PHYSICAL REVIEW D VOLUME 36, NUMBER 8 15 OCTOBER 1987

Hamiltonian formalism for path-dependent Lagrangians

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(Received 16 December 1986; revised manuscript received 20 May 1987)

A presymplectic structure for path-dependent Lagrangian systems is set up such that, when applied to ordinary Lagrangians, it yields the familiar Legendre transformation. It is then applied to derive a Hamiltonian formalism and the conserved quantities for those predictive invariant systems whose solutions also satisfy a Fokker-type action principle.

I. INTRODUCTION

The standard Lagrangians in analytical mechanics depend on the coordinates and velocities for instantaneous configurations (i.e., for a given value of the evolution parameter t, which we shall refer to as "time," although it is not necessarily time measurable by standard clocks). On the contrary, path-dependent Lagrangians exhibit a functional dependence on the trajectories as a whole. That makes these systems more complex than the standard ones; but, since they permit us to consider interaction terms depending on noninstantaneous configurations of particles, path-dependent Lagrangians are especially useful in describing relativistic systems of particles interacting without an intermediate field.

Path-dependent Lagrangians were first used by Fokker,[1] who proposed an action principle for symmetric electrodynamics—half-retarded plus half-advanced—of two charges without an intermediate field. This is the reason why Lagrangians of this kind are also called "Fokker-type Lagrangians."

The symmetric electrodynamics of Wheeler and Feynman[2] is a generalization of Fokker's to the case of more than two charges. It is also based on a path-dependent Lagrangian. This theory, complemented with the "absorber theory of radiation," permits, in an action-at-a-distance framework, accounting for electromagnetic radiation, radiation damping, and the fact that the observed interaction between charges is purely retarded.

Several other relativistic theories of noninstantaneous action at a distance between particles have been set in terms of Fokker-type Lagrangians.[4] This is the case, for instance, of those interactions that are somehow related with a classical field. An exhaustive account of the usefulness of path-dependent Lagrangians, the description of Poincaré-invariant systems of directly interacting particles, can be found in Ref. 5.

The main advantage of Fokker-type Lagrangians in action-at-a-distance relativistic dynamics lies in the fact that they allow noninstantaneous interactions. These are more general than those met by the Komar-Todorov[6] or the Droz-Vincent[7] approach, which can only be used for interactions being instantaneous in some reference frame.[8]

Nevertheless, the Fokker-type Lagrangian formalism presents important drawbacks. One among them is to pose properly the variational principle from where the equations of motion must be derived. The earliest applications of Fokker action principles either seemed not to be fully aware of the problem[1] or solved it in a more or less naive way. A clear statement of the problem and a satisfactory solution can be found in Refs. 9 and 10.

The most important drawback, which is somehow related to the latter, is that the Euler equations, derived from the Fokker action principle, are of functional-differential type (difference-differential equations in the simplest cases). Therefore, the evolution space (space of initial data) is non-Newtonian; that is, the positions and velocities in a given instant of time do not determine uniquely the future evolution of the system. Furthermore, the evolution space has not yet been well determined up to now.[11] As a consequence, it has not been possible to generalize an algorithm as a Legendre transformation to the Fokker-type Lagrangians; nor has an equivalent Hamiltonian formalism been set up yet.

Most of the attempts in the literature to construct a Hamiltonian picture for Fokker-type Lagrangian systems seek a formulation on a phase space of Newtonian style. In our opinion, this makes a possible solution to the problem more involved. Therefore, we are going to consider here separately, two issues that most authors have mixed together: (a) the Hamiltonian formalism for Fokker-type Lagrangian systems, in a phase space suitably chosen for functional Euler equations, and (b) the choice, among all the solutions of these functional equations, of a subfamily of the physically significant ones.

The latter issue has been considered and discussed elsewhere.[12–15] The space of physically significant solutions, depending only on a Newtonian set of initial data, can be obtained by picking out, among the solutions of the functional equations, those that become straight lines when the coupling constants go to zero. In this case, predictive relativistic mechanics[16] provides a technique to obtain the accelerations of each particle in terms of the positions and velocities of them all. (The functional-differential Euler equations are taken as boundary conditions to solve Droz-Vincent's equations.[17] In our opinion, these techniques, which have been widely developed elsewhere,[18] completely solve the issue we have denoted (b).

The main difficulty in approaching the solution of (a) comes from the fact that it is not clear at all what the evolution space is. The problem can be somehow simplified by what Kerner[19] called "the differential form of the Fokker-type Lagrangian". It consists in the following:[20] Suppose that we are only interested in those solutions depending analytically on time, and substitute their corresponding Taylor expansions into the path-dependent Lagrangian. The outcome will be a Lagrangian depending on coordinates, velocities, accelerations, and all the time derivatives of coordinates, corresponding to instantaneous configurations of the system. This infinite-order Lagrangian is somehow conceptually simpler than the path-dependent one, because it depends on instantaneous configurations; but, as a counterpart, it depends infinitely on many variables.

The Euler equations of such a system are obtained by generalizing the Euler equations of an order-n Lagrangian[21,22] (by merely putting ∞ instead of n). Marnelius[10] has proven that these equations are exactly the same as one would have obtained by substituting the Taylor expansions into the functional-differential Euler equations.

The Hamiltonian formalism for infinite-order Lagrangians can be approached by generalizing the Ostrogradski transformation,[21,22] and putting ∞ instead of n, as before. In this infinite-order case, however, the configuration space and evolution space are mixed up. As we shall see in Sec. III, since all time derivatives of coordinates occur in the Euler equations, it is not possible to determine the derivatives of a certain (finite) order in terms of the lower-order ones. Hence, all time derivatives of coordinates at a given instant of time must be given to determine the future evolution of the system. But, once these infinitely many initial data have been given, the Euler equations are no longer necessary to provide supplementary information about the future evolution. Indeed, the latter is fully contained in the initial data, and can be obtained as a power series by merely substituting the initial data into the corresponding Taylor expansion.

However, there is another role left to the infinite-order Euler equations. They must be taken as constraints, to be satisfied by the set of infinitely many initial data and to permit the definition of the evolution space as a submanifold of the space of sequences $(q_\alpha^{(n)})n \in \mathbb{N}$ [each sequence gives the coordinates and all their derivatives at a given time, and the superindex (n) means the derivative order].

In the infinite-order case, the evolution space and configuration space coincide, contrary to what happened in finite-order cases. The phase space does, hence, coincide with configuration space also. Indeed, the generalized Ostrogradski transformation yields the canonical momenta as functions of the configuration-space variables, unlike what happened in finite-order cases, where canonical momenta, together with configuration-space coordinates, constituted a set of independent canonical variables.

The latter coincidence is a new problem in comparison with what we are used to in the standard case. A clue to the solution comes from the following interesting property: if one starts from a first-order Lagrangian $L(q,\dot{q},t)$ and performs and Ostrogradski transformation, as though the Lagrangian was order n, the transformation then turns out to be singular and some constraints occur. Then, it can be easily proven that the canonical formalism, which one finally obtains after eliminating the spurious variables by means of these constraints, and using Dirac brackets,[23] is exactly the same as would have been obtained by performing a standard Legendre transformation.

In this case, some among the equations defining the Ostrogradski transformation act as constraints, and something similar happens in the infinite-order case. Starting from an infinite-order Lagrangian, whose configuration space we denote by E_∞, we define a Hamiltonian system on the cotangent space T^*E_∞. The symplectic structure is the standard one.[24,25] Then, the generalized Ostrogradski transformation, corresponding to the given Lagrangian system, enables us to define a set of constraints, or, taking it in an alternative fashion, an immersion of E_∞ into T^*E_∞, whose rank is a submanifold $\Gamma_\infty \subset T(E_\infty)$, which will be the phase space of our system. The symplectic structure on T^*E_∞ can then be pulled back, thus yielding a presymplectic structure on Γ_∞. All this is displayed in detail in Sec. III.

The differential form of Fokker-type Lagrangians, i.e., infinite-order ones, presents some unpleasant features. Among the most significant ones is the fact that the formalism uses an infinite series without a clear prescription of how to sum them. Furthermore, the assumption of analytical time dependence of trajectories seems to be too strong for physical purposes. However, this must not bother us, because the results obtained in this formalism are merely an intermediate step to the proper results which are derived by dealing directly with Fokker-type Lagrangians (Sec. IV). This implies, of course, a lot of guess work, and the results in Sec. III act as a guidance to it. Thus, the "initial data" already contain all the information about future evolution. While in the discrete case (Sec. III) they consisted in all derivatives, in Sec. IV are the whole trajectories themselves, $q_a(\xi_a)$, $a=1,\ldots,m$; $\xi_a \in \mathbb{R}$. The functional-differential Euler equations also act as constraints on the initial data. Following this analogy, the results obtained in Sec. III are reformulated in a path-dependent formalism by changing discrete indices into continuous ones, and series into integrals. Although this paper has been written in a rather constructive tone, and little attention has been paid to mathematical accuracy, it seems that the lack of rigor in the results in Sec. III could be circumvented in Sec. IV by means of spaces of generalized functions and related methods.[26,27]

In Sec. V, the methods developed in Sec. IV are applied to a standard first-order non-singular Lagrangian, which can be written in a Fokker-type fashion by multiplying it by some δ functions. After applying the methods developed in this paper, the evolution space and the phase space obtained are the same as one would have obtained by the standard procedure. This acts as a proof that the method proposed here is indeed a generalization of the standard Legendre transformation.

