Conformal rotation in perturbative gravity

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The classical Euclidean action for general relativity is unbounded below; therefore Euclidean functional integrals weighted by this action are manifestly divergent. However, as a consequence of the positive-energy theorem, physical amplitudes for asymptotically flat spacetimes can indeed be expressed as manifestly convergent Euclidean functional integrals formed in terms of the physical degrees of freedom. From these integrals, we derive expressions for these same physical quantities as Euclidean integrals over the full set of variables for gravity computed as metric perturbations off a flat background. These parametrized Euclidean functional integrals are weighted by manifestly positive actions with rotated conformal factors. They are similar in form to Euclidean functional integrals obtained by the Gibbons-Hawking-Perry prescription of contour rotation.

I. INTRODUCTION

When searching for a quantum theory to describe a physical system, one in general starts by attempting to quantize its classical counterpart. In the case of gravity, such attempts have met with many difficulties of both a conceptual and technical nature. This has led to proposals for alternate quantum theories for gravity, such as string theories,[1] that hold promise for alleviating at least some of these difficulties. However, although Einstein gravity may not provide the correct theory of quantum gravity, it is the correct theory in the low-energy limit. This means that an understanding of its quantum properties is interesting and useful as a qualitative guide to the low-energy limit of proposed quantum theories for gravity. In addition, quantized Einstein gravity is a useful tool in investigation of quantum cosmology and in minisuperspace models, where one is interested in the qualitative behavior of the theory below the Planck scale.

In order to study the quantum mechanics of a theory, one needs to construct quantities such as wave functions describing the possible states of the system and transition amplitudes between these states. There are two parts to this; first one needs to develop formal expressions for these quantities that incorporate the kinematics of the theory. In general, these expressions will contain ultraviolet divergences. Therefore, next one needs to regulate and renormalize these formal expressions in order to get physical answers. In the case of Einstein gravity, the theory calculated perturbatively is nonrenormalizable.[2] However, quantities can still be computed to a given order in perturbation theory by including the appropriate counterterms; the problem is that in order to compute to all orders, an infinite number of different counterterms must be included. However, in order to begin to regulate and renormalize, one must have the results of the first step; formal expressions for the physical quantities.

One productive method of formulating the kinematics of a theory is to express such quantities as functional integrals, an approach that is especially convenient for further formal manipulation. Functional integrals directly implement the sum over histories formulation of quantum mechanics which connects the quantum amplitudes to the classical action. They are especially useful in the case of theories with local invariances because functional integrals for quantum amplitudes can be formulated to manifestly display these invariances. Lorentzian functional integrals,

$$\int d\phi(x)\exp\{iS[\phi(x)]\} , \tag{1.1}$$

give quantities such as transition amplitudes by summing over the appropriate class of field configurations weighted by the classical action $S[\phi]$. Euclidean functional integrals, which involve sums over field configurations weighted by the classical Euclidean action $I[\phi]$,

$$\int d\phi(x)\exp\{-I[\phi(x)]\} , \tag{1.2}$$

express ground-state wave functions or generating functions in a form useful for actual computations.

One proposed approach to quantizing Einstein gravity is to use Euclidean functional integrals to construct the states of the theory.[3–5] In this approach one forms these integrals using the Euclidean action for general relativity. The appropriate action when the induced three-metric h_{ij} is fixed on the boundary is

$$l^2I[g]=-\int_M d^4x\, g^{1/2}R-2\int_{\partial M} d^3x\, h^{1/2}K , \tag{1.3}$$

where $l=(16\pi G)^{1/2}$ is the Planck length in the units $\hbar=c=1$ and K is the extrinsic curvature of the boundary hypersurface ∂M. Immediately, there is a difficulty; unlike the Euclidean actions for more familiar gauge theories such as electromagnetism, that for gravity is not positive definite. This can be seen by writing the metric $g_{\alpha\beta}$ in terms of a metric $\tilde{g}_{\alpha\beta}$ in the conformal equivalence class of $g_{\alpha\beta}$ and a conformal factor Ω:

$$g_{\alpha\beta}=\Omega^2\tilde{g}_{\alpha\beta} . \tag{1.4}$$

This decomposition is fixed by requiring $\tilde{g}_{\alpha\beta}$ to satisfy a coordinate invariant condition of the form

$$R(\tilde{g})=0 \tag{1.5}$$

and fixing boundary conditions on Ω such as $\Omega=1$ on ∂M. In these variables the action (1.3) becomes

$$l^2 I[\tilde{g},\Omega]=-\int_M d^4x\, \tilde{g}^{1/2}[\Omega^2 R(\tilde{g})+6(\nabla\Omega)^2] -2\int_{\partial M} d^3x\, h^{1/2}K \ . \tag{1.6}$$

It is readily apparent that (1.6) will become arbitrarily negative for Ω that vary rapidly enough. Consequently Euclidean functional integrals for gravity of the form (1.2) weighted by (1.6) will be manifestly divergent.[6] This divergence is one appearing in the kinematical formulation of the theory; it is not related to the ultraviolet divergences. It must be taken care of first before one can regulate and renormalize these quantities.

In order to construct convergent Euclidean functional integrals for the kinematics of the theory, Gibbons, Hawking, and Perry proposed an additional formal manipulation called conformal rotation. First change the variables of integration in the functional integral from $g_{\alpha\beta}$ to Ω and $\tilde{g}_{\alpha\beta}$ which satisfy the condition (1.5). Next distort the contour of the Ω integration to complex values. The action (1.6) then becomes positive definite; the integration over the conformal factor becomes manifestly positive and for asymptotically flat spacetimes the positive action theorem[7] guarantees the positivity of the surface term. Consequently, the resulting Euclidean functional integral is then convergent.

Conformal rotation provides a method of forming convergent Euclidean functional integrals for gravity. However it does so by starting with a divergent Euclidean gravitational integral, a quantity that does not really exist, and manipulating it to produce a convergent one. This manipulation is not needed to construct Euclidean functional integrals for more familiar theories with invariances such as electromagnetism and Yang-Mills theories; the classical Euclidean actions of these theories are manifestly positive. Therefore it would be useful to have a more physically based motivation for Euclidean gravitational integrals in their conformally rotated form. In this paper we will provide such motivation. In doing so we will concentrate on the kinematical formulation of the theory; in the rest of the paper we will not discuss regulating and renormalizing these quantities. Instead we will assume that these procedures can be carried out following the standard methods to handle the ultraviolet divergences in Einstein gravity as needed. Therefore, in this paper the term convergence will refer to the formal properties of the Euclidean functional integrals alone. To begin, we will first construct physical quantities as manifestly convergent Euclidean functional integrals from the fundamental formulation of the quantum theory in terms of its physical degrees of freedom. We expect that we can do so for asymptotically flat spacetimes by virtue of the positive-energy theorem.[8,9] Then starting from these integrals in the physical variables we derive convergent parametrized Euclidean functional integrals with rotated conformal factors for the same physical quantities. First we will do so for the theory of linearized gravity, reviewing previous work done in collaboration with Hartle.[10] We will then discuss how to extend this to perturbative gravity, by which we mean Einstein gravity in asymptotically flat spacetimes when the metric configurations are treated as perturbations on a flat background.

Classically, theories with local invariances such as gauge or parametrized theories are usually given in a form in which not all field configurations are physically distinct. In electromagnetism for example, fields A_μ that differ by a gauge transformation, $A_\mu \to A_\mu + \partial_\mu \Lambda$, are physically equivalent. The classical dynamics of such theories is summarized in an action that is a local functional of the field variables and this is also manifestly invariant under both the local invariance and Lorentz transformations. However, the initial data cannot be freely specified for theories with local invariances because it has to be compatible with the invariance for the evolution of the system to be consistent. This means that the full set of variables contains redundant fields as well as the physical degrees of freedom. The physical components are invariant; their initial data can be freely given. The rest are redundant variables needed to display the symmetry of the theory in a local set of fields. In electromagnetism, the physical fields are the transverse components of the vector potential A_i^T; the longitudinal and time components, A_i^L and A_0, are redundant variables which change under gauge transformations. As the action is manifestly invariant, its physical content can be expressed in terms of the physical variables alone. However, in general this form of the action will be nonlocal in the original potentials and will not display all of the invariances of the theory. For example, the action for electromagnetism can be written in terms of A_i^T alone; however this form is neither local nor manifestly Lorentz invariant.

Although a theory with invariances is most elegantly presented classically using redundant variables, its quantum mechanics is really based on the dynamics as given in terms of the physical degrees of freedom. Physical amplitudes can be constructed as sums over histories in the physical degrees of freedom weighted by the physical action. For example, the ground-state wave functional for electromagnetism can be given as a Euclidean functional integral over A_i^T weighted by the action of the theory expressed in terms of A_i^T. When the physical variables can be explicitly solved for, as in electromagnetism, these integrals are similar in form to those of scalar field theory. These integrals for quantum amplitudes contain the correct physical content of the theory; however, they usually do not display all its invariances in the most transparent form. They also typically do not present the theory in a tractable form for calculation when the physical fields cannot be explicitly isolated. Therefore it is useful to have expressions for these same quantities as functional integrals over the full set of variables. Starting with the integrals in terms of the physical degrees of freedom, one inserts additional integrals over the redundant variables to recover expressions displaying the original invariance and locality of the classical theory. These redundant integrals must be added in a way that leaves the value unchanged, so that the resulting parametrized integrals give the same amplitudes as those in the physical variables. This process can be demonstrated explicitly when the physical degrees of freedom can be explicitly isolated; it can be carried through formally if they cannot. This procedure is most familiar for Lorentzian functional integrals; it provides the correct form and measure for Lagrangian integrals in the full set of variables from the Hamiltonian form given in the physical fields.[11,12] It can be carried out for Euclidean functional integrals as well.

Euclidean functional integrals for both linearized gravity and perturbative gravity are convergent when given in the physical variables. This is manifestly so for linearized gravity; it is implied by the positive-energy theorem for asymptotically flat spacetimes in the interacting case. We shall therefore study how to relate these convergent integrals in the physical degrees of freedom to those over the full set of variables in order to gain a more physically based understanding of conformal rotation. We will show that indeed parametrized Euclidean functional integrals weighted by positive actions for physical quantities can be derived from those given in terms of the physical degrees of freedom.

Before we proceed to treat gravity itself, it is useful to introduce some of the basic techniques in a simpler context. In Sec. II we shall use a simple model to illustrate some basic issues in adding redundant variables to Euclidean functional integrals. Next, in Sec. III we shall derive convergent Euclidean functional integrals for the ground state given in terms of the physical variables for both linearized gravity and perturbative gravity. Because the physical variables can be explicitly solved for in the linearized case, the connection between the parametrized Euclidean functional integral and that in the physical variables can be carried out explicitly. In Sec. IV we shall make this connection and derive the conformally rotated Euclidean functional integral for the ground state of linearized gravity. As the physical variables cannot be explicitly solved for in the case of the interacting theory, it is not practical to parametrize the Euclidean functional integral for the ground state directly. In Sec. V we will discuss how to proceed in this case and derive conformally rotated Euclidean functional integrals for gravity when the metric configurations can be treated as interacting perturbations on a flat background.

II. FUNCTIONAL INTEGRALS FOR THEORIES WITH INVARIANCES

We will first discuss quantizing theories with invariances using a simple model which is a generalization of one discussed by Hartle and Kuchař.[13,14] It allows us to outline the basic techniques in the more familiar context of single-particle quantum mechanics. (1) Beginning with the classical theory expressed in its manifestly gauge-invariant form, one first isolates the physical degrees of freedom and expresses the dynamics in terms of them. (2) One next formulates the quantum theory as functional integrals in the physical variables weighted by the appropriate physical action (the one that gives the dynamics in the physical variables). (3) Finally, one adds in integrations over the redundant variables to recover the manifest invariance expressed in the full set of variables. One does this in such a way that the resulting parametrized functional integrals equal those given in the physical variables. The model will be too simple to adequately illustrate all of the issues, but will be a useful conceptual guide when we turn to the case of gravity.

Let us consider a system consisting of n variables $q^a(t)$ which are the physical degrees of freedom and two variables $\phi(t)$ and $\lambda(t)$ which are the redundant variables. The Lagrangian is

$$L = l(q^a,\dot{q}^a) + l^g(\phi,\dot{\phi},\lambda) , \tag{2.1a}$$

where

$$l(q^a,\dot{q}^a) = \tfrac{1}{2}\delta_{ab}\dot{q}^a\dot{q}^b - V(q^a) , \tag{2.1b}$$

$$l^g(\phi,\dot{\phi},\lambda) = \tfrac{1}{2}\mu(\dot{\phi}-\lambda)^2 - \kappa(\dot{\phi}-\lambda)F(q^a) . \tag{2.1c}$$

L is invariant under the transformation

$$\phi(t) = \phi(t) + \Lambda(t) , \tag{2.2a}$$

$$\lambda(t) = \lambda(t) + \dot{\Lambda}(t) , \tag{2.2b}$$

which serves as the analog of a gauge transformation in the model. Because of the invariance, there will be a constraint on the initial data for the model; it has to be consistent with (2.2) so that the system will evolve preserving it. Of course, gauge theories are not usually written in a set of variables in which the physical degrees of freedom are obvious. More typically they are described in a set of fields in which the gauge-invariant components are not already isolated but in which the invariance is manifest. However, the model does display the basic content of a gauge theory, albeit in a very simple form.

A useful way of displaying the constraint on the initial data and showing how it is preserved by the dynamics is to study the system in its Hamiltonian form. In the following we will briefly discuss constrained Hamiltonian dynamics; we refer the reader to the literature[15-17] for a more thorough and elegant presentation.

