, \tag{6.1}$$

where the upper bound is saturated by the square-well potential and the lower bound is obtained by the Coulomb potential. For shape-invariant potentials it is therefore not unlikely that the structure of E_n be of the form, for large n,

$$E_n \sim \sum_\alpha K_\alpha n^\alpha, \quad -2 \le \alpha \le 2 \,. \tag{6.2}$$

Now for shape-invariant potentials, E_n is given by (2.23):

$$E_n = \sum_{m=1}^{n} C(a_m) \,. \tag{6.3}$$

Hence, it follows that if

$$C(a_m) \sim \sum_\beta m^\beta \tag{6.4}$$

then

$$-3 \le \beta \le 1 \,. \tag{6.5}$$

The real question to understand is how to implement the constraints (6.4) and (6.5)? While we have no rigorous answer to this question, it is easy to see that a fairly general factorizable form of $W(x,a)$ which produces the above m dependence in $C(a_m)$ is given by

$$W(x,a) = [(m+c)g_1(x) + h_1(x)/(m+c) + f_1(x)] + [(n+d)g_2(x) + h_2(x)/(n+d)] + \cdots \,, \tag{6.6}$$

where

$$a = (m,n,\dots), \quad a_1 = (m_1 = m+\alpha; n_1 = n+\beta; \dots) \,. \tag{6.7}$$

Note that the ansatz (6.6) excludes all potentials leading to E_n that contain fractional powers of n, such as the solvable potentials of the class (3.4) which have E_n given by (3.8). We showed that the potentials of class (3.4) are not, in general, shape invariant. We also note that by assuming that the parameters change in the above simple fashion ($m_1 = m+\alpha$, etc.) under reparametrization, we convert the functional differential equation (2.22) into a differential difference equation. This is a start in making the functional differential equation tractable.

On using the above ansatz for W in the shape-invariance condition

$$\begin{aligned} V_+(x,a) &\equiv W^2(x,a) + W'(x,a) \\ &= W^2(x,a_1) - W'(x,a_1) + C(a_1) \\ &\equiv V_-(x,a_1) + C(a_1) \,, \end{aligned} \tag{2.22}$$

one can obtain the conditions to be satisfied by the functions $g_i(x)$, $h_i(x)$, and $f_1(x)$. One condition, of course, is that the only W's admissible are those which give a square-integrable ground-state wave function $\exp[-\int^x W(x')dx']$. We shall now try to classify the different solutions to Eq. (2.22) in terms of the constraints on the functions f_1, g_i, and h_i.

Let us first start with the simplest possibility

$$W(x,a) = (m+c)g_1(x) + h_1(x)/(m+c) + f_1(x) \,. \tag{6.8}$$

Using Eqs. (6.7) and (6.8) in Eq. (2.22) we find the following three solutions.

$$\text{(a)} \quad g_1(x) = h_1(x) = 0, \quad f_1(x) = \tfrac{1}{2}\omega x + b \,. \tag{6.9}$$

In this case $a_1 = a = \omega/2$, $C(a_1) = \omega$ and, hence, $E_n^- = n\omega$. The only potential that belongs to this class is the shifted one-dimensional harmonic-oscillator potential.

$$\text{(b)} \quad h_1(x) = 0 \tag{6.10a}$$

with f_1 and g_1 satisfying the equations

$$\alpha g_1{}^2(x) - g_1'(x) + k_2/2 = 0 \,, \tag{6.10b}$$

$$f_1'(x) - f_1(x)g_1(x) + (k_2/4)(\alpha+2c) - k_1/2 = 0 \,. \tag{6.10c}$$

In this case,

$$C(m) = k_2 m + k_1, \quad k_1, k_2 \text{ const} \tag{6.10d}$$

and, hence, the energy eigenvalue spectrum of H_- is given by

$$E_n^- = \sum_{m=1} C(m) = k_2 n(n+1)/2 + k_1 n, \quad E_0^- = 0 \,. \tag{6.10e}$$

The various solutions to Eqs. (6.10) are as follows.

$$\text{(i)} \quad W = \omega r/2 - (l+1)/r \tag{6.11a}$$

in which case

$$a = l+1, \quad a_1 = l+2, \quad C(a_1) = 2(a_1 - a)\omega \,. \tag{6.11b}$$

This gives rise to the three-dimensional harmonic-oscillator potential. Note that even n-dimensional harmonic-oscillator problems with arbitrary centrifugal barriers of the form $\alpha(\alpha+1)/r^2$ belong to this class and hence are shape invariant.

