PHYSICAL REVIEW D VOLUME 36, NUMBER 8 15 OCTOBER 1987

Abelian bosonization in curved space

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(Received 2 February 1987; revised manuscript received 22 June 1987)

We analyze the massive Thirring model and the massive Schwinger model in curved space and show the relationship between these models and the sine-Gordon theory.

I. INTRODUCTION

The study of field theory in a curved space can be viewed as a first step towards understanding a full theory where quantum gravity is available. One of the central questions in this area is how the nontrivial structure of space modifies the already known flat-space results.

In flat space, two-dimensional field theories have been successfully used for testing theoretical ideas such as confinement and asymptotic freedom. One of the properties that showed up is the possibility of transforming Fermi fields into Bose fields.[1] Bosonization can be obtained either by using the operator formalism[1] or by employing the path-integral approach.[2,3]

In this paper, we are concerned with the study of bosonization in two-dimensional curved space. By analyzing the relationship between bosonic and fermionic models, we can obtain useful information about confinement and/or charge screening. Here, we study the bosonization of the massive Thirring model and the massive Schwinger model in a curved Euclidean manifold which is noncompact; i.e., it is infinite. The analysis of the massless Schwinger model in curved space was made in Ref. 4 and it was shown that even in the presence of a gravitational background the massless Schwinger model is equivalent to a system of massive scalars, a fact which was foreshadowed by an analysis[5] of the model at finite temperature, i.e., on $S^1\times\mathbb{R}^1$. However, when we consider the massive Schwinger model in curved space we see that it is equivalent to a massive sine-Gordon model whose interaction term is position dependent. The analysis of the massive Thirring model shows that it also is isomorphic to a position-dependent sine-Gordon theory.

In our analysis we employ an important property of two-dimensional spaces: any metric tensor may always be cast in the form[6]

$$g_{\mu\nu}(x)=\Omega^2(x)\eta_{\mu\nu}\ , \tag{1.1}$$

where $\eta_{\mu\nu}$ is the flat metric tensor $\eta_{\mu\nu}=\mathrm{diag}(1,1)$. The use of the conformal gauge (1.1) allows us to relate the models in curved space with models in flat space.

This paper is organized as follows. In Sec. II we analyze the equivalence between the massive Thirring model and the sine-Gordon theory. Section III contains the analysis of the massive Schwinger model. Our conclusions are presented in Sec. IV.

II. MASSIVE THIRRING MODEL IN CURVED SPACE

The dynamics of the massive Thirring model in curved space[7] is determined by the Lagrangian density

$$\mathcal{L}_T=\sqrt{g}\left[-\frac{i}{2}e_a^\mu(\bar\psi\hat\gamma^a\overleftrightarrow{\partial}_\mu\psi)-\frac{\lambda^2}{2}J_\mu J^\mu+iZm\bar\psi\psi\right]\ , \tag{2.1}$$

where $J^\mu=e_a^\mu\bar\psi\hat\gamma^a(x)\psi$, $g=\det g_{\mu\nu}$, and Z is a cutoff-dependent constant. The zweibein fields (e_a^μ,e_μ^a) are such that

$$g_{\mu\nu}=e_\mu^a e_\nu^b\eta_{ab}$$

and

$$g^{\mu\nu}=e_a^\mu e_b^\nu\eta^{ab}\ . \tag{2.2}$$

For the gauge (1.1) the zweibein are

$$\begin{aligned}&e_a^\mu=\Omega^{-1}\delta_a^\mu,\quad e_\mu^a=\Omega\delta_\mu^a\ ,\\ &e^{\mu a}=\Omega^{-1}\eta^{\mu a},\quad e_{\mu a}=\Omega\eta_{\mu a}\ .\end{aligned} \tag{2.3}$$

On our conventions the flat Dirac matrices are Hermitian and satisfy $\{\hat\gamma^\mu,\hat\gamma^\nu\}=2\eta^{\mu\nu}$ and γ_5 is defined to be $\gamma_5\equiv i\hat\gamma_0\hat\gamma_1$. It is useful to know that $\hat\gamma_\mu\gamma_5=i\hat\epsilon_{\mu\nu}\hat\gamma^\nu$ with $\hat\epsilon^{01}=\hat\epsilon_{01}=-\hat\epsilon_{10}=-\hat\epsilon^{10}=+1$.