One first solves for the momenta conjugate to the variables in terms of the velocities using $p_a = \partial L/\partial \dot{q}^a$. Immediately, one finds that this Legendre transformation is singular; λ has no conjugate momenta as it occurs in (2.1a) without time differentiation. This is a primary constraint on the system:

$$p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0 . \tag{2.3}$$

It reflects the fact that λ is not a dynamical variable; it is a Lagrange multiplier. The rest of the momenta can be solved for in terms of the velocities. What this means is that the classical phase space of the system is smaller than what one would naively guess; it consists of the variables q^a and ϕ and their respective conjugate momenta p_a and π. One finds that the Hamiltonian corresponding to (2.1a) is

$$H = h(q^a,p_a) + h^g(\lambda,\pi,\phi) + \lambda\pi , \tag{2.4a}$$

$$h(q^a,p_a)=\tfrac{1}{2}\delta^{ab}p_a p_b+V(q^a)\ , \tag{2.4b}$$

$$h^g(\lambda,\pi,\phi,q^a)=\frac{1}{2\mu}[\pi+\kappa F(q^a)]^2\ . \tag{2.4c}$$

Infinitesimal canonical transformations of the phase-space variables are implemented by taking their Poisson brackets with the generator of the transformation. The brackets are defined by

$$\{A,B\}=\sum_\alpha\left[\frac{\partial A}{\partial p_\alpha}\frac{\partial B}{\partial q^\alpha}-\frac{\partial A}{\partial q^\alpha}\frac{\partial B}{\partial p_\alpha}\right]\ , \tag{2.5a}$$

where α labels the n variables q^a and ϕ. The fundamental brackets between the canonical variables is

$$\{p_\alpha,q^\beta\}=-\delta^\beta_\alpha\ . \tag{2.5b}$$

Of special interest are canonical transformations that give the invariances and dynamics of the system. In particular, Hamilton's equations of motion, $\dot p_\alpha=-\partial H/\partial q^\alpha$ and $\dot q^\alpha=\partial H/\partial p_\alpha$, follow from the brackets of the variables with H, which is the generator of time evolution.

The initial conditions for solving Hamilton's equations are given by a point in the phase space of the system (p_α,q^α). However, this point cannot be freely specified; in order for the system to evolve consistently it must stay in the phase space, so (2.3) must be preserved. Consequently, the equations of motion imply that the momentum conjugate to ϕ vanishes:

$$\{p_\lambda,H\}=\pi=0\ . \tag{2.6}$$

How is this constraint related to the invariance (2.2) of the model? It generates the infinitesimal gauge transformations of the canonical variables. The variable ϕ transforms as in (2.2a):

$$\delta_\Lambda\phi=\{\phi,\pi\}\Lambda=\Lambda\ . \tag{2.7}$$

The variables p_a and q^a are unchanged; they are gauge invariant. Again for consistency, Eq. (2.6), called a secondary or dynamical constraint, must also be preserved in time. It is, because $\{\pi,H\}=0$. Therefore we have found all the constraints needed for consistent evolution of the model.

What (2.6) tells us is that not all regions of phase space are allowed by the dynamics. The allowed region consists of configurations of p_a, q^a, and ϕ with $\pi=0$. When restricted to this region, the Hamiltonian

$$h^P=\tfrac{1}{2}\delta^{ab}p_a p_b+V(q^a)+\frac{\kappa^2}{2\mu}F^2(q^a) \tag{2.8}$$

is a function of p_a and q^a only. The variable ϕ does not enter into the dynamics; it is a redundant variable whose value is arbitrary. Therefore the physical phase space is just (p_a,q^a); the physical content of the theory is described in terms of these variables. However, note that the presence of the interaction terms between the redundant and physical variables in the Lagrangian (2.1) has resulted in an added term of $[\kappa^2/(2\mu)]F^2$ to the potential in (2.8). It is the analog in the model of a nonlocal interaction induced when the redundant fields are eliminated in a self-interacting gauge theory. In summary, this analysis tells us that the physical degrees of freedom are p_a and q^a and their evolution is given by the physical Hamiltonian (2.8).

Having isolated the physical degrees of freedom, we see that the Hamiltonian when written as a function of the full set of variables is not unique.[17] We could, for example, add any function of the constraint to it, $\tilde H=H+g(p_\alpha,q^\alpha)\pi$, and it would still generate the same dynamics. This means that the action in the full set of variables, $S=\int dt\,(p_\alpha\dot q^\alpha-H)$, is also not unique if the requirements leading to its form in the original set of variables are relaxed. There are many alternate actions that will generate the same dynamics as that corresponding to (2.1). The constraints themselves can be similarly generalized; they only have to vanish up to constraints as well. This emphasizes that the physical dynamics is not contained in the canonical form of the full Hamiltonian but only in its value when the constraints are satisfied.

We have introduced a lot of analytical machinery to study the dynamics of a simple model. However, the power of the Hamiltonian formulation of constrained systems is that this kind of analysis can be carried through when the physical degrees of freedom cannot be explicitly isolated. (1) Primary constraints arise in systems with invariances because some variables are Lagrange multipliers. (2) Consistent time evolution then requires secondary constraints on the canonical variables. These constraints generate the gauge transformations of the theory. (3) The variables canonically conjugate to these secondary constraints are arbitrary parameters specified by choice of gauge. The dynamics of the system is independent of this choice as it is gauge invariant. (4) The physical phase space is the subspace of the canonical phase space that is orthogonal to both constraints and their canonically conjugate variables. The physical content of the theory is determined by the Hamiltonian on this space. We shall use the extension of this formalism to field theory to isolate the physical degrees of freedom for gravity in Sec. III.

Having explicitly reduced the model to its physical degrees of freedom we can proceed to construct quantum amplitudes as sums over histories in terms of them. We shall take the states of the model to be labeled by $|q^a,t\rangle$. The transition amplitude or propagator is then

$$\langle q'^a,t'|q^a,t\rangle=\int dp_a dq^a \exp\left[i\int_t^{t'}dt[p_a\dot q^a-h^P(p_a,q^a)]\right]\ . \tag{2.9}$$

The sum is over phase-space paths which begin at q^a at t and end at q'^a at t'. The measure (in which we do not display constant factors) is the canonically invariant Liouville measure on paths in the physical phase space: $dp\,dq/2\pi\hbar$. The action in the exponent is the classical action for the physical theory in Hamiltonian form. The integrand of (2.9) as it stands is purely oscillatory; in order to define the functional integral one needs to make it convergent.[18] This can be done by inserting factors of the form $\exp(-\delta\int dt\,p_a{}^2)$ where δ is a positive real constant for all the variables and taking the limit $\delta=0$

after evaluation. This prescription defines these integrals independent of the phase of the integrand. We shall assume that this procedure is implied when we write Lorentzian functional integrals in the rest of the paper.

Equation (2.9) is a formal expression for the transition amplitude; in order to make it concrete we need to spell out how to compute the sum over paths. There are various different methods for doing this. One standard way is by time slicing;[19,14] the interval $t'-t$ is divided up into N discrete time steps of length ϵ where $N\epsilon=t'-t$. By giving the value of the physical phase-space coordinates $(p_a(i),q^a(i))$ at every time step, paths can be described by straight lines connecting $(p_a(i),q^a(i))$ to $(p_a(i+1),q^a(i+1))$ for all N steps. The boundary conditions fix $q^a(0)=q^a$ and $q^a(N)=q'^a$. The functional integral (2.9) is then evaluated for N steps and the transition amplitude results in the limit as $N\to\infty$ while keeping $N\epsilon=t'-t$:

$$\langle q'^a,t' \mid q^a,t\rangle=\lim_{N\to\infty}\int\frac{dp_a(0)}{(2\pi)^n}\prod_{i=1}^{N-1}\left[\frac{dp_a(i)dq^a(i)}{(2\pi)^n}\right]\exp\{iS^p[p_a(i),q^a(i)]\}\ , \tag{2.10a}$$

$$S^p=\epsilon\sum_{i=0}^{N-1}\left[p_a(i)\left[\frac{q^a(i+1)-q^a(i)}{\epsilon}\right]-h^p(p_a(i),q^a(i))\right]\ . \tag{2.10b}$$

The initial $p_a(0)$ is summed over, but the final $p_a(N)$ is not because (2.10b) is independent of it. Another useful way to implement the integral is to expand the variables in a Fourier decomposition and write it as a product of integrations over mode amplitudes. We shall use this method to evaluate functional integrals for linearized gravity. The precise form of the measure in (2.9) will depend on how the sum over paths is specified; different methods contribute different factors to the measure in the same way that changing coordinates in an ordinary functional integral introduces factors of the Jacobian. From now on we will assume that some such prescription for the path integrals is supplied and concentrate instead on the main issue of constructing functional integrals for physical quantities.

Because the physical Hamiltonian of our model is quadratic in the momenta, the momentum integrals can be carried out explicitly. This results in the familiar Lagrangian form of the functional integral for the transition function

$$\langle q'^a,t' \mid q^a,t\rangle=\int dq^a\exp\left[i\int_t^{t'}dt\, l^p(q^a,\dot q^a)\right]\ , \tag{2.11a}$$

$$l^p(q^a,\dot q^a)=\tfrac{1}{2}\delta_{ab}\dot q^a\dot q^b-V(q^a)-\frac{\kappa^2}{2\mu}F^2(q^a)\ . \tag{2.11b}$$

The transition from (2.9) to (2.11) provides the correct form of the measure from the canonically invariant one of Hamiltonian quantum mechanics. It is especially useful in field theories where the correct form of the Lorentzian measure is not obvious.

Euclidean functional integrals provide expressions for certain states of the system; an important example is the ground-state wave function. The Euclidean functional integral for the ground state can be derived from the Feynman-Kac formula[19] and the transition amplitude (2.9) or (2.11). Given a complete set of eigenstates $\Psi_m(q^a)$ with energy E_m of the physical Hamiltonian, the transition amplitude can be written as

$$\langle q^a,0 \mid q'^a,t\rangle=\sum_m\Psi_m(q^a)\Psi_m^*(q'^a)\exp(iE_mt)\ . \tag{2.12}$$

Next take q'^a to be at a minimum of $V(q)$, which for concreteness we take at $q'^a=0$, and rotate $t\to -i\tau$. If the energy spectrum of the physical Hamiltonian is bounded below then the dominant contribution as $\tau\to-\infty$ will be proportional to the ground state; if E_0 is renormalized to zero then one has

$$\lim_{\tau\to-\infty}\langle q^a,0 \mid 0,-i\tau\rangle\sim\Psi_0(q^a)\Psi_0^*(0) \tag{2.13}$$

as all other terms fall off exponentially. If we carry out the same procedure resulting in the Feynman-Kac formula using the path-integral form of the transition amplitude (2.11) instead of (2.12), the result is a Euclidean functional integral for the ground-state wave function up to a normalization:

$$\Psi_0(q^a)=\mathcal{N}\int dq^a\exp(-i^p[q^a])\ , \tag{2.14}$$

where i^p is the Euclidean action

$$i^p=\int_{-\infty}^{0}d\tau\left[\tfrac{1}{2}\delta_{ab}\dot q^a\dot q^b+V(q^a)+\frac{\kappa^2}{2\mu}F^2(q^a)\right]\ . \tag{2.15}$$

The class of paths summed over in (2.14) is all those matching q^a at $\tau=0$ that go to 0 in the infinite past. $\mathcal{N}$ is the normalizing constant which includes the factors needed to renormalize the ground-state energy to zero. We also have, using (2.9),

$$\Psi_0(q^a)=\mathcal{N}\int dp_a dq^a\exp\left[\int_{-\infty}^{0}d\tau[ip_a\dot q^a-h^p(p_a,q^a)]\right]\ . \tag{2.16}$$

[Note that the momenta are not rotated in passing from (2.9) to (2.14) and a divergent expression would result if they were.] The usual configuration space integral (2.14) can be derived from (2.16) by integrating over the momenta. The functional integral over the phase-space variables for the ground state (2.16) is less familiar but it is useful especially when the physical degrees of freedom cannot be isolated explicitly. One sees that if the Hamiltonian in terms of the physical variables is bounded below, then the Euclidean functional integrals of the theory will be convergent. This will be the case in the model if $V+[\kappa^2/(2\mu)]F^2$ is bounded below.

By adding integrations over the redundant variables, the functional integrals can be expressed as integrals over the full set of extended variables. It is easy to see that there are many different ways to do so, corresponding to different ways of forming identities out of these variables. Therefore, what do we want to achieve by this procedure? As stated at the beginning of this section, one would like to recover the manifest gauge invariance in the full set of variables. If possible, one would like to recover a functional integral weighted by an action that is a local functional of the original variables as well. Both of these goals can be achieved for Lorentzian functional integrals by aiming to recover an integral weighted by the classical action. Whether or not this is the case for Euclidean functional integrals will depend on the properties of the classical Euclidean action for the theory.