Section VI is devoted to predictive relativistic systems related to a Fokker-type Lagrangian (e.g., see Ref. 28 for symmetric electrodynamics). As has been already noted, predictive relativistic mechanics provides an algorithm to obtain Newtonian reductions of order[29] of the functional-differential Euler equations (that is a second-order ordinary differential system, whose solutions also satisfy the Euler equations). The phase space for this predictive relativistic system is a Newtonian one; i.e., it has eight times as many dimension as number of particles. By immersing the Newtonian phase space into the Fokker one, and by the corresponding pull back map, the symplectic structure on the latter is taken to the phase space of the predictive system.

II. FOKKER-TYPE AND INFINITE-ORDER LAGRANGIANS

The Fokker-type actions we are going to consider here have the generic form[10]

$$S = S_0 + S_I \ , \tag{2.1}$$

with

$$S_0 = -\sum_{a=1}^{N} m_a \int (-\dot{x}_a^{\,2})^{1/2} dt_a \tag{2.2}$$

and

$$S_I = -\tfrac{1}{2} \sum_{a,b=1}^{N} g_a g_b \int_{\mathbb{R}} dt'' \, dt' \, w_{ab}^{r}(t'',t') \ , \tag{2.3}$$

where g_a, $a = 1, \ldots, N$, are the coupling constants and

$$w_{ab}^{r}(t'',t') = G_{ab}([x_a(t'') - x_b(t')]^2)[-\dot{x}_a^{\,\mu}(t'')\dot{x}_{b\mu}(t')]^r \times [-\dot{x}_a^{\,2}(t'')]^{(1-r)/2}[-\dot{x}_b^{\,2}(t')]^{(1-r)/2} \tag{2.4}$$

($a^2 \equiv a^\mu a_\mu$, for any four-vector a^μ; and $r \in \mathbb{N}$ is a characteristic of the specific interaction considered).

Action integrals such as these have been proposed by several authors[1,2,4,9,10] to derive the equations of motion for a relativistic system of directly interacting particles.

Although the interaction term (2.3) only contains two-body contributions, it could be generalized in order to consider three-body, and n-body, interactions also. This would not imply further conceptual difficulties.

According to Marnelius,[10] we refer the individual parameters t'' and t' in the interaction term (2.3), to a common one t, and a relative one ξ as

$$t'' = t - \alpha\xi, \quad t' = t + (1-\alpha)\xi \ . \tag{2.5}$$

This enables us to write the action S in terms of a path-dependent Lagrangian:

$$S = \int_{\mathbb{R}} dt \, L_\alpha([x_a^\mu(t')], t) \ ,$$

where

$$L_\alpha \equiv -\sum_{a=1}^{N} m_a(-\dot{x}_a^{\,2})^{1/2} - \tfrac{1}{2} \sum_{a,b=1}^{N} g_a g_b \int_{\mathbb{R}} d\xi \, w_{ab}^{r}(t-\alpha\xi, t+(1-\alpha)\xi) \ , \tag{2.6}$$

with α an arbitrary real constant.

If the variational principle $\delta S = 0$ is only applied to trajectories $x_a^\mu(t)$ exhibiting an analytical dependence on t, the path-dependent Lagrangian L can be translated into an infinite-order one $L_\alpha^{(\infty)}$; that is, a Lagrangian depending on positions, velocities, and accelerations of any order, for a given value of t. In the noncovariant formalism, this Lagrangian looks like[19,20]

$$L_\alpha^{(\infty)} = -\sum_{a=1}^{N} m_a(1-\dot{\mathbf{x}}_a^{\,2})^{1/2} - \tfrac{1}{2} \sum_{a,b=1}^{N} g_a g_b \sum_{s=0}^{\infty} \frac{1}{(2s)!} [(1-\alpha)\hat{D} - \hat{D}_a]^{2s} \times [(1-\dot{\mathbf{x}}_a \cdot \dot{\mathbf{x}}_b)^r (1-\dot{\mathbf{x}}_a^{\,2})^{(1-r)/2}(1-\dot{\mathbf{x}}_b^{\,2})^{(1-r)/2} \Phi_{ab}^{s}(r_{ab})] \ , \tag{2.7}$$

where

$$\hat{D} \equiv \sum_{a=1}^{N} \hat{D}_a, \quad \hat{D}_a \equiv \sum_{s=0}^{\infty} \mathbf{x}_a^{(r+1)} \frac{\partial}{\partial \mathbf{x}_a^{(r)}}, \quad r_{ab} \equiv (\mathbf{x}_a - \mathbf{x}_b) \ , \tag{2.8}$$

$$\Phi_{ab}^{s}(r) \equiv \int_{\mathbb{R}} d\theta \, \theta^{2s} G_{ab}(\theta^2 - r^2) \ , \tag{2.9}$$

and $\mathbf{x}^{(r)}$ is the r derivative of $\mathbf{x}$ with respect to t.

We can then derive the Euler equations for both Lagrangians (2.6) and (2.7). For the first one, these equations follow from applying the Fokker action principle:[9]

$$\frac{d}{dt}\left[m_a \frac{\dot{x}_a^\mu}{(-\dot{x}_a^{\,2})^{1/2}} \right] = g_a \sum_{b \neq a} g_b \int_{\mathbb{R}} dt' \left[\frac{d}{dt} \frac{\partial}{\partial \dot{x}_a^{\,\mu}} w_{ab}^{r}(t,t') - \frac{\partial}{\partial x_a^\mu} w_{ab}^{r}(t,t') \right] . \tag{2.10}$$

For the second one, a generalization of the Euler equations for order-n Lagrangians[21,22] is taken, so obtaining

$$\sum_{r=0}^{\infty} (-\hat{D})^r \frac{\partial L_0^{(\infty)}}{\partial \mathbf{x}_a^{(r)}} = 0, \quad a = 1, \ldots, N \ . \tag{2.11}$$

In Ref. 10, a kind of equivalence of both systems (2.10) and (2.11) is proven. Moreover, none of them depends on the arbitrary real constant α occurring in the Lagrangian.

As is well known, Eq. (2.10) is of functional-differential type (difference-differential in the simplest

cases. Therefore, a finite set of initial data is not enough to uniquely determine a solution of them. The evolution space will then have infinitely many dimensions. Similarly, since all derivatives of $\mathbf{x}_b$, $b=1,\ldots,N$, occur in Eqs. (2.11), they do not permit us to obtain one of these derivatives, say, $\mathbf{x}_b^{(n)}$, in terms of those of lower-order (i.e., $\mathbf{x}_a^{(r)}$, $r<n$). Hence, a finite set of initial data will not suffice to uniquely select a solution, and the evolution space will also be infinite dimensional. Thus, little has been gained with the translation from the Fokker-type formalism into the infinite-order one. The latter will reveal itself very useful as an intermediate step to generalize the Ostrogradski transformation of order n to the path-dependent formalism. (To speak more graphically, discrete infinity is halfway between finite and continuous infinity.)

III. HAMILTONIAN FORMALISM FOR INFINITE-ORDER LAGRANGIANS

The infinite-order Lagrangian formalism stems from generalizing the n-order one, when n is taken to be ∞. As a consequence, new difficulties arise that specifically belong to the infinite-order case. On the one hand, we have an infinite series instead of finite sums and, therefore, as long as a summability criterion is not provided, the value of equations will be merely heuristic. On the other hand, configuration space has infinitely many dimensions and it is identical to evolution space. Furthermore, the generalized Ostrogradski transformation is not invertible.

The Euler equations for an order-n Lagrangian $L(q_a,q_b^{(1)},\ldots,q_i^{(n)})$ are

$$\mathcal{L}_a^{(n)}[L]\equiv\sum_{r=0}^{\infty}(-\hat{D})^r\left[\frac{\partial L}{\partial q_a^{(r)}}\right]=0,\quad a=1,\ldots,m\ , \tag{3.1}$$

where

$$\hat{D}\equiv\frac{d}{dt}\equiv\sum_{k=0}^{\infty}\sum_{a=1}^{m}q_a^{(k+1)}\frac{\partial}{\partial q_a^{(k)}}\ . \tag{3.2}$$

The configuration space C_n is labeled by the coordinates $(q_a,q_b^{(1)},\ldots,q_c^{(n-1)})$, and the evolution space E_{2n} by $(q_a,q_b^{(1)},\ldots,q_e^{(2n-1)})$, because (3.1) is an ordinary differential system of order $2n$ with n unknown.

The Hamiltonian formalism for such a Lagrangian system is set up by performing the Ostrogradski transformation

$$p_{a,r}=\sum_{k=r+1}^{n}(-\hat{D})^{k-r-1}\left[\frac{\partial L}{\partial q_a^{(k)}}\right]\ , \tag{3.3}$$

which transforms the evolution space into the phase space

$$(q_a,q_b^{(1)},\ldots,q_c^{(n-1)},\ldots,q_e^{(2n-1)})\mapsto(q_a,q_b^{(1)},\ldots,q_c^{(n-1)};p_{a,0},p_{b,1},\ldots,p_{c,n-1})\ .$$

This transformation is locally invertible if, and only if, the Hessian matrix $(\partial^2L/\partial q_a^{(n)}\partial q_b^{(n)})$ is regular.