Our starting point is the generating functional

$$Z_T=N\int[D\bar\psi][D\psi]\exp\left[-\int d^2x\,\mathcal{L}_T\right]\ . \tag{2.4}$$

In order to reveal the relationship between this model and one in flat space we have to perform a change of the fermionic variables.[8] By defining $\psi=\chi/\sqrt{\Omega}$ and $\bar\psi=\bar\chi/\sqrt{\Omega}$ the generating functional of the system reads

$$Z_T=NJ\int[D\bar\chi][D\chi]\exp\left[-\int d^2x\left[-\frac{i}{2}\bar\chi\hat\gamma_a\overleftrightarrow{\partial}_a\chi-\frac{\lambda^2}{2}(\bar\chi\hat\gamma_a\chi)(\bar\chi\hat\gamma_a\chi)+i\frac{Zm\sqrt{g}}{\Omega}\bar\chi\chi\right]\right]\ . \tag{2.5}$$

The above change of variables possesses a nontrivial Jacobian

$$J=\exp\left[\frac{1}{8\pi}\int d^2x\sqrt{g}\ R\ \ln\Omega\right],$$

which is related to the conformal anomaly.[9] Since J is independent of the fermionic degrees of freedom we absorb it into the normalization factor N. Had we been studying a full theory, which includes quantum gravity, we would have to keep track of factors such as this since it depends on $g_{\mu\nu}$.

Looking at (2.5), we can recognize Z_T as the generating functional of a massive Thirring model in flat space with a position-dependent mass. So this model is equivalent to a bosonic one, whose action is

$$S_B=\int d^2x\left[\tfrac{1}{2}\eta^{ab}\partial_a\phi\partial_b\phi-\frac{m}{\Omega}\sqrt{g}\cos(\beta\phi)\right], \quad (2.6)$$

where $\beta^2/4\pi=1/(1+\lambda^2/\pi)$. S_B can also be written as

$$S_B=\int d^2x\sqrt{g}\left[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{m}{\Omega}\cos(\beta\phi)\right]. \quad (2.7)$$

Therefore, we can say that the massive Thirring model in curved space is equivalent to the sine-Gordon model when we allow the latter to have a position-dependent interaction. This fact is not surprising. This equivalence in flat space can be traced back to the specific form of the free massless fermionic and bosonic propagators in two dimensions. However, these propagators behave differently under a conformal transformation. Once the most general two-dimensional metric is conformally flat, we have to expect that the bosonization procedure had to be modified.

III. MASSIVE SCHWINGER MODEL IN CURVED SPACE

The massive Schwinger model is quantum electrodynamics of a Dirac particle of mass m and charge e in a two-dimensional space. This model is defined by the Lagrangian density (in order to keep the calculation simple we will work the $\theta=0$ sector so one can integrate by parts without paying attention to surface terms)

$$\mathcal{L}_{\mathrm{QED}_2}=\sqrt{g}\left[-e_a^\mu\bar{\psi}\left[\frac{i}{2}\hat{\gamma}^a\overleftrightarrow{\partial}_\mu+e\hat{\gamma}^aA_\mu\right]\psi+\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+im\bar{\psi}\psi\right], \quad (3.1)$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. The generating functional for this model is

$$Z_{\mathrm{QED}_2}=N\int[DA_\mu][D\bar{\psi}][D\psi]\exp\left[-\int d^2x\,\mathcal{L}_{\mathrm{QED}_2}\right]\times(\text{gauge-fixing term}). \quad (3.2)$$

Working in the Landau gauge $\nabla_\mu\cdot A^\mu=0$, we can write

$$A_\mu=-\frac{1}{e}\sqrt{g}\ \hat{\epsilon}_{\mu\nu}\partial^\nu\sigma, \quad (3.3)$$

and the expression for Z_{QED_2} is then

$$Z_{\mathrm{QED}_2}=N'\int[D\sigma][D\bar{\psi}][D\psi]\exp\left\{-\int d^2x\sqrt{g}\left[-e_a^\mu\bar{\psi}\left[\frac{i}{2}\hat{\gamma}^a\overleftrightarrow{\partial}_\mu-\sqrt{g}\,\hat{\gamma}^a\hat{\epsilon}_{\mu\nu}\partial^\nu\sigma\right]\psi+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+im\bar{\psi}\psi\right]\right\}. \quad (3.4)$$

Performing the change of variables $\bar{\psi}=\bar{\chi}/\sqrt{\Omega}$ and $\psi=\chi/\sqrt{\Omega}$ we obtain

$$Z_{\mathrm{QED}_2}=(NJ)\int[D\sigma]\exp\left[-\int d^2x\frac{\sqrt{g}}{2e^2}\sigma\Box\Box\sigma\right]\int[D\bar{\chi}][D\chi]\exp\left[-\int d^2x\left[-\frac{i}{2}\bar{\chi}\hat{\gamma}_a\overleftrightarrow{\partial}_a\chi+\bar{\chi}B_a\hat{\gamma}_a\chi+\frac{im\sqrt{g}}{\Omega}\bar{\chi}\chi\right]\right], \quad (3.5)$$