For our simple model, the parametrization can be carried out directly because the action we want to recover is quadratic in the redundant variables. In order to add in the redundant variables to the transition amplitude (2.11) we will use the following two integrals. First, suppose $f(x)$ is 0 for a unique value of x. Then one has the identity

$$1=\int dx\left|\frac{df}{dx}\right|\delta[f(x)]\ . \tag{2.17}$$

The second integral is a Gaussian:

$$\exp(-ia^2)=\int_{-\infty}^{\infty}\frac{dx}{\sqrt{i\pi}}\exp(ix^2+2iax)\ . \tag{2.18}$$

It can be verified by completing the square in the exponent. Now consider for a moment the form of the physical Lagrangian; it consists of two parts. One is $l(q^a,\dot{q}^a)$, which is the first of the two terms in the original Lagrangian (2.1). The other, $[\kappa^2/(2\mu)]F^2$, comes from the term coupling F to the redundant variables. This suggests combining integrals of the form (2.17) and (2.18) so that we form this extra term. Specifically, what we want is

$$\exp\left[-i\int_t^{t'}dt\frac{\kappa^2}{2\mu}F^2(q^a)\right]=\int d\phi\, d\lambda \det\left|\frac{\delta\Phi}{\delta\phi}\right|\delta[\Phi(\phi)]\exp\left[i\int_t^{t'}dt\, l^g(\phi,\dot{\phi},\lambda)\right]\ , \tag{2.19}$$

where $\Phi(\phi)$ is a function that vanishes for only one value of ϕ. Equation (2.19) can be verified by doing the integration over λ (it is a Gaussian of the form (2.18) and then doing the integration over ϕ using the δ function. It is manifestly gauge invariant; l^g (2.1c) is manifestly invariant and the determinant in (2.19) is the product of factors needed to make the integral over the δ function unity for any Φ. It is the Faddeev-Popov determinant for the analog of the gauge-fixing condition $\Phi(\phi)=0$ in the model. This formula (2.19) can be constructed explicitly using some method of summing over the paths. For example, using time slicing, one implements the determinant and δ function by using the identity (2.17) on each time slice.

If we now substitute the right-hand side of (2.19) for the left in (2.11) we arrive at the manifestly invariant functional integral for the transition amplitude

$$\langle q'^a,t'|q^a,t\rangle=\int dq^a d\phi\, d\lambda \det\left|\frac{\delta\Phi}{\delta\phi}\right|\delta[\Phi(\phi)]\exp(iS[q^a,\phi,\lambda])\ , \tag{2.20}$$

where S is the manifestly gauge-invariant action made from the sum of l and l^g. Thus we have recovered the familiar Faddeev-Popov prescription for the functional integral for the transition amplitude in a gauge theory.[11,12]

The above analysis is not a very general or powerful way to look at adding redundant variables to a theory. A much more general way is to begin with the Hamiltonian form of the functional integral and add in integrations over the redundant phase-space variables using functional δ functions. It is an especially useful approach when the physical degrees of freedom cannot be isolated explicitly. By exponentiating the δ functions of the constraints with the Lagrange multipliers of the theory and then performing the momentum integrations, one can produce the familiar Faddeev-Popov form of functional integrals for gauge theories.[20,21] We will not discuss these methods further here because we will be doing so in the case of gravity in later sections. However it should be emphasized that the fundamental idea behind all these methods is that the quantum amplitudes are given as functional integrals in the physical variables; integrations over the redundant variables are added to recover the manifest invariance of the theory.

The ideas used to find parametrized Euclidean functional integrals from those in the physical variables are directly analogous to those in the Lorentzian case. However there is an additional restriction on the process; in order for Euclidean functional integrals for physical quantities to be well defined they must be convergent. Therefore, beginning with convergent Euclidean integrals in the physical variables, we add in convergent integrals in the redundant variables to recover well-defined parametrized Euclidean integrals that display the manifest invariance of the theory. Again we may add quantities using δ functions as in (2.17) but (2.18) becomes

$$\exp(-a^2)=\int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}\exp[-(x^2-2iax)]\ . \tag{2.21}$$

Note that the integrand is complex on the right-hand side, but the integral results in a real quantity. As an example, we will consider how to derive a manifestly invariant form of the Euclidean functional integral for the ground state (2.14) using these identities. First of all, the classical form of the manifestly gauge-invariant Euclidean action corresponding to (2.1) is

$$I[q^a,\phi,\lambda]=i+I^g\ , \tag{2.22a}$$

$$i=\int d\tau \tfrac{1}{2}\delta_{ab}\dot{q}^a\dot{q}^b+V(q^a)\ , \tag{2.22b}$$

$$I^g=\int d\tau \tfrac{1}{2}\mu(\dot{\phi}-\lambda)^2-2i\kappa(\dot{\phi}-\lambda)F(q^a)\ . \tag{2.22c}$$

It is obtained from the Lorentzian action by rotating both $t\to -i\tau$ and $\lambda\to i\lambda$; the rotation of λ is required in order to preserve gauge invariance. This Euclidean action is uniquely determined by requiring gauge invariance and the same form as the Lorentzian action in the original set of variables. Note that this prescription has resulted in a complex term. Using (2.22c) as a guide, we can construct the analog of (2.19) for Euclidean functional integrals using (2.17) and (2.21):

$$\exp\left[-\int_{-\infty}^{0}\frac{\kappa^2}{2\mu}F^2(q^a)\right] =\int d\phi\, d\lambda\, \det\left|\frac{\delta\Phi}{\delta\phi}\right|\delta[\Phi(\phi)]\exp(-I^g[\phi,\lambda])\ . \tag{2.23}$$

This identity is true when μ is positive. However when μ is negative the Gaussian integration over the redundant variables becomes manifestly divergent. Therefore (2.23) does not exist for $\mu<0$. In that case we cannot recover a parametrized integral for the ground state weighted by the local Euclidean action (2.22a). This does not mean that Euclidean functional integrals do not exist for the theory; as we have already seen, they do in terms of the physical variables if $V+[\kappa^2/(2\mu)]F^2$ is bounded below. Nor does it mean that a manifestly invariant parametrized integral cannot be found.

Instead of adding redundant variables by using I^g we can form an identity which is a manifestly convergent Gaussian integral when $\mu<0$. It is

$$\exp\left[-\int_{-\infty}^{0}\frac{\kappa^2}{2\mu}F^2(q^a)\right] =\int d\phi\, d\lambda\, \det\left|\frac{\delta\Phi}{\delta\phi}\right|\delta[\Phi(\phi)]\exp(-\hat{I}^g[q^a,\lambda,\phi])\ , \tag{2.24a}$$

$$\hat{I}^g=\int_{-\infty}^{0} d\tau[-\tfrac{1}{2}\mu(\dot{\phi}-\lambda)^2+\kappa(\dot{\phi}-\lambda)F(q^a)]\ . \tag{2.24b}$$

Not only has the sign in front of μ been changed but the interaction term has also been modified from its local Euclidean form (2.22c); it is now real. This is necessary in order to get the sign of the F^2 term to be the same as that in the physical Euclidean action. Using this identity in (2.14) we find a manifestly convergent integral for the ground state:

$$\Psi_0(q^a)=\mathcal{N}\int dq^a d\phi\, d\lambda \exp(-\hat{I}[q^a,\phi,\lambda])\ , \tag{2.25a}$$

$$\hat{I}=i[q^a]+\hat{I}^g[q^a,\lambda,\phi]\ . \tag{2.25b}$$

The action that this integral is weighted by is positive, but is no longer of the same form as the Lorentzian action in the original set of variables, as reflected in the sign of μ and the form of the interaction. This change in the form of the action will correspond in the case of gravity to the Euclidean action being nonlocal in the original set of variables. It could be obtained from (2.22a) formally by performing an additional rotation on the variables $\phi\to i\phi$ and $\lambda\to i\lambda$, the analog of conformal rotation.

The action (2.25b) we obtained is not unique; there are many other positive actions that we could have used instead. For example, we could have constructed the manifestly gauge-invariant identity

$$1=\int d\phi\, d\lambda\, \det\left|\frac{\delta\Phi}{\delta\phi}\right|\delta[\Phi(\phi)] \times\exp\left[-\int_{-\infty}^{0} d\tau[-\tfrac{1}{2}\mu(\dot{\phi}-\lambda)^2]\right]\ . \tag{2.26}$$

Using this identity one could recover a manifestly gauge-invariant ground state of the form (2.24a) weighted instead by the action

$$\tilde{I}=i^p[q^a]+\int_{-\infty}^{0} d\tau[-\tfrac{1}{2}\mu(\dot{\phi}-\lambda)^2]\ , \tag{2.27}$$

where i^p is (2.15), the action in terms of the physical variables.

Because the interaction between the physical and redundant variables is simple in the model, appropriate convergent identities are easy to guess. For a more complicated interacting theory, the appropriate modification of the interaction terms may not be so obvious, as is the case in the full theory of gravity. The fundamental requirement is that it be chosen so that the parametrized integral equal that in terms of the physical degrees of freedom when evaluated. However, the model does demonstrate the basic principle; convergent, manifestly invariant Euclidean functional integrals can be found for the theory that equal those given in terms of the physical variables. The quantum mechanics of a system with redundant variables is really given in terms of the physical degrees of freedom. If the Hamiltonian in terms of the physical variables is bounded below, the quantum theory is well behaved. In that case convergent Euclidean functional integrals for the states of the theory can be formulated in terms of the physical variables. We can also find convergent Euclidean integrals for these states that reflect the invariances of the theory, even if the classical Euclidean action is unbounded below. However, if it is unbounded, then we cannot recover integrals that are weighted by a Euclidean action that is a local functional of the variables. They are weighted by nonlocal actions; moreover the form of the nonlocal action is not unique. Which nonlocal action one wants to use depends on what is most convenient.

III. THE GROUND STATE

Physical amplitudes for gravity such as ground-state wave functionals can be expressed as convergent Euclidean functional integrals in terms of the physical degrees of freedom for asymptotically flat spacetimes. This will be possible because the Hamiltonian is positive definite for such metrics by virtue of the positive-energy theorem. To illustrate this we will construct the Euclidean functional integral that describes the ground state for asymptotically flat metrics in the physical fields. We shall give a precise definition of this integral by treating gravity as interacting metric perturbations on a flat background. This will be the starting point for deriving a convergent parametrized Euclidean functional integral for the same state.

The appropriate action for Einstein's theory of relativity when the induced three-metric h_{ij} is fixed on the boundary is

$$l^2S[g]=\int_M d^4x\, g^{1/2}R(g)+2\int_{\partial M} d^3x\, h^{1/2}K \ , \quad (3.1)$$

where K is the extrinsic curvature of the boundary of the manifold ∂M. It is invariant under general coordinate transformations, $x^\alpha \to \bar{x}^\alpha$, which change the metric $g_{\alpha\beta}(x)\to\bar{g}_{\alpha\beta}(\bar{x})$. The invariance of the action means that some metric components are not physical as the form depends on the coordinate system. Thus, as in the model problem, there will be constraints. In order to display these constraints and isolate the physical degrees of freedom, we will write the theory in Hamiltonian form.[22–24]

It is convenient to begin by dividing the spacetime into a family of spacelike hypersurfaces labeled by t, a function constant on each hypersurface. The four metric is decomposed with respect to these surfaces:

$$ds^2=-(N^2-N_iN^i)dt^2+2N_i dx^i dt+h_{ij}dx^i dx^j \ . \quad (3.2)$$

For the spacetime to be asymptotically flat, the metric components must satisfy certain falloff conditions.[23,22] A sufficient behavior is that in an appropriate set of coordinates the metric components fall off like the Schwarzchild metric at spatial infinity:

$$ds^2\sim-\left[1-\frac{2M}{r}\right]dt^2+\left[\delta_{ij}+\frac{2Mx^ix^j}{r^3}\right]dx^idx^j +O\left[\frac{1}{r^2}\right] . \quad (3.3)$$

Thus, also choosing a coordinate system in which the x^i are asymptotically Euclidean, the conditions on the metric components and their derivatives are

$$\begin{aligned} &g_{ij}-\delta_{ij}\sim 1/r, \quad \partial_k g_{ij}\sim 1/r^2 \ ,\\ &N-1\sim 1/r, \quad \partial_k N\sim 1/r^2 \ ,\\ &N^i\sim 1/r, \quad \partial_k N^i \sim 1/r^2 \ , \end{aligned} \quad (3.4)$$

as r approaches infinity.

The Hamiltonian corresponding to (3.1) is

$$H=\int d^3x\, N^\mu \mathcal{H}_\mu + E \ , \quad (3.5a)$$

$$\mathcal{H}_0=l^2G_{ijkl}\pi^{ij}\pi^{kl}-\frac{1}{l^2}hR(h) \ , \quad (3.5b)$$

$$\mathcal{H}_i=-2D_j\pi_i^j \ , \quad (3.5c)$$

$$E=\frac{1}{l^2}\int d^2s_i(\partial_j h_{ij}-\partial_i h_{jj}) \ , \quad (3.5d)$$

where π^{ij} is the momenta conjugate to h_{ij}, the metric on the constant t hypersurfaces, D_j is the corresponding covariant derivative, and $G_{ijkl}=\frac{1}{2}(h_{ik}h_{jl}+h_{il}h_{jk}-h_{ij}h_{kl})$. Indices in (3.5) are lowered using h_{ij}. E is the surface integral needed in order to get the correct equations of motion for asymptotically flat spacetimes from Hamilton's principle.[25] N^0 and N^i have no conjugate momenta. This is because of the general coordinate invariance of the theory; not all components of the metric are dynamical as metrics that differ by coordinate transformations describe the same spacetime. Consequently N^0 and N^i are Lagrange multipliers enforcing the constraints $\mathcal{H}_0$ and $\mathcal{H}_i$. $\mathcal{H}_0$ is a factor of $h^{1/2}$ different from its usual form; this choice simplifies subsequent calculations. Therefore N^0 also differs from the conventional lapse N as given in (3.2); $N=h^{1/2}N^0$. This results in the Hamiltonian density having an overall weight of one as required by the density of the original action (3.1). The action expressed in terms of the phase-space variables is

$$S=\int_t^{t'} dt\left[\int d^3x\, \pi^{ij}\dot{h}_{ij}-H\right] , \quad (3.6)$$

where t and t' label the boundary hypersurfaces of the manifold. The overdot means the derivative with respect to t.