In such a case, the differential system (3.1) is equivalent to the Hamilton equations derived from the Hamiltonian function

$$H=\sum_{k=0}^{n-1}\sum_{a=1}^{m}p_{a,k}q_a^{(k+1)}-L(q,q^{(1)},\ldots,q^{(n)}(q,q^{(1)},\ldots,q^{(n-1)};p_{n-1}))\ , \tag{3.4}$$

with the elementary Poisson brackets

$$\begin{aligned}&\{q_a^{(r)},p_{b,s}\}=\delta_{ab}\delta_s^r\ ,\\ &\{q_a^{(r)},q_b^{(s)}\}=\{p_{a,r},p_{b,s}\}=0\ ,\\ &r,s=1,\ldots,n-1,\quad a,b=1,\ldots,m\ .\end{aligned} \tag{3.5}$$

In going to the limit $n\mapsto\infty$, the Lagrangian is an infinite-order one, and the Euler equations become

$$\mathcal{L}_a[L]\equiv\sum_{r=0}^{\infty}(-\hat{D})^r\left[\frac{\partial L}{\partial q_a^{(r)}}\right]=0\ , \tag{3.6}$$

so containing the t derivatives of any order, $q_b^{(n)}$, $n\in\mathbb{N}$.

Since there are no highest-order derivatives in (3.6), these equations are not, properly speaking, differential equations. One cannot solve them for the derivatives of a certain order, $q_a^{(s)}$, in terms of the lower-order ones, $q_b^{(j)}$, $j<s$. Therefore, a finite set of initial data will not be enough to determine a unique solution of (3.6). On the other hand, if the infinite set of all derivatives of the coordinates, $(q_a^{(n)})$, $a=1,\ldots,m$, $n\in\mathbb{N}$, is taken as initial data, they may not have satisfied Eq. (3.6).

Our proposal is to take (3.5), not as a set of equations ruling the evolution of the system (as is done in the standard Lagrangian case), but as a set of constraints to be satisfied by the initial data. Thus, the evolution space for the infinite-order Lagrangian system is

$$E_\infty\equiv\{(q_a^{(n)})\in(\mathbb{R}^m)^{\mathbb{N}}\mid\mathcal{L}_b[L]=0,b=1,\ldots,m\}\ .$$

This somehow corresponds to a "static" reading of Eq. (3.6). Indeed, since the infinite-order systems we are interested in (i.e., those related to a Fokker-type Lagrangian) have somehow implicit an assumption of analyticity on the variable t, it follows that the initial data already contain all information about the evolution. Indeed, if $z_0=(q_{0b}^{(n)})\in E_\infty$ are the initial coordinates, and derivatives at $t=0$, then the trajectories are given by

$$q_a(t)=\sum_{n=0}^{\infty}q_{0a}^{(n)}\frac{t^n}{n!}\ . \tag{3.7}$$

The convergence of the series on the right-hand side, in a certain neighborhood of $t=0$ would imply some additional requirements on z_0, that should be taken into account in a proper definition of E_∞. However, owing to the merely heuristic value assigned to the results we derive in this section, we shall here leave aside all these

topological issues.

In going to the Hamiltonian formalism, the generalized Ostrogradski transformation, for $n \mapsto \infty$, yields the momenta

$$p_{a,r} = \sum_{k=r+1}^{\infty} (-\hat{D})^{k-r-1} \left[\frac{\partial L}{\partial q_a^{(k)}} \right] , \tag{3.8}$$

$r \in \mathbb{N}$, $a = 1, \ldots, m$, which depend on the configuration-space variables $(q_b^{(n)})$, $n \in \mathbb{N}$, $b = 1, \ldots, n$. Owing to the infinite order of the system, it is not at all clear how this Ostrogradski transformation must be inverted; i.e., which among the derivatives of q_a must be substituted by the momenta.

At this point the situation is fairly similar to that which is met in performing the Legendre transformation for a singular first-order Lagrangian system.[23] In this case, the momenta do not depend on all velocities, and the transformation is not invertible. Hence, the effective phase space is not the one spanned by the coordinates and momenta, as in the regular case, but a submanifold of it, which is defined by some constraints linking coordinates and momenta. In our case, the momenta depend on the configuration-space variables only. Therefore, Eq. (3.8) provides the primary constraints, and secondary ones are obtained by requiring their stability under evolution.

Thus, performing the generalized Ostrogradski transformation according with this underlying viewpoint, we have that the generalization of the Hamiltonian (3.4) for $n \mapsto \infty$ is

$$H = \sum_{a=1}^{m} \sum_{k=0}^{\infty} p_{a,k} q_a^{(k+1)} - L(q, q^{(1)}, \ldots, q^{(n)}, \ldots) , \tag{3.9}$$

and the elementary Poisson brackets are

$$\begin{aligned} &\{q_a^{(r)}, p_{b,s}\} = \delta_{ab} \delta_s^r , \\ &\{q_a^{(r)}, q_b^{(s)}\} = \{p_{a,r} p_{b,s}\} = 0 . \end{aligned} \tag{3.10}$$

The Hamilton equations that follow from (3.9) and (3.10), in the phase space $T^*[(R^m)^{\mathbb{N}}]$ spanned by the q's and p's are

$$\frac{dq^{(k)}}{dt} \equiv \{q_a^{(k)}, H\} = q_a^{(k+1)} , \tag{3.11a}$$

$$\frac{dp_{a,k}}{dt} \equiv \{p_{a,k}, H\} = -p_{a,k-1} + \frac{\partial L}{\partial q_a^{(k)}} ,$$

$$a = 1, \ldots, m, \quad k \in \mathbb{N} , \tag{3.11b}$$

where, for the sake of a compact notation, it must be understood that $p_{a,-1} \equiv 0$ in (3.11b)

The set of equations (3.11a) can be easily integrated (under and additional assumption of summability of the MacLaurin series), so yielding, for a given set $(q_{0a}^{(n)}, p_{0b,r})$ of initial data,

$$q_a^{(k)}(t) = \sum_{n=k}^{\infty} q_{0a}^{(n)} \frac{t^{n-k}}{(n-k)!} . \tag{3.12}$$

Then, the second set of Eqs. (3.11b) can be integrated also, after substituting (3.12), and by an iterative procedure, starting from $k = 0$.

At this stage, however, this Hamiltonian formalism has little to do with the starting Lagrangian system. Indeed, as yet there is no way of deriving the Euler equations (3.6) in this Hamiltonian framework.

As noted above, Eqs. (3.8), defining the generalized Ostrogradski transformation, must be considered as primary constraints on the q's and p's. The effective phase space Γ_∞ is then defined by the minimal set of constraints such that (a) it contains the primary ones and (b) is stable by the time evolution generated by the Hamilton equations (3.11).

The secondary constraints are therefore obtained by differentiating the primary ones (3.8) with respect to time, according to the Hamilton equations (3.11); the first differentiation yields

$$\mathcal{L}_a[L] = 0, \quad a = 1, \ldots, m , \tag{3.13}$$

only involving the coordinates and their derivatives $q_b^{(r)}$. Further differentiations yield new secondary constraints, involving no momenta, that have been already considered as constraints defining the evolution space E_∞.

Summarizing, and using a little bit more geometric language, the generalized Ostrogradski transformation defines an immersion

$$\begin{aligned} &\psi : E_\infty \mapsto T^*(\mathbb{R}^m)^{\mathbb{N}} , \\ &z = (q_a^{(n)}) \mapsto \psi(z) = (q_a^{(n)}, p_{b,r}) , \end{aligned} \tag{3.14}$$

with $p_{b,r} = p_{b,r}(q, q^{(1)}, \ldots, q^{(s)}, \ldots)$ given by (3.8).

Since the constraints defined by the Euler equations (3.16) and all their time derivatives are satisfied on E_∞, it turns out that the Jacobian map ψ_* transforms the infinitesimal generators of time in E_∞:

$$\hat{D} \equiv \sum_{a=1}^{m} \sum_{k=0}^{\infty} q_a^{(k+1)} \frac{\partial}{\partial q_a^{(k)}} , \tag{3.15}$$

into the Hamiltonian differential system

$$\hat{H} = \sum_{a=i}^{m} \sum_{k=j}^{\infty} \left[\{q_a^{(k)}, H\} \frac{\partial}{\partial q_a^{(k)}} + \{p_{a,k}, H\} \frac{\partial}{\partial p_{a,k}} \right] , \tag{3.16}$$

that is,

$$\psi_*(\hat{D}) = \hat{H} \quad \text{on } \Gamma_\infty \equiv \psi(E_\infty) . \tag{3.17}$$

Indeed, $\psi_*(\hat{D})$ is a vector field tangent to $\Gamma_\infty \subset T^*(\mathbb{R}^m)^{\mathbb{N}}$ whose components in terms of the coordinates $(q_a^{(k)}, p_{b,l})$, $a, b = 1, \ldots, m$; $k, l \in \mathbb{N}$, are

$$\begin{aligned} &\psi_*(\hat{D}) q_a^{(k)} = \hat{D} q_a^{(k)} = q_a^{(k+1)} , \\ &\psi_*(\hat{D}) p_{a,k} = \hat{D}[p_{a,k}(q_b^{(n)})] \\ &\qquad = \hat{D} \sum_{r=k+1}^{\infty} (-\hat{D})^{r-k-1} \left[\frac{\partial L}{\partial q_a^{(r)}} \right] \\ &\qquad = -\sum_{r=k+1}^{\infty} (-\hat{D})^{r-k} \left[\frac{\partial L}{\partial q_a^{(r)}} \right] \\ &\qquad = -p_{a,k-1}(q_b^{(n)}) + \frac{\partial L}{\partial q_a^{(n)}} . \end{aligned}$$

Then, using (3.11), we have

$$\psi_*(\hat{D})q_a^{(k)}=\hat{H}q_a^{(k)},\quad \psi_*(\hat{D})p_{a,k}=\hat{H}p_{a,k}\quad \text{on } \Gamma_\infty\ , \tag{3.18}$$

which proves (3.17).