where $B_a=\hat{\epsilon}_{ab}\partial_b\sigma$. Now the fermionic part of (3.5) is a model in flat space in the presence of a gauge field $B_a(x)$ and whose mass is position dependent. As has already been shown in Ref. 10, this fermionic system is equivalent to a bosonic one and we can write

$$Z_{\mathrm{QED}_2}=(NJ)\int[D\sigma][D\varphi]\exp\left[-\int d^2x\left[\frac{\sqrt{g}}{2e^2}\sigma\Box\Box\sigma+\frac{1}{2\pi}(\partial_a\varphi)^2+\frac{1}{\pi}\hat{\epsilon}_{ab}\partial_a\varphi B_b-\frac{ecm}{2\pi^{3/2}}\frac{\sqrt{g}}{\Omega}\cos(2\varphi)+\frac{1}{4\pi}\frac{m^2g}{\Omega^2}\right]\right]. \quad (3.6)$$

At this point we absorb all the φ- and σ-independent terms into the normalization N and we make the change of variables $\sigma\to\sigma$ and $\varphi\to\varphi+\sigma$ which yields

$$Z_{\mathrm{QED}_2}=N'\int[D\sigma][D\varphi]\exp\left[-\int d^2x\sqrt{g}\left[\frac{1}{2e^2}\sigma\Box\Box\sigma+\frac{1}{2\pi}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-\frac{ecm}{2\pi^{3/2}\Omega}\cos[2(\varphi+\sigma)]\right]\right]. \quad (3.7)$$

This last expression can be analyzed perturbatively by making an expansion in a power series of $\cos[2(\varphi+\sigma)]$. The propagator $P_F(x)$ used in this expansion for the σ field satisfies

$$\left[\frac{1}{e^2}\Box\Box - \frac{1}{\pi}\Box\right] P_F(x) = \sqrt{g}\,\delta^2(x)\ , \tag{3.8}$$

whose solution is

$$P_F(x) = -\pi \lim_{m\to 0}\left[\Delta_F\left[\frac{e}{\sqrt{\pi}},x\right] - \Delta_F(m,x)\right]\ , \tag{3.9}$$

where $\Delta_F(\mu,x)$ is the propagator of a free scalar of mass μ. From (3.9) we can see that the σ field is equivalent to two scalar fields: one has mass $e/\sqrt{\pi}$ and the other is a massless field quantized with indefinite metric. This last one is a manifestation in the path-integral approach of the zero-mass gauge excitation which appears in the Lowenstein-Swieca solution for the massless Schwinger model.[1] After rescaling the fields by $\sqrt{\pi}$ we obtain

$$Z_{\mathrm{QED}_2} = N' \int [D\eta][D\Sigma][D\varphi]\exp\left[-\int d^2x\sqrt{g}\left[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\Sigma\partial_\nu\Sigma - \tfrac{1}{2}g^{\mu\nu}\partial_\mu\eta\partial_\nu\eta + \tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \frac{e^2}{2\pi}\Sigma^2 - \frac{ecm}{2\pi^{3/2}\Omega}\cos[2\sqrt{\pi}(\Sigma+\eta+\varphi)]\right]\right]\ . \tag{3.10}$$

Therefore, in curved space, the massive Schwinger model is equivalent to a sine-Gordon theory whose interaction is position dependent. By taking $m=0$ we recover the result of Barcelos-Neto and Das in Ref. 4. If we set $\Omega(x)=1$ in the last expression, we reobtain the flat-space result.[11]

IV. CONCLUSIONS

We have shown the equivalence between the massive Thirring model and the sine-Gordon theory in curved space. We have also studied the relationship between the massive Schwinger model and the massive sine-Gordon theory. We saw that the equivalence between those models remain true even in the presence of a classical gravitational background. However, there is a change when we go from flat to curved space. In order for the equivalence to remain true, we must allow the interaction in the sine-Gordon model to be position dependent.

We can understand this fact through the following argument. At each point we can locally erect inertial reference frames where the equivalence holds. However, when we compare two distinct points we have to take into account the change of scales which is built in $g_{\mu\nu}=\Omega^2(x)\eta_{\mu\nu}$. Since the mass of the fermion is a dimensional parameter and considering this change of scales, we are naturally lead to a position-dependent sine-Gordon model.

In the Schwinger model we were able to integrate out the fermions and obtain a bosonic theory. This is a good sign that this model in curved space continues to exhibit screening or confinement of the charges associated to the electromagnetic fields.[12] This should be expected on physical grounds since gravitation is an attractive force so it helps confinement.

ACKNOWLEDGMENTS

I am grateful to Professor R. Jackiw for discussions and a critical reading of the manuscript. Thanks are also due to A. J. da Silva, A. Lima-Santos, and P. Orland for useful conversations. This work was supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under Contract No. DE-AC02-76ER03069, and by Conselho Nacional de Desenvolvimento Científico e Tecnológico.

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