As in the model problem, the invariances and dynamics of the theory can be conveniently displayed using Poisson brackets. The constraints are the generators of the infinitesimal gauge transformations of the canonical variables:

$$\delta_F h_{ij}(x)=\left\{h_{ij}(x),\int d^3x' F^\mu(x')\mathcal{H}_\mu(x')\right\} , \quad (3.7a)$$

$$\delta_F \pi^{ij}(x)=\left\{\pi^{ij}(x),\int d^3x' F^\mu(x')\mathcal{H}_\mu(x')\right\} , \quad (3.7b)$$

computed using the fundamental brackets

$$\begin{aligned} &\{\pi^{ij}(x),h_{kl}(x')\}=-\delta^{ij}_{kl}(x,x') \ ,\\ &\delta^{ij}_{kl}(x,x')=\tfrac{1}{2}(\delta^i_k\delta^j_l+\delta^i_l\delta^j_k)\delta(x,x') \ . \end{aligned} \quad (3.7c)$$

$\mathcal{H}_i$ generates infinitesimal diffeomorphisms by F^i in the spacelike hypersurface and $\mathcal{H}$ generates the infinitesimal transformations of the variables caused by deformations of the hypersurface by F^0 in the direction of its normal. The algebra of the constraints closes which means the constraints are first class:

$$\{\mathcal{H}_\mu(x),\mathcal{H}_\nu(x')\}=\int d^3x''\, U^\rho_{\mu\nu}(x,x';x'')\mathcal{H}_\rho(x'') \ . \quad (3.8)$$

$U^\rho_{\mu\nu}$ are the structure functions of the theory which depend on the three-metric; the nonvanishing ones are[24]

$$U^i_{00}(x,x';x'') =hh^{ij}(x'')\delta_{,j}(x,x')[\delta(x,x'')+\delta(x',x'')] \ , \quad (3.9a)$$

$$U^0_{i0}(x,x';x'')=\delta_{,i}(x,x')[\delta(x,x'')+\delta(x',x'')] \ , \quad (3.9b)$$

$$U^k_{ij}(x,x';x'')=\delta^k_i\delta_{,j}(x,x')\delta(x',x'') +\delta^k_j\delta_{,i}(x,x')\delta(x,x'') \ , \quad (3.9c)$$

where the spatial derivative is with respect to the first coordinate in the δ function. The action (3.6) is invariant under the gauge transformation (3.7) provided that N^μ transforms as

$$\delta_F N^\mu(x) = \dot{F}^\mu(x) - \int d^3x' d^3x'' U^\mu_{\nu\rho}(x',x'';x) N^\nu(x') F^\rho(x'') \tag{3.10}$$

and the initial and final hypersurfaces are not deformed, $F^0(x,t)=F^0(x',t')=0$. Therefore the classical action is in completely parametrized form; it is manifestly diffeomorphism invariant.

What does this mean? The physical content of the theory is independent of the parametrization of the family of hypersurfaces or the choice of coordinates in them. Because the constraints generate these gauge transformations the physical degrees of freedom must have vanishing Poisson brackets with them. Given h_{ij} and π^{ij} that commute with the constraints on an initial hypersurface, the closure of the algebra and the form of the Hamiltonian (3.5) ensures that they evolve such that they do on future slices too. Therefore it is sufficient to isolate the physical degrees of freedom on the initial data.

The physical variables are identified as those canonical variables (h_{ij},π^{ij}) that identically satisfy the constraints on the initial constant t hypersurface. The constraints can be viewed as fixing four conditions on the six metric components and their conjugate momenta. Suppose we could make a change of variables to a set of canonical variables for gravity in which the constraints were four of the momenta. Then as in the model problem, the fields conjugate to these momenta are also no longer dynamical. They are arbitrary parameters in the theory corresponding to a choice of coordinates. Therefore, four gauge conditions can be given to fix this choice. The same holds true when the constraints are written in terms of (h_{ij},π^{ij}). Four functions fixing gauge must also be specified on the canonical variables in addition to the constraints to reduce the system to its physical degrees of freedom. Counting these degrees of freedom, the twelve canonical fields have been reduced to four: two metric components and their conjugate momenta. This is what is expected for a massless spin-2 field.

The identification of the physical variables can be carried out explicitly for the theory of linearized gravity;[26] they are the transverse trace-free components of the metric perturbation and its conjugate momenta. The Hamiltonian for linearized gravity is obtained from that of the full theory by expanding the metric components in perturbations around flat space, $g_{ij}=\delta_{ij}+l\gamma_{ij}$, $N^0=1+l(n^0-\frac{1}{2}\gamma^i_i)$, $N^i=ln^i$, and truncating the result at quadratic order. The γ^i_i contribution to the expansion of N^0 is from it being a density of weight -1. The leading term in the expansion of $N^0\mathcal{H}_0$ cancels the surface term E; the result is

$$H_2=\frac{1}{4}\int d^3x\,[\,4(\pi^{ij})^2-2(\pi^i_i)^2+(\partial_k\gamma_{ij})^2-(\partial_k\gamma^i_i)^2 -2\partial_k\gamma^{ki}(\partial_j\gamma^j_i-\partial_i\gamma^j_j)+n^\mu\mathcal{H}^{(1)}_\mu]\;, \tag{3.11a}$$

$$\mathcal{H}^{(1)}_0=\partial_i\partial_j\gamma^{ij}-\partial^2\gamma^i_i\;, \tag{3.11b}$$

$$\mathcal{H}^{(1)}_i=-2\partial_j\pi^j_i\;, \tag{3.11c}$$

where we have used the convention that $(\pi^{ij})^2=\delta_{ik}\delta_{jl}\pi^{ij}\pi^{kl}$ with similar contractions holding for any other tensor written in this notation. Also, in expressions for linearized gravity, indices are raised and lowered using the flat metric δ_{ij}. The physical degrees of freedom can be isolated using the Fourier decomposition of the perturbations on an initial hypersurface. Labeling the Fourier transform of the spatial dependence of the tensor fields by the wave vector k^i, the perturbations can be written as Fourier components on the tensor space. There are six different types of components for each value of k^i: three which are parallel to k^i, two that are transverse to k^i and trace-free, and one that is the trace of the transverse part of the tensor. The constraints (3.11b) and (3.11c) fix the longitudinal components of the momenta and the trace of the transverse component of the metric perturbation. The variables conjugate to these will no longer be dynamical and therefore depend on the choice of gauge. Thus the physical part of the three-metric and its conjugate momenta are the two independent Fourier components of each that lie in the tensor subspace that is both transverse to k^i and trace-free, that is, γ^{TT}_{ij} and π^{ij}_{TT}.

Formally, the physical variables for general relativity are isolated using the same method illustrated above for the linearized theory; however, unlike linearized gravity, these variables cannot be explicitly constructed because of the full theory's nonlinear interactions. This means that it is very useful and necessary to be able to express quantities in terms of the full set of fields that are equivalent to those in the physical variables. An example of how this can be done will be discussed for the case of the transition functional for the interacting theory.

Having identified the physical degrees of freedom, the quantum mechanics of the theory can be formulated in terms of them. A basic unit in this is the transition functional which gives the evolution of the metric configuration specified on an initial hypersurface to that fixed on the final hypersurface; it is the generalization of the transition amplitude or propagator in quantum mechanics. When the physical configurations can be explicitly found then this integral is the direct analog of (2.9). This is the case for the theory of linearized gravity:

$$G[h'^{TT}_{ij},t';h^{TT}_{ij},t]=\int d\gamma^{TT}_{ij}d\pi^{ij}_{TT}\exp\left[i\int dt\left[\int d^3x\;\pi^{ij}_{TT}\dot{\gamma}^{TT}_{ij}-h_2[\gamma^{TT},\pi_{TT}]\right]\right]\;, \tag{3.12a}$$

$$h_2=\int d^3x\,[(\pi^{ij}_{TT})^2+\tfrac{1}{4}(\partial_k\gamma^{TT}_{ij})^2]\;. \tag{3.12b}$$

The sum is over all paths in the physical phase space which consists of the field configurations $(\pi^{ij}_{TT},\gamma^{TT}_{ij})$ where the transverse traceless metric perturbations take on the values h'^{TT}_{ij} and h^{TT}_{ij} fixed on the constant t' and t hypersurfaces, respectively. The measure is the functional generalization of the Liouville measure of the model problem. Again constant factors will not be displayed in the measure in general, although we will when providing an explicit formulation of the functional integral for linearized gravity in the Appendix.

When the physical degrees of freedom cannot be found explicitly, functional integrals that are equivalent to those in terms of the physical variables can be formulated in terms of the full set of canonical variables. This is done by using functional δ functions of the constraints and gauge degrees of freedom in the measure to eliminate all but the physical variables from the sum. Let us consider this first for linearized gravity, where we can add back in the other phase-space variables explicitly. In this case, π^{ij}_L the longitudinal components of the momenta are fixed by the momentum constraints (3.11c). Their conjugate variables γ^L_{ij} are fixed by specifying three gauge-fixing conditions. Similarly, γ^T_{ij}, the trace of the transverse component of the metric is fixed by the linearized Hamiltonian constraint (3.11b); its conjugate variable π^{ij}_T is also fixed by a choice of gauge. Writing the gauge-fixing functions as G^ν, the following identity is true:

$$1=\int d\mathcal{H}^{(1)}_\mu dG^\nu \delta[\mathcal{H}^{(1)}_\mu(\pi_L,\gamma^T)]\delta[G^\nu(\gamma^L,\pi_T)] \ . \qquad (3.13)$$

Next change the variables of functional integration from $\mathcal{H}^{(1)}_\mu$ and G^ν to the redundant phase space variables; the Jacobian of this transformation is simply the determinant of the Poisson brackets of the constraints with the gauge-fixing functions: $|\{\mathcal{H}^{(1)}_\mu,G^\nu\}|$. If we add the resulting form of the identity into the transition functional (3.12), we arrive at the following functional integral:

$$G[h'^{TT}_{ij},t';h^{TT}_{ij},t]=\int d\gamma_{ij}d\pi^{ij}\delta[\mathcal{H}^{(1)}_\mu]\delta[G^\nu]\,|\{\mathcal{H}^{(1)}_\mu,G^\nu\}|\exp\left[i\int_t^{t'}dt\left[\int d^3x\,\pi^{ij}\dot{\gamma}_{ij}-H_2\right]\right] . \qquad (3.14)$$

Because the constraints are enforced by δ function, the action for linearized gravity in terms of the extended variables can be used in (3.14). This integral over the full set of canonical variables is manifestly identical to that written in the physical degrees of freedom.

This same procedure can be carried out when the physical degrees of freedom cannot be explicitly isolated. The resulting transition functional for the full theory of gravity is

$$G[h'_{ij},t';h_{ij},t]=\int dh_{ij}d\pi^{ij}\delta[\mathcal{H}_\mu]\delta[G^\nu]\,|\{\mathcal{H}_\mu,G^\nu\}|\exp\left[i\int_t^{t'}dt\left[\int d^3x\,\pi^{ij}\dot{h}_{ij}-E\right]\right] . \qquad (3.15)$$

This integral is weighted by the classical action (3.6) with the constraints set to zero. $G^\nu(\pi^{ij},h_{ij})$ are four functions of the canonical variables fixing the gauge. In order to do this, they must have nonzero Poisson brackets with the constraints. The sum is over all metric configurations matching the initial and final data on the boundary hypersurfaces t and t'. Note that the gauge choice must be consistent with this data. The determinant $|\{\mathcal{H}_\mu,G^\nu\}|$ included in the measure is precisely the factor needed to make it equal to the Liouville measure in the physical degrees of freedom. In order to make this path integral meaningful, the class of metric configurations summed over must be specified. In this paper we shall make this integral concrete by defining it to be over metric perturbations. One first selects a one-parameter family of constant t hypersurfaces and chooses coordinates on those surfaces that are asymptotically Euclidean. The metric of each hypersurface is then taken to be of the form $h_{ij}=\delta_{ij}+l\gamma_{ij}$ where γ_{ij} is a bounded function with the falloff behavior (3.4). Its corresponding momentum π^{ij} is also bounded with falloff behavior $\pi^{ij}\sim 1/r^2$ at spatial infinity. The sum over geometries matching the boundary data is then done by summing over all fields with this behavior on each intermediate hypersurface between the initial and final hypersurface. This sum over geometries does not include those geometries that cannot be written in this form in this coordinate system or those that develop singularities. However the transition functional will be useful for cases where the initial and final metrics are almost flat because then one expects the major contribution to its value to come from nearby configurations that are also almost flat.

Both sides of Eq. (3.15) really depend on the physical degrees of freedom only; however, the specification of the path integral depends on the gauge chosen. This is because we cannot express the true (gauge-invariant) degrees of freedom explicitly but only in a particular gauge. The two components of h_{ij} that we choose as physical are free data on the $t=$const boundary hypersurface; the other components of h_{ij} must agree with the gauge choice.

From transition functionals in the physical degrees of freedom we can derive Euclidean functional integrals for the ground state by continuing $t\to -i\tau$ and then taking the leading behavior as $\tau\to-\infty$ in analogy to the Feynman-Kac prescription in quantum mechanics. These integrals will be convergent and therefore this procedure is allowed if the physical Hamiltonian is positive. In linearized gravity, the Hamiltonian (3.12b) in terms of the physical variables is manifestly positive; implementing the construction procedure used in Sec. II on the transition functional (3.12a) one obtains the Euclidean functional integral for the ground-state wave functional for linearized gravity:

$$\Psi_0[h^{TT}_{ij}]=\mathcal{N}\int d\gamma^{TT}_{ij}d\pi^{ij}_{TT}\times\exp\left[\int_{-\infty}^0 d\tau\left[\int d^3x\, i\pi^{ij}_{TT}\dot{\gamma}^{TT}_{ij}-h_2\right]\right]. \qquad (3.16)$$

The sum is over all transverse traceless tensors matching the data h_{ij}^{TT} given at the boundary $\tau=0$ that fall off fast enough at Euclidean infinity so that the action is finite. $\mathcal{N}$ is a normalization factor independent of the perturbations. The equivalent configuration-space functional integral can be derived by doing the quadratic momentum integrals:

$$\Psi_0[h_{ij}^{TT}]=\mathcal{N}\int d\gamma_{ij}^{TT}\exp(-i_2)\ , \tag{3.17a}$$

$$i_2=\frac{1}{4}\int_{-\infty}^{0} d^4x[(\dot{\gamma}_{ij}^{TT})^2+(\partial_k\gamma_{ij}^{TT})^2]\ . \tag{3.17b}$$

When evaluated, this integral results in a wave functional composed of the product of harmonic-oscillator ground states whose arguments are the two independent amplitudes of the transverse traceless Fourier components for each wave vector k^i (Refs. 27 and 28). Therefore the Feynman-Kac procedure is explicitly seen to give the ground-state wave functional in this case.