Therefore, we have a one-to-one map from E_∞ (the Lagrangian space of initial data) into $\Gamma_\infty \subset T^*(\mathbb{R}^m)^{\mathbb{N}}$ (the phase space for the constrained Hamiltonian system) which maps the Lagrangian infinitesimal generator $\hat{D}$ for time evolution into the Hamiltonian one $\hat{H}$. Hence, we are entitled to state the equivalence between the infinite-order Lagrangian system (3.1) and the constrained Hamiltonian system defined by (3.8)–(3.10).

Furthermore, the natural injection of Γ_∞ into $T^*(\mathbb{R}^m)^{\mathbb{N}}$ enables us to pull back the symplectic structure, defined by (3.10) on the latter, to the effective phase space Γ_∞. However, we are not going to develop the consequences of this possibility here.

IV. HAMILTONIAN FORMALISM FOR PATH-DEPENDENT LAGRANGIANS

Throughout this section, Lagrangians of the functional type

$$L_t=\int_{\mathbb{R}^m} d\xi_1\cdots d\xi_m \mathcal{L}(q_a(t+\xi_a),\dot{q}_b(t+\xi_b),\xi_c)\ , \tag{4.1}$$

will be considered. All Lagrangians (2.6) can be cast into this general form, by using as many δ functions as necessary.

As has been commented in Sec. II, the Euler equations, derived from the Fokker action principle,[1,2,9] are of the functional-differential type, and the space of initial data for such a system has remained an open issue up until now.[11]

Our proposal is to deal with this problem from the same static viewpoint as we have done in the previous case with infinite-order Lagrangian systems. There, the initial data, $(q_b^{(k)})$, $b=1,\ldots,m$, $k\in\mathbb{N}$, already contained all the information about the time evolution (3.7) of the system; and the Euler equations acted as constraints on the initial data.

Similarly, a set of initial data will now consist of m curves: $q_a(\xi_a)$, $\xi_a\in\mathbb{N}$, $a=1,\ldots,m$, satisfying the constraints defined by the Euler equations, derived from the Fokker action principle. We shall denote this initial data space by E.

To describe this evolution space in more precise terms, we start from the space E_0 of n-tuples of curves $[q_a(\xi_a)]$, $a=1,\ldots,m$, $\xi_a\in\mathbb{R}$, where a time-evolution operator $\hat{\theta}_t$ is defined, according to

$$\hat{\theta}_t q_a(\xi_a)=q_a(\xi_a+t)\ . \tag{4.2}$$

[Realize the similarity of this evolution law and Eq. (3.7) in the discrete case.]

The infinitesimal generator of time evolution in E_0 is then given by

$$\hat{D}=\sum_{a=1}^{m}\int_{\mathbb{R}} d\xi\,\dot{q}_a(\xi)\frac{\delta}{\delta q_a(\xi)}\ , \tag{4.3}$$

where the functional derivative notation is used. In particular, it is obvious that

$$\hat{D}q_a(\xi)=\dot{q}_a(\xi)\ . \tag{4.4}$$

The constraints defining E as a submanifold of E_0 are then derived from an action principle. We take the functional Lagrangian

$$L([q_a(\xi_a)])=\int_{\mathbb{R}^m}\prod_{a=1}^{m} d\xi_a \mathcal{L}(q_b(\xi_b),\dot{q}_c(\xi_c);\xi_d)\ , \tag{4.5}$$

which, under time evolution, becomes

$$\hat{\Theta}_t L=L_t\ , \tag{4.6}$$

where L_t is given by Eq. (4.1). The action principle then reads $\delta S=0$, where

$$S\equiv\int_{\mathbb{R}} dt\, L_t\ , \tag{4.7}$$

and the variations $\delta q_a(\xi_c)$ are taken so that they have compact support [i.e., there exists $M>0$ such that $|\xi_a|>M\Rightarrow\delta q_a(\xi_a)=0$].

After some calculations, which are displayed in Appendix A, we find that the solutions of $\delta S=0$ are also the solutions of the functional equations

$$\int_{\mathbb{R}} d\xi[f_a(\tau-\xi,\xi)-\partial_\tau g_a(\tau-\xi,\xi)]=0\ ,\qquad a=1,\ldots,m\ , \tag{4.8}$$

being

$$f_a(t,\xi_a)=\int_{\mathbb{R}^{m-1}}\prod_{a\neq b} d\xi_b\frac{\partial\mathcal{L}}{\partial q_a(t+\xi_a)} \tag{4.9a}$$

and

$$g_a(t,\xi_a)=\int_{\mathbb{R}^{m-1}}\prod_{a\neq b} d\xi_b\frac{\partial\mathcal{L}}{\partial\dot{q}_a(t+\xi_a)}\ . \tag{4.9b}$$

It can be easily checked that these equations are equivalent to those obtained in Refs. 9 and 10.

Thus, *the evolution space E is the submanifold of E_0 defined by the functional constraints (4.8) completed with those resulting from requiring their stability by time evolution.*

Let us now go on to the Hamiltonian formalism. By a mere analogy with (3.9), and changing the infinite series into integrals (as is usual in analogies of this kind), we take the Hamiltonian

$$H\equiv\sum_{a=1}^{m}\int_{\mathbb{R}} d\lambda\, p_a(\lambda)\dot{q}_a(\lambda)-L([q_b(\xi_b)])\ , \tag{4.10}$$

defined on a phase space Γ_0 labeled by

$$q_b(\xi_b),\quad p_a(\xi_a),\quad a,b=1,\ldots,m,\quad \xi_b,\xi_a\in\mathbb{R}\ ,$$

where the elementary Poisson brackets are

$$\{q_a(\xi),p_b(\xi')\}=\delta_{ab}\delta(\xi-\xi')\ ,\qquad \{q_a(\xi),q_b(\xi')\}=\{p_a(\xi),p_b(\xi')\}=0\ . \tag{4.11}$$

The Hamilton equations derived from (4.10) and (4.11) are

$$\hat{H}q_a(\xi)=\{q_a(\xi),H\}=\dot{q}_a(\xi)\ , \tag{4.12a}$$

$$\hat{H}p_a(\xi)=\{p_a(\xi),H\}=\dot{p}_a(\xi)+\frac{\delta L}{\delta q_a(\xi)}\ . \tag{4.12b}$$

These equations define the infinitesimal generator $\hat{H}$ for time evolution on Γ_0.

The set of Eqs. (4.12a) are not coupled to the second one (4.12b), and they can therefore be solved separately; then, for a given set of initial data

$$(q_a(\xi_a),p_b(\xi_b))\in\Gamma_0\ ,$$

we arrive at

$$\hat{\Theta}_t q_a(\xi)=q_a(\xi+t)\ , \tag{4.13}$$

which gives the action of the time-evolution operator on the coordinates $q_a(\xi)$.

Then, using (4.5) and (4.9), we can write Eq. (4.12b) as

$$\hat{H}p_a(\lambda)=\dot{p}_a(\lambda)+f_a(0,\lambda)-\partial_\lambda g_a(0,\lambda)\ , \tag{4.14}$$

where ∂_λ means partial derivative with respect to λ.

By substituting (4.13) into $f_a(0,\lambda)$ and $g_a(0,\lambda)$, Eq. (4.14) becomes a nonhomogeneous linear equation on $p_a(\lambda)$ whose integration would yield $\hat{\Theta}_t p_0(\lambda)$. The latter, completed with $\hat{\Theta}_t q_b(\lambda)$ given by (4.13), would yield the integral curves of $\hat{H}$ on Γ.

So far there is no equivalence at all between the Fokker-type Lagrangian system defined by (4.1) and the Hamiltonian system (4.12). The evolution space E cannot be mapped one to one into the phase space Γ_0. (Roughly speaking, the dimension of the latter is "twice as big" as the former.) Similarly as in Sec. III, we are going to introduce some primary constraints between $q_a(\lambda_a)$ and $p_b(\lambda_b)$—now they will be of the functional type, of course—defining a submanifold $\Gamma\subset\Gamma_0$, such that (i) it is isomorphical to E and (ii) it is invariant under $\hat{\Theta}_t$ [i.e., $\hat{\Theta}_t(\Gamma)\subset\Gamma$] or, equivalently, $\forall z\in\Gamma$, $\hat{H}_z$ is tangent to Γ.

In the previous case (Sec. III) the suitable primary constraints were obtained by a straightforward generalization of the Ostrogradski transformation. In the present case, some additional guess work will be necessary, and details are left to Appendix B where, by exploiting the analogy between Fokker-type and infinite-order Lagrangians, we maintain that a good guess is

$$p_a(\lambda)=A_a([q_b(\xi_b)],\lambda)\ , \tag{4.15}$$

with

$$A_a([q_b(\xi_b)],\lambda)=g_a(0,\lambda)+\int_{\mathbb{R}} d\xi[f_a(\lambda-\xi,\xi)-\partial_\lambda q_a(\lambda-\xi,\xi)][Y(\lambda)Y(\xi-\lambda)-Y(-\lambda)Y(\lambda-\xi)]\ ,$$

where $Y(\lambda)$ is the Heaviside step function.