$$\text{(ii)} \quad W = A - Be^{-\alpha x} \tag{6.12a}$$

in which case

$$a=A,\quad a_1=A-\alpha,\quad C(a_1)=a^2-a_1^2 \tag{6.12b}$$

which gives rise to the Morse potential.[1]

(iii) $W=A\tanh(\alpha x)+B\,\mathrm{sech}(\alpha x)$ (6.13a)

in which case

$$a=A,\quad a_1=A-\alpha,\quad C(a_1)=a^2-a_1^2\ . \tag{6.13b}$$

(iv) $W=A\coth(\alpha r)-B\,\mathrm{csch}(\alpha r),\quad B>A$ (6.14a)

in which case

$$a=A,\quad a_1=A-\alpha,\quad C(a_1)=a^2-a_1^2\ . \tag{6.14b}$$

(v) $W=A\cot(\alpha x)-B\csc(\alpha x),\quad A>B$ (6.15a)

in which case

$$a=A,\quad a_1=A+\alpha,\quad C(a_1)=a_1^2-a^2\ . \tag{6.15b}$$

Note that for this potential $0\le\alpha x\le\pi$ and one has to further demand that $\Psi(x=0)=\Psi(x=\pi/\alpha)=0$.

(c) $f_1(x)=0,\quad h_1(x)=k_1,\quad k_1$ const (6.16a)

while $g_1(x)$ satisfies the Ricatti equation

$$\alpha g_1^2(x)-g_1'(x)+k_2/2=0\ . \tag{6.16b}$$

In this case

$$C(m)=k_2m+(\alpha+2c)k_2/2 \\ -k_1^2[1/(m+c+\alpha)^2-1/(m+c)^2] \tag{6.16c}$$

and, hence, the energy eigenvalues of H_- are

$$E_n^-=\tfrac{1}{2}kn(n+1)+\tfrac{1}{2}(\alpha+2c)k_2n \\ +k_1^2[1/(m+c)^2-1/(m+c+n\alpha)^2]\ . \tag{6.16d}$$

The various admissible solutions to (6.16b) lead to the following.

(i) $W=A\tanh(\alpha x)+B/A$ (6.17a)

in which case

$$a=A,\quad a_1=A-\alpha\ , \tag{6.17b}$$

$$C(a_1)=a^2-a_1^2+B^2(1/a^2-1/a_1^2)\ . \tag{6.17c}$$

This W gives rise to the Rosen-Morse potential.[1] In the special case of $B=0$ this reduces to the Poschl-Teller potential.[1]

(ii) $W=-(l+1)/r+e^2/[2(l+1)]$ (6.18a)

in which case

$$a=l+1,\quad a_1=l+2\ , \\ C(a_1)=(e^4/4)(1/a^2-1/a_1^2)\ . \tag{6.18b}$$

This gives rise to the Coulomb potential in three dimensions. It is easily seen that even the n-dimensional Coulomb problem with arbitrary centrifugal barrier of the type $\alpha(\alpha+1)/r^2$ belongs to this type.

(iii) $W=-A\coth(\alpha r)+B/A$ (6.19a)

in which case

$$a=A,\quad a_1+A+\alpha\ , \\ C(a_1)=a^2-a_1^2+B^2(1/a^2-1/a_1^2)\ , \tag{6.19b}$$

which gives rise to the Eckart potential.[1] We believe that this exhausts all acceptable solutions to the shape invariance condition (2.22) when $W(x,a)$ is as given by Eq. (6.8).

(iv) Let us now consider the more complicated case when $W(x,a)$ is given by

$$W(x,a)=(m+c)g_1(x)+(n+d)g_2(x)+f_1(x)\ . \tag{6.20}$$

When both $g_1(x)$ and $g_2(x)$ are nonzero one can show by using condition (2.22) that $h_1(x)=h_2(x)=0$ and, hence, we have not included them in (6.20). On using Eqs. (6.7) and (6.20) in the shape-invariance condition (2.22) we find that the functions g_1, g_2, and f_1 satisfy the conditions

$$U^2(x)-U'(x)+(\alpha k_1+\beta k_2)/2=0\ , \tag{6.21a}$$

where

$$U(x)=\alpha g_1(x)+\beta g_2(x) \tag{6.21b}$$

and, furthermore,

$$g_1'(x)-U(x)g_1(x)-k_1/2=0\ , \tag{6.21c}$$

$$g_2'(x)-U(x)g_2(x)-k_2/2=0\ , \tag{6.21d}$$

$$f_1'(x)-f_1(x)U(x) \\ +\{[k_1(\alpha+2c)+k_2(\beta+2d)]/4-k_3/2\}=0\ . \tag{6.21e}$$