For asymptotically flat spacetimes, the positive-energy theorem[8,9] states that E is positive for nonsingular vacuum spacetimes where h_{ij} and π^{ij} satisfy the constraints. This means that we can formally construct convergent Euclidean functional integrals over the physical variables. However, the analog of the ground-state wave functional (3.16) for linearized gravity cannot be written explicitly because the physical degrees of freedom cannot be explicitly isolated. Formally this state can be derived for asymptotically flat spacetimes by first computing the transition functional and then using the Feynman-Kac prescription. By the positive-energy theorem, the minimum energy metric configuration is flat space; this means the analog of setting q^a at the minimum of the potential is to fix the initial geometry to be flat. The final geometry is taken to be the argument of the wave functional. For the transition functional (3.15) one has

$$\Psi_0[\gamma_{ij}]=\lim_{\tau\to-\infty}(\mathcal{N}G[\delta_{ij}+l\gamma_{ij},0;\delta_{ij},-i\tau])\ , \tag{3.18}$$

where t is continued to $-i\tau$ in (3.15) and the data given on the $\tau=0$ boundary is almost flat. In (3.18) we have written the argument of the wave functional in terms of the perturbation from flat space γ_{ij} for convenience. Because the functional integration is carried out before the analytic continuation, the constraints are enforced and the Hamiltonian is positive. Thus (3.18) will be a well-defined procedure for obtaining the ground state; however it has several disadvantages. It does not exploit the simplicity of the boundary conditions as (3.17) does because it cannot be explicitly written in terms of the physical variables. It cannot be easily written as a configuration-space path integral because it is necessary to exponentiate the constraints to make it quadratic in the momenta before performing the momentum integrations. The transition functional itself is not parametrized and thus does not display the manifest invariances of the theory. Equation (3.18) is our starting point; our aim is to derive a corresponding convergent parametrized Euclidean functional integral for this state. Before proceeding to do this for perturbative gravity, it is instructive to study how the ground-state wave functional for the linearized theory (3.15) is parametrized. We shall do so in Sec. IV and then treat the interacting case in Sec. V.

IV. LINEARIZED GRAVITY

The transition from Euclidean functional integrals over the physical variables to those over the extended variables can be explicitly worked out for the linearized version of Einstein's theory. This is because the physical degrees of freedom can be explicitly identified and the action in terms of them (3.17) is a quadratic functional. In this section we shall review work done in a previous paper[10] showing how this connection is made for the ground-state wave functional for linearized gravity. Boulware has also obtained this result in the linearized theory.[29] The results of this section will be a useful guide to carrying out the construction of the parametrized Euclidean integral for the ground-state wave functional for asymptotically flat spacetimes calculated perturbatively.

Our starting point is the ground-state wave functional for the linearized theory (3.17). What we want to do is to add integrations over the redundant variables to this expression until we arrive at an expression for Ψ_0 that is manifestly gauge invariant under the linearized diffeomorphisms of the full theory:

$$\gamma_{\alpha\beta}=\gamma_{\alpha\beta}+2\partial_{(\alpha}f_{\beta)} \tag{4.1}$$

and O(4) invariant. What form of the parametrized action can we expect to get? An O(4)-invariant, gauge-invariant Euclidean action that is also local in the perturbation $\gamma_{\alpha\beta}$ is the linearized version of (1.3):[27]

$$I_2=-\tfrac{1}{2}\int_M d^4x\ \gamma^{\alpha\beta}\dot{G}_{\alpha\beta}-\tfrac{1}{2}\int_{\partial M} d^3x\ \gamma^{ij}(\dot{K}_{ij}-\delta_{ij}\dot{K})\ , \tag{4.2a}$$

$$\dot{G}_{\alpha\beta}=-\tfrac{1}{2}(-\partial^2\bar{\gamma}_{\alpha\beta}-\delta_{\alpha\beta}\partial_\gamma\partial_\eta\bar{\gamma}^{\gamma\eta}+\partial_\alpha\partial_\gamma\bar{\gamma}^{\gamma}_{\beta}+\partial_\beta\partial_\gamma\bar{\gamma}^{\gamma}_{\alpha})\ , \tag{4.2b}$$

$$\dot{K}_{ij}=-\tfrac{1}{2}(\dot{\gamma}_{ij}-\partial_i\gamma_{0j}-\partial_j\gamma_{0i})\ , \tag{4.2c}$$

where $\bar{\gamma}_{\alpha\beta}$ is the trace reversed metric perturbation

$$\bar{\gamma}_{\alpha\beta}=\gamma_{\alpha\beta}-\tfrac{1}{2}\delta_{\alpha\beta}\gamma^{\delta}_{\delta}\ . \tag{4.3}$$

We cannot end up with a functional integral for Ψ_0 involving this action because, as in the full theory, it is not positive definite. In particular, on perturbations of the form $\gamma_{\alpha\beta}=2\chi\delta_{\alpha\beta}$,

$$I_2=-6\int_M d^4x\,(\partial_\alpha\chi)^2\ . \tag{4.4}$$

However (4.2a) is not the only gauge- and O(4)-invariant action for linearized gravity. As in the model, there are many others if the action is not required to be local in the original set of variables. We shall find such an action by adding in integrations over the redundant variables to the physical functional integral; our aim will be to construct one as close as possible in form to (4.2a).

To add back the redundant integrations it is useful to decompose $\gamma_{\alpha\beta}$ into pieces corresponding to the physical variables and pieces corresponding to the redundant ones. As the result (4.4) suggests, it is convenient to begin by decomposing $\gamma_{\alpha\beta}$ into conformal equivalence classes

$$\gamma_{\alpha\beta}=\phi_{\alpha\beta}+2\chi\delta_{\alpha\beta} \tag{4.5}$$

This decomposition is fixed by the O(4)-invariant, gauge-invariant condition

$$R^{(1)}(\phi)=\partial_\alpha\partial_\beta\phi^{\alpha\beta}-\partial^2\phi^\beta_\beta=0\ . \tag{4.6}$$

χ is now related to $\gamma_{\alpha\beta}$ by the second-order equation

$$R^{(1)}(\gamma)=-6\partial^2\chi \tag{4.7}$$

subject to the boundary conditions that $\chi=0$ on the $\tau=0$ hypersurface and at Euclidean infinity.

The perturbation $\phi_{\alpha\beta}$ is then decomposed as

$$\phi_{\alpha\beta}=t_{\alpha\beta}+l_{\alpha\beta}+\phi^T_{\alpha\beta}+\phi^L_{\alpha\beta}\ , \tag{4.8}$$

where the components are defined as follows: let n^α be the unit vector orthogonal to the surfaces of constant τ. Consider the subspaces of the space of tensor functions whose elements $t_{\alpha\beta}$, $l_{\alpha\beta}$, $\phi^T_{\alpha\beta}$, and $\phi^L_{\alpha\beta}$ satisfy the conditions

$$\partial^\alpha t_{\alpha\beta}=0,\quad n^\alpha t_{\alpha\beta}=0,\quad t^\alpha_\alpha=0\ , \tag{4.9a}$$

$$\partial^\alpha l_{\alpha\beta}=0,\quad l^\alpha_\alpha=0,\quad \int_M d^4x\, t^{\alpha\beta}l_{\alpha\beta}=0\ , \tag{4.9b}$$

$$\partial^\alpha\phi^T_{\alpha\beta}=0,\quad n^\alpha\phi^T_{\alpha\beta}=0,\quad \int_M d^4x\, t^{\alpha\beta}\phi^T_{\alpha\beta}=0\ , \tag{4.9c}$$

$$\int_M d^4x\, t^{\alpha\beta}\phi^L_{\alpha\beta}=0,\quad \int_M d^4x\, l^{\alpha\beta}\phi^L_{\alpha\beta}=0\ ,\quad \int_M d^4x\, \phi^{T\alpha\beta}\phi^L_{\alpha\beta}=0\ . \tag{4.9d}$$

If the orthogonality conditions are required to hold for all tensors in the subspaces, then the decomposition of $\phi_{\alpha\beta}$ (4.8) is unique. A more explicit version of the decomposition will be given in the Appendix. The condition (4.6) is seen to fix $\phi^T_{\alpha\beta}=0$. The tensors $t_{\alpha\beta}$ correspond to the physical variables γ^{TT}_{ij}; the rest are redundant.

The utility of this decomposition is that under gauge transformations (4.1) $t_{\alpha\beta}$, $l_{\alpha\beta}$, and χ are unchanged. Since the action (4.2) is a gauge invariant, it can be expressed in terms of these variables:

$$I_2=\tfrac{1}{4}\int_M d^4x\,[(\partial_\gamma t_{\alpha\beta})^2+(\partial_\gamma l_{\alpha\beta})^2-24(\partial_\gamma\chi)^2]$$
$$-\tfrac{1}{4}\int_{\partial M} d^3x\, n^\alpha\partial_\alpha[2(n^\beta l_{\beta\gamma})^2-\tfrac{3}{2}(n^\beta n^\gamma l_{\beta\gamma})^2]\ . \tag{4.10}$$

Using this decomposition of the metric we can proceed to add in the redundant degrees of freedom by inserting in (3.17) identities composed of Gaussian integrals over the gauge-invariant quantities and gauge-fixing δ functions over the noninvariant ones. Although the final answer is independent of the gauge choice, the following arguments will be clearer if a particular gauge is fixed. The choice

$$\Phi^\alpha(\phi)=\partial_\beta\bar{\phi}^{\alpha\beta}=0 \tag{4.11a}$$

when combined with the condition (4.5) fixes the $\phi^L_{\alpha\beta}$ components up to a gauge transformation (4.1) where f_α also satisfies

$$-\partial^2 f_\alpha=0\ . \tag{4.11b}$$

This remaining gauge freedom can be eliminated by fixing the value of these components on the boundaries. Boundary conditions on the other redundant variables must also be specified in order to define the class of tensor field configurations that will be summed over. A simple and convenient set of boundary conditions that corresponds to those given for ground state in the physical variables (3.17) and satisfies the above requirements on the redundant ones is to (1) take $t_{\alpha\beta}$ to match the argument h^{TT}_{ij} of the wave function at $\tau=0$, (2) require that $\chi=0$ and the spatial part γ_{ij} of the remaining components vanish there, (3) require that the gauge condition (4.11a) be satisfied on the boundary, and (4) take all components of $\phi_{\alpha\beta}$ and χ to vanish rapidly enough at Euclidean infinity so that the action is finite. On configurations satisfying these four conditions, the surface term in the action (4.10) vanishes.

In terms of the decomposition, the action i_2 (3.17b) in the physical degrees of freedom is

$$i_2=\tfrac{1}{4}\int_M d^4x\,(\partial_\gamma t_{\alpha\beta})^2\ . \tag{4.12}$$

Next, we want a positive-definite action quadratic in the redundant variables that is gauge invariant and O(4) invariant to use in forming Gaussian integrals. If, in addition, we require this integral to be at most quadratic in the derivatives, the most general form of the action that satisfies these requirements is

$$I^g_2=\tfrac{1}{4}\int_M d^4x\,[(\partial_\gamma l_{\alpha\beta})^2+a(\partial_\gamma\chi)^2]\ , \tag{4.13}$$

where a is an arbitrary positive constant. O(4) rotations that mix n^α with the spatial unit vectors $\hat{\mathbf{x}}^i$ will mix the components $t_{\alpha\beta}$ and $l_{\alpha\beta}$; therefore the coefficients of these terms must be the same if O(4) invariance is to be preserved in the full action. This does not similarly restrict the coefficient of $(\partial_\alpha\chi)^2$ as it transforms as a scalar. The constant a must be positive, however, for the action to be positive definite.

As in the model problem, integrals over the action (4.13) and the gauge-fixing condition (4.11a) can be added to the Euclidean functional integral for the ground-state wave function by forming the appropriate combinations of Gaussians and δ functions. In this case we want the identities

$$1=\int dl_{\alpha\beta}d\phi^L_{\alpha\beta}d\chi\,\delta[\Phi^\alpha(\phi)]\left|\frac{\delta\Phi^\alpha}{\delta f^\beta}\right|\exp(-I^g_2[l,\chi])\ , \tag{4.14a}$$

$$1=\int d\phi^T_{\alpha\beta}\delta[R^{(1)}(\phi)]\left|\frac{\delta R^{(1)}}{\delta\omega}\right| . \qquad (4.14b)$$

In (4.14) the functional integrations are over the configurations that we have specified by the decomposition and the boundary conditions. A specific measure is required for these identities to be true. This will be given explicitly in the Appendix for calculating these integrals using mode amplitudes. The determinant in (4.14a) is the Faddeev-Popov determinant of the operator constructed by varying the gauge-fixing term Φ^α (4.11a) with respect to the gauge parameter f^α (4.1). The boundary conditions for this operator are determined by those on the gauge parameter. In order to keep the boundary conditions fixed, $f^\alpha=0$ at $\tau=0$ and at Euclidean infinity. The determinant is calculated using the spectrum of this operator subject to these same boundary conditions. The determinant in (4.14b) is of the operator constructed by varying the condition fixing conformal equivalence classes (4.6) by an infinitesimal conformal transformation

$$\phi_{\alpha\beta}=\phi_{\alpha\beta}+2\delta_{\alpha\beta}\omega . \qquad (4.15)$$

As for the Faddeev-Popov determinant, this one is computed using the spectrum of the operator determined for the boundary conditions that ω vanishes at $\tau=0$ and at Euclidean infinity.