Additional constraints (i.e., the secondary ones) must be added to ensure the stability of Γ under time evolution. In Appendix B, we also arrive at the secondary constraints

$$\int_{\mathbb{R}} d\xi[f_a(\lambda-\xi,\xi)-\partial_\lambda g_a(\lambda-\xi,\xi)]=0\ . \tag{4.8'}$$

The primary constraints (4.15) enable us to map the evolution space E into the phase space Γ_0:

$$\psi:E\to\Gamma\subset\Gamma_0,\quad z\to\psi(z)\ , \tag{4.16}$$

$z=[q_a(\lambda_a)]$ satisfying the functional constraints (4.8) and

$$\psi(z)=(q_a(\lambda_a),A_b([q_c(\lambda_c)],\lambda_b))\ . \tag{4.17}$$

It is obvious that ψ is a one-to-one map from E into $\Gamma=\psi(E)$.

An Appendix B it is also proven that

$$\psi_*(\hat{D}_z)=\hat{H}_{\psi(z)},\quad \forall z\in E\ , \tag{4.18}$$

ψ_* being the Jacobian map associated with ψ.

Then, ψ is a one-to-one map from E, the evolution space for the Lagrangian system, into Γ, the phase space for the constrained Hamiltonian system, which maps the generator of Lagrangian time evolution $\hat{D}$ into the Hamiltonian one $\hat{H}$. This is what entitles us to state the equivalence between the Fokker-type Lagrangian system (4.1) and the constrained Hamiltonian system defined by (4.10) and between the Poisson brackets (4.11) and the primary constraints (4.15).

Moreover, the symplectic form defined on Γ by the Poisson brackets (4.11) can be pulled back onto E by ψ, so endowing it with a presymplectic structure.

V. AN ESPECIALLY INTERESTING PARTICULAR CASE

Standard first-order Lagrangian systems can be considered as particular cases of Fokker-type ones by merely taking a Lagrangian density as

$$\mathcal{L}(q_a(\xi_a),\dot{q}_b(\xi_b))\equiv L(q_a(\xi_a),\dot{q}_b(\xi_b))\prod_{a=1}^{m}\delta(\xi_a)\ . \tag{5.1}$$

Substituting this Lagrangian density into (4.9), we arrive, after a short calculation, at

$$f_a(t,\xi)=\left[\frac{\partial L}{\partial q_a}\right]_{(q_b(t),\dot{q}_c(t))}\delta(\xi) \tag{5.2a}$$

and

$$g_a(t,\xi)=\left[\frac{\partial L}{\partial \dot{q}_a}\right]_{(q_b(t),\dot{q}_c(t))}\delta(\xi)\ , \tag{5.2b}$$

whence the constraints (4.8) read

$$\frac{\partial L}{\partial q_a}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a}=0\ . \tag{5.3}$$

That is, the Euler-Lagrange equations would have been derived in the standard way from $L(q,\dot{q})$.

In the present case, the space E of initial data is the space of curves $[q_a(\xi_a)]$, $a=1,\ldots,m$, satisfying the constraints (5.3). These are, in this special case, an ordinary differential system. If the Hessian matrix $\partial^2 L/\partial\dot{q}_a\partial\dot{q}_b$ is regular, and some additional requirements about continuity and differentiability are met, then, the existence and uniqueness theorems enable us to label E with $2m$ variables: namely, the initial values for the coordinates q_{0a} and velocities $\dot{q}_{0a}$. Indeed, given any set of initial data $(q_{0a},\dot{q}_{0b})$, $a,b=1,\ldots,m$, there is a unique solution $[q_a(\xi_a]$, $a=1,\ldots,m$, such that $q_a(0)=q_{0a}$ and $\dot{q}_b(0)=\dot{q}_{0b}$.

Then, in passing to the Hamiltonian formalism, the phase space Γ must be introduced. Each element of this space is labeled by $(q_a(\xi_a),p_b(\xi'_b))$, $a,b\in\mathbb{N}$. The momenta $p_b(\xi'_n)$ are related to the coordinates $q_a(\xi_a)$ by the constraints (4.15), which in our case, using (5.2), can be set as

$$p_a(\lambda)=\left[\frac{\partial L}{\partial\dot{q}_a}\right]_{(0)}\delta(\lambda)+Y(\lambda)\int_\lambda^\infty d\xi\,\delta(\xi)\left[\left[\frac{\partial L}{\partial q_a}\right]_{(\lambda-\xi)}-\frac{d}{d\lambda}\left[\frac{\partial L}{\partial\dot{q}_a}\right]_{(\lambda-\xi)}\right]$$
$$-Y(-\lambda)\int_{-\infty}^\lambda d\xi\,\delta(\xi)\left[\left[\frac{\partial L}{\partial q_a}\right]_{(\lambda-\xi)}-\frac{d}{d\lambda}\left[\frac{\partial L}{\partial\dot{q}_a}\right]_{(\lambda-\xi)}\right], \tag{5.4}$$

where

$$\left[\frac{\partial L}{\partial q_a}\right]_{(\lambda)}\equiv\left[\frac{\partial L}{\partial q_a}\right]_{(q_b(\lambda),\dot{q}_c(\lambda))}. \tag{5.5}$$

Since the curves $q_a(\xi_a)$, $a=1,\ldots,m$, satisfy the secondary constraints (5.3), Eq. (5.4) becomes

$$p_a(\lambda)=\delta(\lambda)\frac{\partial L(q_{0b},\dot{q}_{0c})}{\partial\dot{q}_{0a}}. \tag{5.6}$$

We shall then define

$$\pi_a(q_0,\dot{q}_0)=\frac{\partial L(q_0,\dot{q}_0)}{\partial\dot{q}_{0a}}. \tag{5.7}$$

If the Hessian matrix is regular, the velocities $\dot{q}_{0b}$ can be obtained from (5.6) in terms of q_{0c} and τ_a. Hence, the latter remain as new elementary variables on E. In order to obtain an expression for the presymplectic structure pulled back from Γ onto E, we write

$$q_{0a}=\int_{\mathbb{R}}d\lambda\,q_a(\lambda)\delta(\lambda),\quad \pi_b=\int_{\mathbb{R}}d\lambda\,p_b(\lambda). \tag{5.8}$$

Calculating their mutual Poisson brackets, according to the elementary ones given by (4.11), we arrive at

$$\{q_{0a},\pi_b\}=\delta_{ab},\quad \{q_{0a},q_{0b}\}=\{\pi_a,\pi_b\}=0, \tag{5.9}$$

Similarly, by substituting (5.8) into (4.10), we obtain the Hamiltonian function

$$H=\sum_{a=1}^m\pi_a\dot{q}_{0a}-L(q_0,\dot{q}_0). \tag{5.10}$$

That is, the constrained Hamiltonian system one would obtain by applying the procedure proposed here to a standard Lagrangian, is equivalent to the standard canonical formalism one would have obtained by the usual Legendre transformation. We can thus say that the method proposed is a generalization of the latter.

VI. PREDICTIVE RELATIVISTIC MECHANICS AS A REDUCTION OF ORDER 2 OF A FOKKER-TYPE LAGRANGIAN SYSTEM

As has been said elsewhere,[10,12–15,19] the functional equations (2.10) have too many solutions besides the physically relevant ones. Consequently, a selection rule must be provided to remove the unphysical solutions. Underlying the formulation of the different selection rule, there has been the search for a phase space with $6N$ dimensions in the noncovariant formalism (respectively, $8N$ in the covariant one) and a set of Newtonian equations of motion.

A commonly accepted selection rule, for systems described by an action integral as (2.1), is that physical solutions must become free motions (uniform and rectilinear) when the coupling constants go to zero. For some interactions described by these action integrals, predictive relativistic mechanics supplies a procedure to obtain a reduction of order 2 (Ref. 29) of the Euler equations, that is, a set of Newtonian equations of motion such that their solution is also a solution of the differential-functional Euler equations (2.10).

Let us now consider a system of N interacting particles, whose action integral is (2.1). The latter can be set in the form (2.12) by taking the "Lagrangian density"

$$\mathcal{L}_\alpha(x_a(\xi_a),\dot{x}_b(\xi_b),\xi_c)=-\sum_{a=1}^N m_a[-\dot{x}_a{}^2(\xi_a)]^{1/2}\delta(\xi_1)\cdots\delta(\xi_N)$$
$$-\tfrac{1}{2}\sum_{a,b=1}^N g_a g_b w^r_{ab}(\xi_a,\xi_b)\prod_{c=a,b}\delta(\xi_c)\delta\left(\frac{\xi_a}{\alpha}-\frac{\xi_b}{1-\alpha}\right)\frac{1}{|\alpha|\,|1-\alpha|}. \tag{6.1}$$

Let us now assume the predictive invariant system

$$\frac{d^2x_a^\mu}{dt^2}=a_a^\mu(x_b^\nu,\dot{x}_c^\rho)\ , \tag{6.2}$$

to be a reduction of order 2 (Ref. 29) for the Euler equations (2.10). That is, any solution of (6.2) is also a solution of the functional system (2.10).

In this case, the space of initial data TM_4^N, of the predictive invariant system, can be mapped into E, the space of initial data of the functional system, according to

$$\phi: TM_4^N \mapsto E,\quad (x,\dot{x}) \mapsto (\phi_a^\mu(\xi_a;x_b^\nu,\dot{x}_c^\rho))\ , \tag{6.3}$$

where $\phi_a^\mu(t,x_b^\nu,x_c^\rho)$ is the solution of (6.2) determined by the initial data $(x_b^\nu,\dot{x}_c^\rho)$.