In this case, $C(a_1)$ is given by

$$C(a_1)=k_1m+k_2n+k_3\ . \tag{6.21f}$$

Two well-known solutions to Eq. (6.21) are

(i) $f_1(r)=0,\quad W(r)=A\tanh(\alpha r)-B\coth(\alpha r)\ ,$

$A>B>0$ (6.22a)

in which case

$$\{a\}=(A,B),\quad \{a_1\}=(A-\alpha,B+\alpha)\ , \tag{6.22b}$$

$$C(a_1)=(A-B)^2-(A-B-2\alpha)^2\ , \tag{6.22c}$$

and

(ii) $f_1(x)=0,\quad W(x)=A\tan(\alpha x)-B\cot(\alpha x)\ ,$

$A,B>0$ (6.23a)

in which case

$$\{a\}=(A,B),\quad \{a_i\}=(A+\alpha,B+\alpha)\ , \tag{6.23b}$$

$$C(a_1)=(A+B+2\alpha)^2-(A+B)^2\ . \tag{6.23c}$$

It may be noted that, in this case, $0\le\alpha x\le\pi/2$ and $\Psi(x=0)=\Psi(x=\pi/2)=0$.

One can easily generalize (6.20) and consider the general form

$$W(x,a)=f_1(x)+(m+c)g_1(x)+(n+d)g_2(x) \\ +\cdots+(p+e)g_i(x)\ , \tag{6.24a}$$

where

$$\{a\}=(m,n,\ldots,p)\;,$$
$$\{a_1\}=(m_1=m+\alpha,n_1=n+\beta,\ldots,p_1=p+\gamma)\;. \qquad (6.24b)$$

On using this form in the shape-invariance condition (2.22) one can show as in Eq. (6.21) that the combination

$$U=\alpha g_1(x)+\beta g_2(x)+\cdots+\gamma g_i(x) \qquad (6.25a)$$

again satisfies the Riccati equation

$$U^2(x)-U'(x)+(\alpha k_1+\beta k_2+\cdots+\gamma k_i)/2=0 \qquad (6.25b)$$

while $f_1,g_1,g_2,\ldots,g_i$ satisfy equations similar to those given by Eqs. (6.21c)–(6.21e). In this case, $C(a_1)$ is given by

$$C(a_1)=k_1m+k_2n+\cdots+k_ip+K\;. \qquad (6.25c)$$

We have failed to find any solution to these equations. In fact, knowing the general structure of the solutions to the Ricatti equation, it is hard to imagine that a solution exists for these equations.

It must be emphasized here that all of the above discussions concern themselves with functions $W(x,a)$ which are separable, i.e., the variables x and $\{a\}$ are of the form $\sum f_i(a)g_i(x)$. We have nothing to say in the general case when $W(x,a)$ is not factorizable.

VII. DISCUSSIONS

In this paper we have used factorization and SUSY to study the hierarchy of Hamiltonians and wave functions generated from the solvable Hamiltonians of the six-parameter Natanzon class and the restricted two-parameter subclass of Ginocchio potentials. We proved that the Ginocchio class of solvable potentials is not, in general, shape invariant and gave strong evidence that the more general Natanzon class is also not shape invariant, although certain special cases of the Natanzon potentials are shape invariant. It, thus, appears that the solvable potentials of the Natanzon class form a very tiny subclass among the solvable potential family. By using SUSY and factorization we have generated a whole class of new solvable potentials which have a very different form than that given by Natanzon and for which the eigenfunctions are a linear combination of hypergeometric (confluent hypergeometric) functions. We have also shown that, in general, the hierarchy of Hamiltonians along with the supersymmetry operators that generate them form a graded Lie algebra sl(1/1)⊗SU(2). Finally, we have found a scheme which allows us to classify some of the solutions to the shape-invariance conditions [Eq. (2.22)].

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