These determinants can be conveniently represented using functional integration over Grassmann variables.[18] Let $-i\overline{\mathcal{C}}_\mu$ and $\mathcal{C}^\mu$ be eight real anticommuting Grassmann fields. Then the Faddeev-Popov determinant in (4.14a) becomes

$$\left|\frac{\delta\Phi^\mu}{\delta f^\nu}\right|=\int d\overline{\mathcal{C}}_\mu d\mathcal{C}^\mu \exp(-I_2^{\rm gh}[\overline{\mathcal{C}},\mathcal{C}]) , \qquad (4.16a)$$

$$I_2^{\rm gh}=\int_M d^4x(-i\overline{\mathcal{C}}_\mu\delta_c^{(2)}\Phi^\mu) , \qquad (4.16b)$$

where $\delta_c^{(2)}\Phi^\mu$ is the value of the gauge-fixing function on the linearized gauge transformation (4.1) with $\mathcal{C}^\mu$ as the gauge parameter. The boundary conditions needed to compute the spectrum of the operator are implemented by requiring that $\overline{\mathcal{C}}_\mu$ and $\mathcal{C}^\mu$ vanish at $\tau=0$ and Euclidean infinity. Again, a particular measure is needed for this identity to hold. For the gauge-fixing choice (4.11a),

$$\delta_c^{(2)}\Phi^\mu=-\partial^2\mathcal{C}^\mu . \qquad (4.17)$$

The determinant in (4.14b) can also be exponentiated in a similar manner; however, we will choose to leave it in the measure.

Inserting these identities into the Euclidean functional integral and exponentiating the Faddeev-Popov determinant using (4.16) we arrive at the following expression for the ground-state wave function:

$$\Psi_0[h_{ij}^{TT}]=\int d\phi_{\alpha\beta}d\chi\, d\overline{\mathcal{C}}_\mu d\mathcal{C}^\mu\delta[\Phi^\alpha(\phi)]\delta[R^{(1)}(\phi)]\left|\frac{\delta R^{(1)}}{\delta\omega}\right|\exp\{-(\hat{I}_2[\phi,\chi]+I_2^{\rm gh}[\overline{\mathcal{C}},\mathcal{C}])\} . \qquad (4.18)$$

Here $\hat{I}_2$ is the sum of i_2 and I_2^g

$$\hat{I}_2=\tfrac{1}{4}\int_M d^4x[(\partial_\gamma t_{\alpha\beta})^2+(\partial_\gamma l_{\alpha\beta})^2+a(\partial_\gamma\chi)^2] , \qquad (4.19)$$

where a is any positive constant. The integral is over all ten components of $\phi_{\alpha\beta}$ and over the linearized piece of the conformal factor χ in the class of configurations described previously. Thus the integral is of the form of one over all gauge inequivalent metrics in a conformal equivalence class fixed by $R^{(1)}=0$ plus an integration over the conformal factor.

The action (4.19) is gauge invariant, O(4) invariant and, for positive a it is positive definite so that the integral (4.18) converges. This action cannot be made to agree with the action I_2 (4.10) because this would require that a be negative and lead to a divergent functional integral. The action $\hat{I}_2$ when a is set to 24 is exactly what would be obtained formally from I_2 by conformal rotation; that is, rotating $\chi\to i\chi$.

As suggested before, the action $\hat{I}_2$ is not local in the original metric perturbation $\gamma_{\alpha\beta}$; however, it can be expressed in terms of it

$$\hat{I}_2[\gamma]=I_2[\gamma]+\frac{a+24}{144}\int d^4x\, R^{(1)}(\gamma)\frac{1}{-\partial^2}R^{(1)}(\gamma) . \qquad (4.20)$$

It is physically equivalent to I_2, gauge and O(4) invariant, and positive definite for $a>0$. Thus at the expense of locality, one can construct convergent Euclidean functional integrals for gravity that manifestly display the invariances of the theory. There are many different forms of these convergent Euclidean functional integrals as the nonlocal action needed is not unique; a is not fixed by the process. These integrals are very similar to those obtained by the Gibbons, Hawking, and Perry prescription of conformal rotation applied to the linearized theory. However, the Gibbons, Hawking, and Perry prescription omits the Faddeev-Popov determinant $|\delta R^{(1)}/\delta\omega|$ of the decomposition fixing δ function in the measure. Our analysis shows that this factor is needed for the manifestly invariant Euclidean integral to equal that given in terms of the physical fields.

We have derived (4.18) directly from the Euclidean functional integral for the ground state given in terms of its physical variables. This is not the only way to arrive at this integral. Alternatively, we could have begun by first parametrizing the Lorentzian transition functional (3.12a); however, instead of aiming for its usual form weighted by the local action S_2 we construct one weighted by the Lorentzian version of $\hat{I}_2$. This parametrized Lorentzian functional integral can be derived along the same lines as the Euclidean one; the result is

$$G[h'^{TT}_{ij},t';h^{TT}_{ij},t]=\int d\phi_{\alpha\beta}d\chi\, d\bar{\mathcal{C}}_\mu d\mathcal{C}^\mu \delta[\Phi^\alpha(\phi)]\delta[R^{(1)}(\phi)]\left|\frac{\delta R^{(1)}}{\delta\omega}\right|\exp(i\hat{S}_2[\phi,\chi]+iS_2^{\rm gh}[\bar{\mathcal{C}},\mathcal{C}]) , \tag{4.21}$$

where the class of configurations summed over is specified by the Lorentzian analogs of the decompositions (4.6) and (4.7), the gauge-fixing condition (4.11), and boundary conditions on both the initial and final hypersurface. The action $\hat{S}_2$ is the Lorentzian version of (4.19)

$$\hat{S}_2=-\tfrac{1}{4}\int_M d^4x\,[(\partial_\gamma t_{\alpha\beta})^2+(\partial_\gamma l_{\alpha\beta})^2+a(\partial_\gamma\chi)^2] . \tag{4.22}$$

The ghost term $S_2^{\rm gh}$ is the analog of (4.16b) where the gauge transformation is done using the Lorentzian signature. The usual form of the Lorentzian functional integral weighted by the action S_2 can be recovered by setting $a=-24$ and doing the integral over χ using the δ function of $R^{(1)}$. By choosing a positive, the analytic continuation of $t\to -i\tau$ can be carried out on (4.21) because the corresponding Euclidean action is positive. Taking the initial data to vanish and the limit $\tau\to-\infty$ results in precisely the same Euclidean functional integral for the ground state (4.18) obtained directly.

V. PERTURBATIVE GRAVITY

In the previous section we showed how convergent Euclidean functional integrals for linearized gravity could be derived by appropriately adding integrals over the redundant variables to the Euclidean functional integral in terms of the physical variables. We found that in order to maintain the convergence of the integral we were led to a parametrized action that was not local in the original metric perturbation. This motivates us to look for a convergent Euclidean functional integral weighted by a nonlocal action for the ground-state functional (3.18) of the full theory, where it is not possible to parametrize the physical functional integral directly. We will derive this in three steps. (1) We will parametrize the Lorentzian transition functional (3.15) in phase space and perform the momentum integrations to get the parametrized local Lorentzian transition functional in configuration space. This will be carried out using Becchi-Rouet-Stora (BRS) invariance following the method of Fradkin and Vilkovisky. (2) Then, as suggested at the end of the previous section, we will find an alternate Lorentzian functional with a nonlocal action that equals the first term by term in perturbation theory. This new Lorentzian functional integral will be constructed so as to have manifestly positive Euclidean action. (3) We shall then use this alternate transition functional in the definition of the ground state (3.18). Consequently the rotation of $t\to -i\tau$ can be carried out term by term. Doing this will result in a manifestly convergent parametrized Euclidean functional integral for the ground state.

The basic idea is to parametrize the Lorentzian functional integral (3.15) in phase space[30] by introducing extra fields to exponentiate the functional δ functions and determinants. The gauge conditions used in (3.15) are given on the canonical variables; one would like to generalize these to include the additional variables so that calculations can be done in other gauges. For this generalization to be correct, the parametrized integral must be manifestly independent of gauge choice and must equal the original transition functional in a canonical gauge. Fradkin and Vilkovsky proved that this is indeed the case for gauge theories and gravity in a series of papers.[21,31,32] It is useful to review how this connection is made for asymptotically flat spaces and to discuss the boundary conditions needed in the parametrized transition functional.

The canonical phase space (h_{ij},π^{ij}) is extended by adding the lapse and shift N^μ and their conjugate momenta P_μ. In addition, eight real anticommuting Grassmann fields $\mathcal{C}^\mu$ and $-i\bar{\mathcal{C}}_\mu$ and their conjugate momenta, $\bar{\mathcal{P}}_\mu$ and $i\mathcal{P}^\mu$ are also added. The Grassmann parity, σ, of the anticommuting variables is odd and that of the commuting variables is even. There is a new structure on this extended phase space, the ghost number. $\mathcal{P}^\mu$ and $\mathcal{C}^\mu$ have ghost number 1 and $\bar{\mathcal{P}}_\mu$ and $\bar{\mathcal{C}}_\mu$ have ghost number -1. The rest of the variables h_{ij}, π^{ij}, N^μ, and P_μ have ghost number 0. Quantities composed of products of these fields have ghost number equal to the sum of that of their components. The corresponding Poisson brackets are

$$\begin{aligned}\{\bar{\mathcal{P}}_\mu(x),\mathcal{C}^\nu(x')\}&=-\delta_\mu^\nu(x,x') ,\\ \{\mathcal{P}^\nu(x),\bar{\mathcal{C}}_\mu(x')\}&=-\delta_\mu^\nu(x,x') ,\\ \{P_\mu(x),N^\nu(x')\}&=-\delta_\mu^\nu(x,x') ,\end{aligned} \tag{5.1}$$

where the Poisson brackets of the Grassmann fields are anticommuting, $\{\bar{\mathcal{P}}_\mu,\mathcal{C}^\nu\}=\{\mathcal{C}^\nu,\bar{\mathcal{P}}_\mu\}$. The Poisson brackets of quantities containing both commuting and anticommuting variables can be computed using the relation

$$\{A,BC\}=\{A,B\}C+(-1)^{\sigma_A\sigma_B}B\{A,C\} , \tag{5.2}$$

where the parity of a quantity, for example σ_A of A, is odd if it contains an odd number of Grassmann variables and even otherwise.

The next step in constructing the transition functional on the extended phase space is to define the BRS transformation on the extended variables. It generalizes the local gauge transformations of general relativity to a global transformation that mixes the commuting and anticommuting variables. The generator of this transformation is

$$\Omega=\int[-i\mathcal{P}^\mu(x)P_\mu(x)+\mathcal{C}^\mu(x)\mathcal{H}_\mu(x)-\tfrac{1}{2}\mathcal{C}^\mu(x)\mathcal{C}^\nu(x')U^\rho_{\nu\mu}(x',x;x'')\bar{\mathcal{P}}_\rho(x'')] , \tag{5.3}$$

where we have introduced the convention to be used in this section that all spatial variables x, x', etc, *repeated* under the integral sign are to be integrated over. The $U^{\alpha}_{\nu\mu}$ are the first-order structure functions (3.9). Ω has ghost number 1 and it follows by the algebra of the constraints and (5.2) that it is nilpotent, $\{\Omega,\Omega\}=0$. The BRS transformations of the variables are given by their Poisson brackets with Ω:

$$\{h_{ij}(x),\Omega\epsilon\}=\int \mathcal{C}^{\mu}(x')\{h_{ij}(x),\mathcal{H}_{\mu}(x')\}\epsilon\ , \tag{5.4a}$$

$$\{\pi^{ij}(x),\Omega\epsilon\}=\int [\mathcal{C}^{\mu}(x')\{\pi^{ij}(x),\mathcal{H}_{\mu}(x')\}-\tfrac{1}{2}\mathcal{C}^{\mu}(x')\mathcal{C}^{\nu}(x'')\{\pi^{ij}(x),U^{\rho}_{\nu\mu}(x'',x';\bar{x})\}\overline{\mathcal{P}}_{\rho}(\bar{x})]\epsilon\ ,$$

$$\{N^{\mu}(x),\Omega\epsilon\}=i\mathcal{P}^{\mu}(x)\epsilon,\quad \{P_{\mu}(x),\Omega\epsilon\}=0\ , \tag{5.4c}$$

$$\{\mathcal{C}^{\mu}(x),\Omega\epsilon\}=\int \tfrac{1}{2}\mathcal{C}^{\nu}(x')\mathcal{C}^{\rho}(x'')U^{\mu}_{\rho\nu}(x'',x';x)\epsilon\ , \tag{5.4d}$$

$$\{\overline{\mathcal{P}}_{\mu},\Omega\epsilon\}=-\mathcal{H}_{\mu}(x)\epsilon+\int \mathcal{C}^{\nu}(x')U^{\rho}_{\nu\mu}(x',x;x'')\overline{\mathcal{P}}_{\rho}(x'')\epsilon\ , \tag{5.4e}$$

$$\{\overline{\mathcal{C}}_{\mu}(x),\Omega\epsilon\}=iP_{\mu}(x)\epsilon,\quad \{\mathcal{P}^{\mu}(x),\Omega\epsilon\}=0\ . \tag{5.4f}$$

The constant anticommuting parameter ϵ is introduced so that the transformation preserves the Grassmann parity of the variables. From its form, one sees that these transformations will contain those of the canonical variables (3.7) with $\mathcal{C}^{\mu}\epsilon$ as the transformation parameter.