The product of both mappings, ϕ and ψ, defined by (6.3) and (4.17), respectively, maps TM_4^N into Γ according to

$$j\equiv\psi\circ\phi: TM_4^N\mapsto\Gamma\ ,$$
$$(x,\dot{x})\mapsto(\phi_a^\mu(\xi_a,x,\dot{x}),p_{b\nu}(\xi_b,x,\dot{x}))\ , \tag{6.4}$$

where the functions $p_{b\nu}(\xi_b,x,\dot{x})$ are obtained by substitution of $\phi_a^\mu(\xi_a,x,\dot{x})$ into Eq. (4.15):

$$p_{b\nu}(\xi_b,x,\dot{x})=p_{b\nu}(\xi_b,[\phi_a(\xi_a,x,\dot{x})])\ . \tag{6.5}$$

The symplectic form Ω defined on Γ by the Poisson brackets (4.11) can then be pulled back onto TM_4^N, by means of mapping (6.4). We thus obtain a closed differential two-form

$$\omega=j^*\Omega\in\Lambda^2(TM_4^N)\ , \tag{6.6}$$

which will provide a presymplectic structure on TM_4^N, whose expression in terms of the variables $(x_a^\mu,\dot{x}_b^\nu)$ is

$$\omega={}^1\omega_{\mu\nu}^{ab}dx_a^\mu\wedge dx_b^\nu+{}^2\omega_{\mu\nu}^{ab}dx_a^\mu\wedge d\dot{x}_b^\nu+{}^3\omega_{\mu\nu}^{ab}d\dot{x}_a^\mu\wedge d\dot{x}_b^\nu\ , \tag{6.7}$$

where

$${}^1\omega_{\mu\nu}^{ab}=\sum_{c=1}^N\int_{\mathbb{R}}d\lambda\left[\frac{\partial\phi_c^\rho}{\partial x_a^\mu}\frac{\partial p_{c\rho}}{\partial x_b^\nu}-\frac{\partial\phi_c^\rho}{\partial x_b^\nu}\frac{\partial p_{c\rho}}{\partial x_a^\mu}\right]\ , \tag{6.7a}$$

$${}^2\omega_{\mu\nu}^{ab}=\sum_{c=1}^N\int_{\mathbb{R}}d\lambda\left[\frac{\partial\phi_c^\rho}{\partial x_a^\mu}\frac{\partial p_{c\rho}}{\partial\dot{x}_b^\nu}-\frac{\partial\phi_c^\rho}{\partial\dot{x}_b^\nu}\frac{\partial p_{c\rho}}{\partial x_a^\mu}\right]\ , \tag{6.7b}$$

$${}^3\omega_{\mu\nu}^{ab}=\sum_{c=1}^N\int_{\mathbb{R}}d\lambda\left[\frac{\partial\phi_c^\rho}{\partial\dot{x}_a^\mu}\frac{\partial p_{c\rho}}{\partial\dot{x}_b^\nu}-\frac{\partial\phi_c^\rho}{\partial\dot{x}_b^\nu}\frac{\partial p_{c\rho}}{\partial\dot{x}_a^\mu}\right]\ . \tag{6.7c}$$

Deriving the coefficients of the presymplectic form (6.6) for a predictive invariant system involves a rather lengthy calculation, and coupling-constant expansions are necessary. The actual calculations for specific physically interesting cases as Wheeler-Feynman electrodynamics, or short-range scalar interaction, are left to a forthcoming paper.

VII. POINCARÉ TRANSFORMATIONS AND CONSERVED QUANTITIES

The presymplectic framework is the most suitable to describe the continuous symmetry transformations of the dynamical systems under consideration, and the associated conserved quantities. Here, we are interested only in Fokker-type Lagrangians such as (2.2)–(2.4), that clearly exhibit Poincaré invariance, and their relationship with predictive invariant systems, as presented in the previous section.

The action of the Poincaré group G on the several evaluation and phase spaces involved in our scheme (i.e., TM_4^N, E, and Γ) are, respectively, defined by

$$\forall g=(A^\mu,\Lambda_\nu^\mu)\in G\ ,$$
$$k_g: TM_4^N\mapsto TM_4^N\ ,\quad (x,\dot{x})\mapsto(x',\dot{x}')\ ,\quad \left.\begin{aligned}x'^\mu&=\Lambda^\mu{}_\nu(x^\nu-A^\nu),\\ \dot{x}'^\mu&=\Lambda^\mu{}_\nu\dot{x}^\nu\end{aligned}\right\}\ , \tag{7.1}$$

$$K_g: E\mapsto E\ ,\quad (q_a^\mu(t_a))\mapsto(q_a'^\mu(t_a))\ ,\quad \left.q_a'^\mu(t_a)=\Lambda^\mu{}_\nu[q_a^\nu(t_a)-A^\nu]\right\}\ , \tag{7.2}$$

$$\tilde{K}_g:\Gamma\mapsto\Gamma\ ,\quad (q_a^\mu(t_a),p_{b\nu}(t_b))\mapsto(q_a'^\mu(t_a),p_{b\nu}'(t_b))\ ,\quad \left.\begin{aligned}q_a'^\mu(t_a)&=\Lambda^\mu{}_\nu[q_a'^\nu(t_a)-A^\nu],\\ p_{b\nu}'(t_b)&=\Lambda^\mu{}_\nu p_{b\mu}(t_b)\end{aligned}\right\}\ . \tag{7.3}$$

These actions are compatible with the immersions

$$TM_4^N\overset{\phi}{\mapsto}E\overset{\psi}{\mapsto}\Gamma\ ,$$
$$(x_a,\dot{x}_b)\mapsto(\phi_a^\mu(t_a,x,\dot{x}))\mapsto(\phi_a^\mu(t_a,x,\dot{x}),p_{b\nu}(t_a,x,\dot{x}))\ ,$$

defined by (6.3) and (4.17), respectively. That is, the diagram

$$\begin{array}{ccccc} TM_4^N & \overset{\phi}{\longmapsto} & E & \overset{\psi}{\longmapsto} & \Gamma \\ {\scriptstyle k_g}\downarrow & & \downarrow{\scriptstyle K_g} & & \downarrow{\scriptstyle \tilde{K}_g} \\ TM_4^N & \overset{\phi}{\longmapsto} & E & \overset{\psi}{\longmapsto} & \Gamma \end{array}\ ,$$

is commutative, $\forall g\in G$.

The infinitesimal generators for these actions of Poincaré group are therefore connected by the Jacobian maps ϕ_* and ψ_*. Namely, if the infinitesimal generators on TM_4^N, E, and Γ are, respectively, denoted by X_I,Y_I,Z_I, $I=1,\ldots,10$, we then have $\phi_*(X_I)=Y_I$, $\psi_*(Y_I)=Z_I$, and

$$j_*(X_I)=Z_I,\quad j\equiv\psi\circ\phi\ . \tag{7.4}$$

We can immediately check that the symplectic form Ω on Γ is invariant under the action (7.3) of the Poincaré group on Γ: $\tilde{K}_g^*\Omega=\Omega$, $\forall g\in G$.

In terms of the infinitesimal generators, this is equivalent to

$$\pounds(Z_I)\Omega=0 , \tag{7.5}$$

where $\pounds$ means "Lie derivative."

The Poincaré lemma[24] then guarantees the local existence of 10 generating functions, $\tilde{J}_I$, $I=1,\ldots,10$, on Γ, defined by

$$i(Z_I)\Omega=-d\tilde{J}_I , \tag{7.6}$$

where $i(\ )$ means "inner product."

In the special case we are considering, (i) $\Gamma=T^*E$, (ii) Ω can be defined in terms of a canonical one-form Ξ (T^*E) such that

$$\Omega=-d\Xi , \tag{7.7}$$

and (iii) the action (7.3) is pointlike and preserves Ξ.

Hence, 10 globally defined that generating functions $\tilde{J}_I$ do exist, such that

$$\tilde{J}_I=i(Z_I)\Xi . \tag{7.8}$$

In our particular case, it can be easily shown that the generating functions associated to space translations are

$$\tilde{P}_\mu=\sum_{a=1}^{N}\int_{\mathbb{R}} d\lambda\, p_{b\mu}(\lambda) , \tag{7.9}$$

and those associated to Lorentz transformations are

$$\tilde{J}_{\mu\nu}=\sum_{a=1}^{N}\int_{\mathbb{R}} d\lambda(q_{a\mu}(\lambda)p_{a\nu}(\lambda)-q_{a\nu}(\lambda)p_{a\mu}(\lambda)) \tag{7.10}$$

(that is, total linear momentum and total angular momentum, respectively).

Using together (7.4), (7.5), and (6.6), it easily follows that

$$\pounds(X_I)\omega=0 .$$

That is, the presymplectic form ω, defined on the Newtonian evolution space, is invariant under the action (7.1) of the Poincaré group.

The associated conserved quantities can then be obtained according to

$$\begin{aligned} d(J_I)&=i(X_I)(j^*\Omega)=j^*(i(j_*X_I)\Omega)\\ &=j^*(i(Z_I)\Omega)\\ &=-j^*(d\tilde{J}_I)=-d(\tilde{J}_I\circ j) , \end{aligned}$$

where well-known results from differential geometry have been used.

Moreover, if (7.8) is taken into account, the conserved quantities can be written as

$$J_I=i(X_J)(j^*\Xi) , \tag{7.11}$$

that, according to (7.9) and (7.10), and taking (6.4) and (6.5) into account, yields

$$P_\mu=\sum_{a=1}^{N}\int_{\mathbb{R}} d\lambda\, p_{b\mu}(\lambda,x,\dot{x}) \tag{7.12}$$

and

$$J_{\mu\nu}=\sum_{a=1}^{N}\int_{\mathbb{R}} d\lambda(\phi_{a\mu}(\lambda,x,\dot{x})P_{a\nu}(\lambda,x,\dot{x}) -\phi_{a\nu}(\lambda,x,\dot{x})P_{a\mu}(\lambda,x,\dot{x})) , \tag{7.13}$$

for the total linear and angular momenta, respectively, in terms of the Newtonian initial data.

The derivation of specific results for physically interesting interactions is also left to a forthcoming paper.

VIII. CONCLUSION AND OUTLOOK

The aim of this paper has been to obtain a Hamiltonian formalism for those predictive invariant systems related with a Fokker-type Lagrangian system (FLS).[1,2,9] In all those cases the predictive invariant system is a reduction of order 2 for the FLS,[29] that is, the former is a second-order ordinary differential system whose solutions are also solutions of the latter. Since the initial data space for the predictive invariant system can be easily mapped into the initial data space of the FLS, then whatever symplectic (or presymplectic) structure we have on the latter, can be pulled back onto the former, so yielding the result we are interested in.

In this paper a Hamiltonian formalism for FLS has been developed. The main problem has been the definition of the initial data space. We have solved it by changing the point of view that is usual in classical mechanics. Since the Euler equations derived from a path-dependent Lagrangian are of functional differential type, the initial data space for a FLS has infinitely many dimensions. In our case, a whole trajectory of the system has been taken as "initial datum." In doing this, the Euler equations do not rule the evolution anymore (all information about it is already contained in the initial datum), and they must be merely considered as constraints on the initial data.

This approach corresponds to a static point of view. The situation is similar to what happens in dealing with a static standard Lagrangian (i.e., one depending on coordinates only): the initial data (q_{0a}), $a=1,\ldots,m$, can in principle, be picked out from an m-dimensional continuum, but the physically significant ones are only those satisfying

$$\left[\frac{\partial L}{\partial q_a}\right]_{(q_{01},\ldots,q_{0m})}=0 .$$

Similarly, in our approach to path-dependent Lagrangian systems, one can in principle, take any m-tuple of curves $(q_1(t_1),\ldots,q_m(t_m))$ as initial data, but these will be physically significant only if we have been lucky enough as to have chosen a set of curves satisfying the Euler functional-differential equations, now considered as constraints.

As an intermediate step in solving our problem, we have considered the infinite-order Lagrangian associated to the given path-dependent Lagrangian. The conceptual difficulties arising in this case are exactly the same: the initial data and configuration spaces do coincide, and Euler equations are to be considered as constraints on

the configuration space. However, put in this way, the problem looks simpler. Indeed, although the initial data space has also infinitely many dimensions, dealing with discrete infinity is more intuitive than dealing with continuous infinity. Moreover, the results given by Ostrogradski for order n Lagrangian systems can then be extrapolated for $n \mapsto \infty$.

After that, we have turned back onto path-dependent Lagrangian systems and, by way of analogy with the previous case, have set up a Hamiltonian formalism for them.

The outcome has been (i) a phase space Γ, (ii) a Hamiltonian H (4.10), (iii) the elementary Poisson brackets (4.11), and (iv) the immersion (4.17) of the initial data space E into Γ. This immersion is stable by the "time evolution" generated by H on Γ.

The standard case of first-order Lagrangian can also be put as a path-dependent Lagrangian (by adding as many δ functions as necessary). Working it out in our framework we have obtained that (i) the Euler equations taken as constraints tell us that the initial data space has $2m$ dimensions (m being the number of degrees of freedom) and (ii) the phase space is the same as one would have obtained by the standard method.

We have not studied here how the Hamiltonian formalism with constraints does actually work on E, or whether E is a second-class submanifold, or the specific expressions for Dirac brackets. We are not actually interested in the dynamics on E, but, as we have commented at the beginning of this section on some reductions of order 2, those represented by predictive invariant systems. Their initial data space TM_4^N is then endowed with a symplectic (presymplectic, as far as it is proven here) structure, by a mere pull back.

We have also studied the case when the Lagrangian density is Poincaré invariant, and the conserved quantities (namely, linear and angular momenta) have been obtained.

We have left to a forthcoming paper the specific calculation of the symplectic form, the canonical coordinates, and the conserved quantities on TM_4 (the initial data space of the predictive invariant system) for some physically relevant interactions.

Another task left aside has been the accurate mathematical formulation of the ideas presented here. As it can be easily noticed, the action S is defined as an integral over the whole history of the system. Put in this way, S is infinite and therefore, it is not well defined. Nevertheless, since the equation $\delta S=0$, (4.8), and the momenta $p_a(t_a)$, (4.15), are altogether finite and well defined for those physically interesting Lagrangian densities, as will be seen in our future paper, it seems plausible that there must be a more accurate formulation based on the same ideas, leading to the same results, and avoiding the above-mentioned ambiguities.

ACKNOWLEDGMENTS

This work has been partially supported by Comisión Asesora de Investigación Científica y Técnica under Contract No. 0649-84. One of us (R.J.) acknowledges the hospitality from the Universitat de Barcelona as well as the partial support of Consejo Nacional de Ciéncia y Tecnológia (México) and Ministerio de Educacion y Ciencia, Instituto de Cooperación Iberoamericana (España).

APPENDIX A: EXTREMAL CURVES FOR A FOKKER-ACTION PRINCIPLE

Let us consider the action integral

$$S=\int_{\mathbf{R}} dt \int_{\mathbf{R}^m} \prod_{a=1}^{m} d\xi_a \, \mathcal{L}(q_b(t+\xi_b),\dot{q}_c(t+\xi_c);\xi_d) \tag{A1}$$

and seek for the curves $q_a(\xi_a)$, $a=1,\ldots,m$, $\xi_a \in \mathbb{R}$, such that

$$\delta S=0 , \tag{A2}$$

when variations $\delta q_a(\xi)=\eta_a(\xi)$, with compact support, are taken.

The variation of (A1) yields

$$\delta S=\int_{\mathbf{R}^{m+1}} dt\, d\xi_1 \cdots d\xi_m \sum_{a=1}^{m} \left[\frac{\partial \mathcal{L}}{\partial q_a(t+\xi_a)} \eta_a(t+\xi_a)+\frac{\partial \mathcal{L}}{\partial \dot{q}_a(t+\xi_a)} \dot{\eta}_a(t+\xi_a) \right] , \tag{A3}$$

where $\eta_a(t)$, $a=1,\ldots,m$ are arbitrary functions. Then, defining

$$f_a(t,\xi_a) \equiv \int_{\mathbf{R}^{m-1}} \prod_{b\neq a} d\xi_b \frac{\partial \mathcal{L}}{\partial q_a(t+\xi_a)} , \tag{4.9a}$$

$$g_a(t,\xi_a) \equiv \int_{\mathbf{R}^{m-1}} \prod_{b\neq a} d\xi_b \frac{\partial \mathcal{L}}{\partial \dot{q}_a(t+\xi_a)} , \tag{4.9b}$$

and substituting them into (A3), we arrive at

$$\delta S=\int_{\mathbf{R}^2} dt\, d\xi \sum_{a=1}^{m} [f_a(t,\xi)\eta_a(t+\xi)+g_a(t,\xi)\dot{\eta}_a(t+\xi)] . \tag{A4}$$

Now, introducing the new variable $\tau=t+\xi$, (A4) becomes

$$\delta S=\int_{\mathbf{R}^2} d\tau\, d\xi \sum_{a=1}^{m} [f_a(\tau-\xi,\xi)\eta_a(\tau)+g_a(\tau-\xi,\xi)\dot{\eta}_a(\tau)] ,$$

which, after integrating by parts, and taking into account that $\eta_a(\tau)$ has compact support, leads to

$$\delta S=\int_{\mathbf{R}} d\tau \sum_{a=1}^{m} \eta_a(\tau) \int_{\mathbf{R}} d\xi [f_a(\tau-\xi,\xi)-\partial_\tau g_a(\tau-\xi,\xi)] .$$

Finally, the requirement $\delta S=0$, for every variation $\eta_a(\tau)$ with compact support, implies

$$\int_{\mathbb{R}} d\xi [f_a(\tau-\xi,\xi)-\partial_\tau g_a(\tau-\xi,\xi)]=0 ,$$
$$a=1,\ldots,m, \quad \tau\in R . \tag{4.8}$$

APPENDIX B: PRIMARY CONSTRAINTS

According to the analogy between the infinite-order systems and the Fokker-type ones, the relationship between the variables $(q_a^{(n)})$, $a=1,\ldots,m$, $n\in\mathbb{N}$, and $[q_a(\lambda_a)]$, $a=1,\ldots,m$; $\lambda_a\in R$ of the respective evolution spaces, is

$$q_a(\lambda)=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}q_a^{(n)} . \tag{B1}$$

Similarly, the relationship between the respective momenta $p_{a,n}$, $a=1,\ldots,m$, $n\in\mathbb{N}$, and $p_a(\lambda)$ must be such that

$$\int_{\mathbb{R}} d\lambda\, p_a(\lambda)\dot{q}_a(\lambda)=\sum_{n=0}^{\infty} p_{a,n}q_a^{(n+1)} . \tag{B2}$$

Using (B1) and taking into account that $(q_a^{(n)})$, $n\in\mathbb{N}$ is an arbitrary sequence, we arrive at

$$p_{a,n}=\frac{1}{n!}\int_{\mathbb{R}} d\lambda\,\lambda^n p_a(\lambda) . \tag{B3}$$