The action needed to form the transition functional on the extended phase space is the generalization of the action (3.6) to the extended variables. It is found by requiring it to be BRS invariant and have ghost number 0. The general form of this action is

$$S_{\mathrm{FV}}=\int_{t}^{t'} dt([\pi^{ij}(x)\dot{h}_{ij}(x)+P_{\mu}(x)\dot{N}^{\mu}(x)+\dot{\mathcal{P}}^{\mu}(x)\overline{C}_{\mu}(x)+\dot{C}^{\mu}(x)\overline{\mathcal{P}}_{\mu}(x)]-E+\{\Phi,\Omega\})\ . \tag{5.5}$$

Φ is an arbitrary functional of any of the variables on the extended phase space such that it has ghost number -1. It is the analog of the gauge choice in the usual Hamiltonian form. This is more apparent if we take a specific form for Φ,

$$\Phi=\int [i\overline{\mathcal{C}}_{\mu}(x)\varphi^{\mu}(x)+\overline{\mathcal{P}}_{\mu}(x)N^{\mu}(x)]\ , \tag{5.6}$$

where φ^{μ} is an arbitrary function with ghost number 0 of the extended set of variables. It is convenient to restrict it to be independent of the momenta $\mathcal{P}^{\mu}$ and $\overline{\mathcal{P}}_{\mu}$. The Poisson brackets of Φ with Ω is then

$$\begin{aligned}\{\Phi,\Omega\}=\int (&-N^{\mu}(x)[\mathcal{H}_{\mu}(x)-\mathcal{C}^{\nu}(x')U^{\rho}_{\nu\mu}(x',x;x'')\overline{\mathcal{P}}_{\rho}(x'')]-i\overline{\mathcal{P}}_{\mu}(x)\mathcal{P}^{\mu}(x)-P_{\mu}(x)\varphi^{\mu}(x)\\ &+i\overline{\mathcal{C}}_{\mu}(x)\{\varphi^{\mu}(x),\mathcal{H}_{\nu}(x')\}\mathcal{C}^{\nu}(x')+\overline{\mathcal{C}}_{\mu}(x)\{\varphi^{\mu}(x),P_{\nu}(x')\}\mathcal{P}^{\nu}(x')\\ &-\tfrac{1}{2}i\overline{\mathcal{C}}_{\mu}(x)\mathcal{C}^{\nu}(x')\mathcal{C}^{\rho}(x'')\{\varphi^{\mu}(x),U^{\lambda}_{\rho\nu}(x'',x';x''')\}\overline{\mathcal{P}}_{\lambda}(x'''))\ .\end{aligned} \tag{5.7}$$

With this choice, the action (5.5) contains terms corresponding to the functional δ functions of the constraints and gauge-fixing condition and measure as well as additional terms that give dynamics to the extra fields. The integrand of (5.5) will transform into itself under BRS transformations. This is because (1) the factor of the form *pdq* is canonically invariant, (2) E commutes with Ω, and (3) $\{\{\Phi,\Omega\},\Omega\}=0$ by the Jacobi identity because Ω is nilpotent.

The BRS transformation will also act on the values of the variables fixed on the t and t' constant boundary hypersurfaces. In general, initial and final data for half of the variables is specified to determine the classical evolution of a Hamiltonian system. For the corresponding classical action (5.5) to be BRS invariant, a consistent BRS-invariant set of such data must be selected. We shall choose a set of boundary conditions that generalize those used in the physical transition functional (3.15). The three-metric h_{ij} equals its values given on the t and t' constant hypersurfaces as before. Next, note that under an infinitesimal BRS transformation (5.4a), components proportional to $\mathcal{C}^{\mu}$ and $\overline{\mathcal{C}}_{\mu}$ are added to h_{ij}. Therefore $\mathcal{C}^{\mu}$ and $\overline{\mathcal{C}}_{\mu}$ must vanish on the boundary to preserve the initial and final values of the three-metric. In order that $\overline{\mathcal{C}}_{\mu}=0$ be unchanged by BRS transformation, the condition that $P_{\mu}=0$ on the boundaries is also required as seen by (5.4f). The remaining variables π^{ij}, $\mathcal{P}^{\mu}$, $\overline{\mathcal{P}}_{\mu}$, and N^{μ} are not fixed on the boundaries and will be integrated over on the initial hypersurface.

The transition functional on the extended phase space is

$$G[h'_{ij},t';h_{ij},t]=\int D\mu \exp\left[i\int dt\, S_{\mathrm{FV}}\right]\ , \tag{5.8a}$$

where the measure is

$$D\mu=d\pi^{ij}dh_{ij}dN^{\mu}dP_{\mu}d\mathcal{C}^{\mu}d\overline{\mathcal{C}}_{\mu}d\overline{\mathcal{P}}^{\mu}d\mathcal{P}_{\mu} \tag{5.8b}$$

and the sum is over all phase-space paths subject to the boundary conditions already discussed previously. The measure in (5.8b) is the usual canonically invariant Liouville measure on paths in the extended phase space. Because all the variables are dynamical there are no divergences arising from summing over equivalent field configurations. This is because the BRS transformation is not a local gauge transformation but a global one. The transition functional (5.8) is BRS invariant because both the action and measure are invariant. As seen in the case of the physical transition functional (3.15), the gauge choice φ^μ must be compatible with the data fixed on the boundaries.

How is the transition functional on the extended set of variables related to (3.15)? Fradkin and Vilkovisky proved a general theorem[31] that for theories with first-class constraints such as Yang-Mills theories and Einstein gravity, the value of this BRS-invariant transition functional is independent of the choice of Φ. In addition, by appropriately choosing φ^μ (5.8) reduces to (3.15) when the extra phase-space variables are integrated over. Specifically, one takes $\varphi^\mu = 1/\beta G^\mu$ where β is an arbitrary constant and rescales $\mathcal{C}^\mu = \beta \mathcal{C}^\mu$, $P_\mu = \beta P_\mu$. The measure of (5.8) is unchanged by this scaling because one variable is Grassmann and the other is not. Because of the Fradkin-Vilkovisky theorem, the functional integral is independent of the value of β. This allows one to set $\beta=0$. Doing the integrations over the extended variables, one now obtains the expression for the transition functional in the physical variables (3.15). Therefore the transition functional (5.8) is equal to the physical transition functional for arbitrary choices of gauge including those involving the redundant variables. Thus, (5.8) displays the equivalence of canonical and covariant gauge choices for gravity.

We have obtained a general parametrized path integral for the physical transition amplitude. Because it is quadratic in the momenta, the integrations over these variables can be performed to arrive at the equivalent parametrized configuration-space transition functional. This is most easily discussed by restricting φ^μ to be independent of the momenta π^{ij} although it can be shown for more general cases.[21,31] One shifts the momenta π^{ij}, $\mathcal{P}^\mu$, and $\overline{\mathcal{P}}_\mu$ in the action (5.6) by the appropriate combination of variables to bring it into quadratic form:

$$\pi^{ij}(x)=\pi^{ij}(x)+h^{1/2}[K^{ij}(x)-h^{ij}K(x)]-\frac{i}{N^0(x)}\overline{\mathcal{C}}_\mu(x)\int \frac{\delta\varphi^\mu(x)}{\delta h_{ij}(x')}\mathcal{C}^0(x')\ , \tag{5.9a}$$

$$\mathcal{P}^\mu(x)=\mathcal{P}^\mu(x)-i\dot{\mathcal{C}}^\mu(x)+\int[-\tfrac{1}{2}\overline{\mathcal{C}}_\lambda(\bar{x})\mathcal{C}^\nu(x')\mathcal{C}^\rho(x'')\{\varphi^\lambda(\bar{x}),U^\mu_{\rho\nu}(x'',x';x)\}-iN^\nu(x'')\overline{\mathcal{C}}^\rho(x')U^\mu_{\rho\nu}(x',x'';x)]\ , \tag{5.9b}$$

$$\overline{\mathcal{P}}_\mu(x)=\overline{\mathcal{P}}_\mu(x)+i\dot{\overline{\mathcal{C}}}_\mu(x)+\int \overline{\mathcal{C}}_\nu(x')\{\varphi^\nu(x'),P_\mu(x)\}\ , \tag{5.9c}$$

where K_{ij} is the extrinsic curvature

$$K_{ij}=-\frac{1}{2N^0h^{1/2}}(\dot{h}_{ij}-2D_{(i}N_{j)})\ . \tag{5.9d}$$

(K_{ij} differs from the usual form because N^0 is a density of weight -1.) The additional term in (5.9a) arises because the Poisson brackets $\{\varphi^\nu,\mathcal{H}_\mu\}$ in (5.7) is linear in the momenta. After this shift the action becomes

$$S_{\rm FV}=\int_t^{t'} dt\,[-i\overline{\mathcal{P}}_\mu(x)\mathcal{P}^\mu(x)-N^0G_{ijkl}\pi^{ij}(x)\pi^{kl}(x)]+S_L\ , \tag{5.10a}$$

$$\begin{aligned} S_L=&\int_M d^4x\, N^0h\,[K^{ij}K_{ij}-K^2+R(h)]-\int_t^{t'} dt\, E\\ &+\int dt(\,P_\mu(x)\Phi^\mu(x)-i\overline{\mathcal{C}}_\mu(x)\{\Phi^\mu(x),\mathcal{H}_\nu(x')\}\,|_{\rm cl}\mathcal{C}^\nu(x')\\ &\qquad -i\overline{\mathcal{C}}_\mu(x)\{\Phi^\mu(x),P_\nu(x')\}[\dot{\mathcal{C}}^\nu(x')-N^\rho(x'')\mathcal{C}^\lambda(\bar{x})U^\nu_{\lambda\rho}(x'',\bar{x};x')])\ . \end{aligned} \tag{5.10b}$$

The notation $\{\Phi^\mu(x),\mathcal{H}_\nu(x')\}\,|_{\rm cl}$ means to evaluate the Poisson brackets at the classical value of π^{ij}, $\pi^{ij}_{\rm cl}=-h^{1/2}(K^{ij}-h^{ij}K)$. S_L is the sum of the classical Lorentzian action for Einstein's theory, the integral over the Arnowitt-Deser-Misner (ADM) energy and contributions from ghost terms. The gauge choice

$$\Phi^\mu=\dot{N}^\mu-\varphi^\mu \tag{5.11}$$

has been redefined to simplify the form of the ghost term. Performing the integrations over π^{ij}, $\mathcal{P}^\mu$, and $\overline{\mathcal{P}}_\mu$ in (5.8) using the action (5.10a) results in a contribution of the determinant $|N^0G_{ijkl}|^{-1/2}$ to the measure.

The transition functional is now over the configuration space variables. However the part of (5.10b) that gives the Faddeev-Popov determinant still appears in the Hamiltonian form; the infinitesimal gauge transformation of the gauge-fixing term is implemented by the Poisson brackets with the constraints. In order to convert this term into Lorentzian form, where infinitesimal gauge transformations can be implemented by Lie derivatives, we need to make another change of variables. This change can be found by comparing the result of performing an infinitesimal Lorentzian gauge transformation parametrized by the vector f^μ,

$$\delta_f g_{\alpha\beta}=f^\gamma\partial_\gamma g_{\alpha\beta}+g_{\alpha\gamma}\partial_\beta f^\gamma+g_{\beta\gamma}\partial_\alpha f^\gamma\ , \tag{5.12}$$

to the result of performing a Hamiltonian one (3.7) parametrized by the vector F^μ. What choice of F^μ will give the same transformation of the metric using (3.7) as f^μ does using (5.12)? The answer is

$$F^i=f^i+N^if^0,\quad F^0=N^0f^0\ . \tag{5.13}$$

Therefore, in order to rewrite the Poisson brackets in (5.10b) as a Lorentzian gauge transformation the variable $\mathcal{C}^\mu$ has to be changed as indicated in (5.13); this change of variables will introduce a factor of N^0 in the measure. The final result for the parametrized transition functional is

$$G[h'_{ij},t';h_{ij},t]=\int D\bar{\mu}\,\delta[\Phi^\mu]\exp(iS_L)\ , \qquad (5.14a)$$

where

$$l^2S_L=S[g]-\int_t^{t'}dt\,E+S^{\rm gh}[g,\overline{\mathscr{C}},\mathscr{C}]\ , \qquad (5.14b)$$

$$S^{\rm gh}=\int dt[-i\overline{\mathscr{C}}_\mu(x)\delta_{\mathscr{C}}\Phi^\mu(x)]\ . \qquad (5.14c)$$

E is given by (3.5d) and $S(g)$ is the Einstein action (3.1). The measure is

$$D\bar{\mu}=\frac{1}{N^0}\,|N^0G_{ijkl}|^{-1/2}dh_{ij}dN^\mu d\mathscr{C}^\mu d\overline{\mathscr{C}}_\mu$$
$$=g^{00}(g)^{-3/2}dg_{\alpha\beta}d\mathscr{C}^\mu d\overline{\mathscr{C}}_\mu\ . \qquad (5.14d)$$

The notation $\delta_{\mathscr{C}}\Phi^\mu$ means to perform the infinitesimal transformation (5.12) on the gauge-fixing term with $\mathscr{C}^\mu$ as the vector field. The action and measure have been rewritten in terms of the metric $g_{\alpha\beta}$. The integration over P^μ has resulted in a gauge-fixing δ function in the measure of (5.14a). The boundary conditions on the remaining variables are $\mathscr{C}^\mu=\overline{\mathscr{C}}_\mu=0$ and h_{ij} matches the values on the t and t' constant hypersurfaces; the N^μ are integrated over on the initial hypersurface.