The expression (B2) suggests to regard $p_a(\lambda)$ as a generalized function,[26,27] acting on a certain space of test functions $\phi(\lambda)$, according to

$$\int_{\mathbb{R}} d\lambda\, p_a(\lambda)\phi(\lambda)=\sum_{n=0}^{\infty}\phi^{(n)}(0)\int_{\mathbb{R}} d\lambda\, p_a(\lambda)\lambda^n=\sum_{n=0}^{\infty} p_{a,n}\phi^{(n)}(0) , \tag{B4}$$

where

$$\phi^{(n)}(0)\equiv\left[\frac{d^n\phi}{d\lambda^n}\right]_{\lambda=0} .$$

The analogy between the Fokker-type Lagrangian systems and infinite-order ones, suggests that the $p_{a,n}$ in the right-hand side of (B4) are given by

$$p_{a,n}=\sum_{l=n+1}^{\infty}(-\hat{D})^{l-n-1}\left[\frac{\partial\tilde{L}}{\partial q_a^{(l)}}\right] , \tag{B5}$$

that is, the generalized Ostrogradski transformation, corresponding to the infinite-order Lagrangian $\tilde{L}$ associated to the Fokker-type one (4.5). Following Marnelius,[10] it can be easily shown that

$$\frac{\partial\tilde{L}}{\partial q_a^{(l)}}=\int_{\mathbb{R}^m} d\xi_1\cdots d\xi_m\left[\frac{\partial\mathcal{L}}{\partial q_a(\xi)}\frac{\partial q(\xi)}{\partial q_a^{(l)}}+\frac{\partial\mathcal{L}}{\partial\dot{q}_a(\xi)}\frac{\partial q_a(\xi)}{\partial q_a^{(l)}}\right] ,$$

or, taking (4.9) and (3.7) into account

$$\frac{\partial\tilde{L}}{\partial q_a^{(l)}}=\int_{\mathbb{R}} d\xi\left[f_a(0,\xi)\frac{\xi^l}{l!}+g_a(0,\xi)\frac{\xi^{l-1}}{(l-1)!}\right] . \tag{B6}$$

Then, by substituting (B5) and (B6) into (B4) and commuting integrals and infinite series, we obtain

$$\int_{\mathbb{R}} d\lambda\, p_a(\lambda)\phi(\lambda)=\int d\xi\sum_{n,l=0}^{\infty}(-\hat{D})^l\left[f_a(0,\xi)\frac{\xi^{n+l+1}}{(n+l+1)!}+g_a(0,\xi)\frac{\xi^{n+1}}{(n+l)!}\right]\phi^{(n)}(0) . \tag{B7}$$

To add up the series

$$u(\xi)\equiv\sum_{n,l=0}^{\infty}(-\hat{D})^l f_a(0,\xi)\frac{\xi^{n+l+1}}{(n+l+1)!}\phi^{(n)}(0) \tag{B8a}$$

and

$$v(\xi)\equiv\sum_{n,l=0}^{\infty}(-\hat{D})^l g_a(0,\xi)\frac{\xi^{n+l}}{(n+l)!}\phi^{(n)}(0) , \tag{B8b}$$

we shall take into account that the operator $\hat{D}$ only acts on $f_a(0,\xi)$ and $g_a(0,\xi)$—the only ones depending on $q_b(\lambda)$—according to

$$\hat{D}f_a(t,\xi)=\partial_t f_a(t,\xi), \quad \hat{D}g_a(t,\xi)=\partial_t g_a(t,\xi) , \tag{B9}$$

as it easily follows from (4.9).

Let us now consider the auxiliary series

$$U(t,\xi)\equiv\sum_{l,s=0}^{\infty}\frac{\xi^{l+s+1}}{(l+s+1!)}(-\partial_t)^l h(t)\phi^{(s)}(0) \tag{B10}$$

and

$$V(t,\xi)\equiv\sum_{l,s=0}^{\infty}\frac{\xi^{l+s}}{(l+s)!}(-\partial_t)^l h(t)\phi^{(s)}(0)$$
$$=\partial_\xi U(t,\xi) . \tag{B11}$$

By differentiating (B10) with respect to t, we obtain

$$\partial_t U(t,\xi)=-\partial_\xi U(t,\xi)+h(t)\phi(\xi) , \tag{B12}$$

and $U(t,\xi)$ can be obtained by integrating this linear partial differential equation, with the initial data

$$U(t,0)=0 ,$$

which follows immediately from taking $\xi=0$ in (B10).

So, we finally arrive at

$$U(t,\xi)=\int_0^{\xi} d\lambda\, h(t+\lambda-\xi)\phi(\lambda) , \tag{B13}$$

which, differentiating with respect to ξ and taking (B11) into account, also yields

$$V(t,\xi)=h(t)\phi(\xi)-\int_0^{\xi} d\lambda\,\partial_t h(t-\xi+\lambda)\phi(\lambda) . \tag{B14}$$

Then, realizing that the shape of our series $u(\xi)$ $[v(\xi)]$ is similar to that of $U(0,\xi)$ $[V(0,\xi)]$, taking $f_a(t,\,)$ $[g_a(t,\,)]$ instead of $h(t)$, we obtain

$$u(\xi)=\int_0^{\xi} d\lambda\, f_a(\lambda-\xi,\xi)\phi(\lambda) , \tag{B15}$$

$$v(\xi)=g_a(0,\xi)\phi(\xi)-\int_0^{\xi} d\lambda(\partial_\lambda g_a(\lambda-\xi,\xi))\phi(\lambda) , \tag{B16}$$

which, substituted in (B7) and according to (B8), yield

$$\int_{\mathbb{R}} d\lambda\, p_a(\lambda)\phi(\lambda)=\int_{\mathbb{R}} d\xi \left[g_a(0,\xi)\phi(\xi)+\int_0^{\xi} d\lambda\,\phi(\lambda)[f_a(\lambda-\xi,\xi)-\partial_\lambda g_a(\lambda-\xi,\xi)]\right] . \tag{B17}$$

Finally, ordering the integrations and taking into account that (B17) holds for any $\phi(\lambda)$ in the space of test functions, we have

$$\begin{aligned} p_a(\lambda)&=A_a([q_b(\xi_b)],\lambda)\\ &=g_a(0,\lambda)+\int_{\mathbb{R}} d\xi[f_a(\lambda-\xi,\xi)-\partial_\lambda g_a(\lambda-\xi,\xi)][Y(\lambda)Y(\xi-\lambda)-Y(-\lambda)Y(\lambda-\xi)] . \end{aligned} \tag{4.15'}$$

These are the primary constraints to define the effective phase space.

To prove the stability of these constraints under time evolution, it suffices to show that

$$\psi_*(\hat{D})=\hat{H} . \tag{B18}$$

(That is, ψ_* maps the generator of time evolution on E into the generator of time evolution on Γ.)

From (4.12a) we easily obtain that

$$\hat{H}q_a(\lambda)=\dot{q}_a(\lambda)=\hat{D}q_a(\lambda) . \tag{B19}$$

Then, using (4.14), we can write

$$\hat{H}p_a(\lambda)=\frac{d}{d\lambda}p_a(\lambda)+f_a(0,\lambda)-\partial_\lambda g_a(0,\lambda) .$$

On the constrained phase space Γ, we have

$$\begin{aligned} \frac{d}{d\lambda}p_a(\lambda)=\frac{d}{d\lambda}A_a([q_b(\xi_b)],\lambda)&=\partial_\lambda g_a(0,\lambda)-f_a(0,\lambda)+[\partial_\lambda g_a(\lambda-\xi,\xi)]_{\xi=\lambda}\\ &+\int_{\mathbb{R}} d\xi\partial_\lambda[f_a(\lambda-\xi,\xi)-\partial_\lambda g_a(\lambda-\xi,\xi)][Y(\lambda)Y(\xi-\lambda)-Y(-\lambda)Y(\lambda-\xi)]\\ &+\delta(\lambda)\int_{\mathbb{R}} d\xi[f_a(\lambda-\xi,\xi)-\partial_\lambda g_a(\lambda-\xi,\xi)] . \end{aligned} \tag{B20}$$

Moreover, since $\Gamma=\psi(E)$, and $[q_b(\xi_b)]_{b=1,\ldots,m}\in E$, the conditions (4.8) are met. Using it in (B20) we obtain that

$$\hat{H}p_a(\lambda)=[\partial_\lambda g_a(\lambda-\xi,\xi)]_{\xi=\lambda}+\int_{\mathbb{R}} d\xi\,\partial_\lambda[f_a(\lambda-\xi,\xi)-\partial_\lambda g_a(\lambda-\xi,\xi)][Y(\lambda)Y(\xi-\lambda)-Y(-\lambda)Y(\lambda-\xi)] . \tag{B21}$$

On the other hand, we have

$$\psi_*(\hat{D})p_a(\lambda)\equiv\hat{D}A_a([q_b(\xi_b)],\lambda)=\hat{D}g_a(0,\lambda)+\int_{\mathbb{R}} d\xi\,\hat{D}[f_a(\lambda-\xi,\xi)-\partial_\lambda g_a(\lambda-\xi,\xi)][Y(\lambda)Y(\xi-\lambda)-Y(-\lambda)Y(\lambda-\xi)] ,$$

that, taking (B9) into account, can also be written as

$$\psi_*(\hat{D})p_a(\lambda)\equiv[\partial_t g_a(t,\lambda)]_{t=0}+\int d\xi\,\partial_\lambda[f_a(\lambda-\xi,\xi)-\partial_\lambda g_a(\lambda-\xi,\xi)][Y(\lambda)Y(\xi-\lambda)-Y(-\lambda)Y(\lambda-\xi)] . \tag{B22}$$

Finally, comparing (B21) and (B22), we arrive at $\psi_*(\hat{D})=\hat{H}$ on $\Gamma=\psi(E)$. Q.E.D.

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