By the series of steps sketched above we have derived the Lorentzian transition functional in its fully parametrized form. It is local in the metric variables $g_{\alpha\beta}$ and manifestly invariant. To make this path integral definite, we shall take the class of tensor field configurations summed over to be defined in terms of metric perturbations as discussed in Sec. III for the transition functional (3.15). Choosing an asymptotically flat coordinate system, the metric is written as $g_{\alpha\beta}=\eta_{\alpha\beta}+l\gamma_{\alpha\beta}$. The initial and final data are taken to be of the form $h_{ij}=\delta_{ij}+l\gamma_{ij}$ and the sum is taken over all bounded perturbations $\gamma_{\alpha\beta}$ with the fall-off behavior (3.4) that match this data at t and t'.

The transition functional (5.14a) defined on this class of metric configurations cannot be computed exactly but we can compute its asymptotic expansion in powers of l using perturbation theory. To do this, the action (5.14b) is expanded in the metric perturbation $l\gamma_{\alpha\beta}$ and separated into a quadratic piece and an interaction piece which contains the higher-order terms. One finds that

$$S_L=S_2[\gamma]+S_2^{\rm gh}[\overline{\mathscr{C}},\mathscr{C}]+S_I[\gamma,\overline{\mathscr{C}},\mathscr{C}]\ , \qquad (5.15a)$$

$$S_I=\sum_{k=3}^{\infty}S_k[\gamma]+S_k^{\rm gh}[\gamma,\overline{\mathscr{C}},\mathscr{C}]\ , \qquad (5.15b)$$

where S_2 is the Lorentzian action for linearized gravity corresponding to the linearized Euclidean action (4.1) and $S_2^{\rm gh}$ is the quadratic Lorentzian ghost term corresponding to (4.16b). The S_k and $S_k^{\rm gh}$ are the contributions to S_I of order $k-2$ in l that come from the expansion of the curvature and ghost terms in (5.14b) to the appropriate order in the metric perturbation. The exponential of the interaction term is then written as its power series

$$G[\gamma'_{ij},t';\gamma_{ij},t]=\int D\bar{\mu}\,\delta[\Phi^\mu]\exp[i(S_2+S_2^{\rm gh})]\left[\sum_{j=0}^{\infty}\frac{1}{j!}(S_I)^j\right] \qquad (5.16)$$

and the order of functional integration and integration over spacetime points is interchanged. The terms involving $g_{\alpha\beta}$ in the measure $D\bar{\mu}$ (5.14d) are also expanded in powers of l. For simplicity we will assume that Φ^μ is linear in $g_{\alpha\beta}$. The δ function of the gauge-fixing condition will then be linear in $\gamma_{\alpha\beta}$. If Φ^μ is not linear in the metric, then its δ function will be more complicated to evaluate; basically additional terms will enter into the measure of the functional integral from the change of variables needed to make it linear.

The leading-order contribution to this Lorentzian path integral for the transition functional is simply that for the linearized theory. The changes to the transition functional introduced by the interactions are computed perturbatively to the desired order in l by including the appropriate contributions to the transition functional for the linearized theory from Gaussian integrations over $S_2+S_2^{\rm gh}$ weighted by the interaction terms. In order to explicitly carry out these computations to arrive at physical quantities, the theory needs to be regulated and renormalization counterterms need to be introduced. However, as stated in the Introduction, we are concentrating on finding formal integrals expressing the kinematics of the theory and therefore will assume that the standard procedures for handling the divergences in these quantities can be implemented as needed.

Given the transition functional as computed in perturbation theory the next step is to construct an alternate transition functional that is (1) identical to (5.16) order by order in perturbation theory and (2) convergent when $t\to -i\tau$. The observation that the leading-order term in (5.16) is linearized gravity and the discussion of Sec. IV suggest the following approach to finding a nonlocal action for the full theory that is physically equivalent to the local invariant one (5.14b). First decompose the metric perturbation into the equivalence classes of the linearized theory (4.5), $\gamma_{\alpha\beta}=\phi_{\alpha\beta}+2\chi\eta_{\alpha\beta}$ where $\chi=0$ on the boundaries. Using this set of variables (5.16) becomes

$$G[\gamma'_{ij},t';\gamma_{ij},t]=\int D\mu'\delta[\Phi^\mu]\delta[R^{(1)}(\phi)]\exp(iS_2[\phi,\chi]+iS_2^{\rm gh}[\overline{\mathscr{C}},\mathscr{C}])\left[\sum_{j=0}^{\infty}\frac{1}{j!}(S_I[\phi,\chi,\overline{\mathscr{C}},\mathscr{C}])^j\right]\ , \qquad (5.17)$$

where S_2 and S_I are now written in terms of the decomposition (4.5) and

$$D\mu'=[g^{00}(g)^{-3/2}]\left|\frac{\delta R^{(1)}(\phi)}{\delta\omega}\right|d\phi_{\alpha\beta}d\chi\,d\mathscr{C}^\mu d\overline{\mathscr{C}}_\mu\ . \qquad (5.18)$$

The factors of g in (5.18) are also written in terms of the decomposition (4.5). This form of the transition amplitude is identical to (5.16) as it differs only by a change of variables. To lowest order in l, (5.18) is the transition functional for linearized gravity weighted by the local action S_2. In Sec. IV we argued that the alternate transition functional (4.21) weighted by the nonlocal action $\hat{S}_2$ (4.22) is physically equivalent. Moreover, when a is positive, the Feynman-Kac procedure implemented using this alternate functional produces a manifestly convergent Euclidean functional integral for the ground state. Therefore we will construct the alternate transition functional by using $\hat{S}_2(\phi,\chi)$ instead of $S_2(\phi,\chi)$ in the exponent of (5.17) and then appropriately modifying the interaction terms so as to obtain the same result. It is convenient to set $a=24$ in $\hat{S}_2$ to match the absolute value of the coefficient of the corresponding term in S_2. Then we will demonstrate the equivalence.

The next step is to find the correct interaction terms for the alternate transition functional. This can be done by studying the form of the Gaussian integrations over χ in (5.17). An arbitrary interaction term in (5.17) can be written as

$$(S_I[\phi,\chi,\bar{\mathcal{C}},\mathcal{C}])^j=\int \chi(x_1)\chi(x_2)\cdots\chi(x_n)\times F(\phi,\bar{\mathcal{C}},\mathcal{C};x_1,x_2,\ldots,x_n)\,, \quad (5.19)$$

where F is a function of the other variables and will in general contain δ functions of the coordinates and partial derivatives. The integrations over χ in (5.17) are then of the form

$$F[\phi,\bar{\mathcal{C}},\mathcal{C}]=\int d\chi \exp\left[+6i\int_M d^4x\,(\partial_\alpha\chi)^2\right]\times\int[\chi(x_1)\chi(x_2)\cdots\chi(x_n)\times F(\phi,\bar{\mathcal{C}},\mathcal{C};x_1,x_2,\ldots,x_n)]\,. \quad (5.20)$$

The sign of the χ term in S_2 has opposite sign from that of $\hat{S}_2$. Because $\chi=0$ on the boundaries, the classical contribution to this integral vanishes. The only contributions to this integral will come from the fluctuations around the classical path; this means that only interaction terms even in χ will contribute as Gaussian integrals over an odd number of variables vanishes. If n is even then the integration over χ in (5.20) will give a factor of $(-1)^{n/2}$ relative to integration over the same interaction term weighted by an exponential with opposite sign. If we now modify the interaction terms by taking χ to $i\chi$, as well as changing the sign in the exponent then the added factor of $(i)^n$ will give the same contribution; consequently

$$F[\phi,\bar{\mathcal{C}},\mathcal{C}]=\int d\chi\exp\left[-6i\int_M d^4x\,(\partial_\alpha\chi)^2\right]\int (i)^n\chi(x_1)\chi(x_2)\cdots\chi(x_n)F(\phi,\bar{\mathcal{C}},\mathcal{C};x_1,x_2,\ldots,x_n)\,. \quad (5.21)$$

A transition functional weighted by a nonlocal action that is physically equivalent to (5.17) in perturbation theory is thus

$$G[\gamma'_{ij},t';\gamma_{ij},t]=\int D\mu'\delta[\Phi^\mu]\delta[R^{(1)}(\phi)]\exp(i\hat{S}_2[\phi,\chi]+iS_2^{\mathrm{gh}})\left[\sum_{j=0}^{\infty}\frac{1}{j!}(S_I[\phi,i\chi,\bar{\mathcal{C}},\mathcal{C}])^j\right], \quad (5.22)$$

where the factors of χ that appear in the measure (5.18) are also taken to $i\chi$. The modified action in (5.22) is complex; however, the resulting transition functional is the same as that weighted by the real action because only terms even in χ contribute to it.

Now the rotation of the time coordinate can be carried out before functional integration order by order in l because the resulting quadratic action $\hat{I}_2$ is manifestly positive definite. The resulting parametrized Euclidean functional integral for the ground-state wave functional (3.18) is

$$\Psi_0[\gamma_{ij}]=\int D\mu'\delta[\Phi^\mu]\delta[R^{(1)}(\phi)]\exp(-\hat{I}_2-I_2^{\mathrm{gh}})\left[\sum_{j=0}^{\infty}\frac{1}{j!}(I_I[\phi,i\chi,\bar{\mathcal{C}},\mathcal{C}])^j\right], \quad (5.23)$$

where I_I are the Euclideanized interaction terms corresponding to (5.15b). This integral is convergent and O(4) invariant; it does not manifestly display the coordinate invariance of the theory because the conformal factor was isolated using the linearized scalar curvature. However, its action is simply related to the manifestly gauge invariant one of the full theory.

Can we recover a parametrized, convergent Euclidean functional integral that is manifestly diffeomorphism invariant? The answer is a qualified yes. One can do so by using an alternate Lorentzian perturbation theory to evaluate (5.14); however, because this new theory is very nonlocal in terms of the original perturbations, it does not obviously reproduce the same physical integral. To get this alternate Lorentzian perturbation theory one first forms the identity

$$1=\int d\Omega\left|\frac{\delta R(\Omega^{-2}g)}{\delta\Omega}\right|\delta[R(\Omega^{-2}g)]\,, \quad (5.24)$$

where the scalar curvature is evaluated on $\Omega^{-2}g_{\alpha\beta}$. The integration over Ω is defined to be over configurations of the form $\Omega=1+l\chi$ where $\chi\sim 1/r$ at spatial infinity and vanishes on the boundaries. The metric is assumed to be perturbative, $g_{\alpha\beta}=\eta_{\alpha\beta}+lh_{\alpha\beta}$, with appropriate falloff behavior as before. This identity is true if the scalar curvature of $g_{\alpha\beta}$ is sufficiently small. One then inserts (5.24) into (5.14) and then changing variables to $g_{\alpha\beta}=\Omega^2\tilde{g}_{\alpha\beta}$ one obtains

$$G[h'_{ij},t';h_{ij},t]=\int D\tilde{\mu}\,\delta[\Phi^\mu]\delta[R(\tilde{g})]\exp(iS_L)\,, \quad (5.25a)$$

where

$$D\tilde{\mu}=\tilde{g}^{\,00}(\tilde{g})^{-3/2}\Omega^4\left|\frac{\delta R}{\delta\Omega}\right|d\tilde{g}_{\alpha\beta}d\chi\, d\mathscr{C}^{\mu}d\overline{\mathscr{C}}_{\mu}\ . \tag{5.25b}$$

S_L is (5.14b) evaluated with $g_{\alpha\beta}=\Omega^2\tilde{g}_{\alpha\beta}$. One can in principle define this integral in a perturbation expansion in l using $\tilde{g}_{\alpha\beta}=\eta_{\alpha\beta}+l\phi_{\alpha\beta}$, $\Omega=1+l\chi$; however, in practice evaluating it is difficult because one has to solve a nonlinear equation for $\phi_{\alpha\beta}$ in order to integrate over the decomposition fixing δ function. If one proceeds by solving this equation for one component of $\phi_{\alpha\beta}$ in terms of the others as a power series in l, one can then similarly expand the action and measure by performing this substitution. The leading order of (2.25a) is found to be the transition amplitude for linearized gravity weighted by its action $S_2[\phi,\chi]$. One again can make the same arguments of (5.18)–(5.22) to derive the corresponding nonlocal action. The resulting Euclidean functional integral in this case is then

$$\Psi_0[\gamma_{ij}]=\int D\tilde{\mu}\,\delta[\Phi^{\mu}]\delta[R(\tilde{g})]\exp(-\hat{I})\ , \tag{5.26a}$$

where

$$\hat{I}=I[\tilde{g},\tilde{\Omega}]-\int d\tau\, E[\phi,i\chi]+I^{\rm gh}[\phi,i\chi,\overline{\mathscr{C}},\mathscr{C}]\ . \tag{5.26b}$$

$I[\tilde{g},\Omega]$ is the Euclidean action for Einstein gravity (1.6) evaluated for $\Omega=1+il\chi$. E and $I^{\rm gh}$ are the appropriate forms of (3.5d) and (5.14c) and again, factors of χ are taken to $i\chi$ in the measure. The above functional integral is to be evaluated perturbatively in the same manner as (5.23). The difficulty with this form is checking that it equals (5.23) order by order in perturbation theory; that is, that the interactions obtained by solving the decomposition-fixing δ function perturbatively in $\phi_{\alpha\beta}$ give the same results as the local Lorentzian perturbation theory. That it does can be verified to the next order in l after linearized gravity explicitly; the equivalence was checked with the help of the MACSYMA ITENSR package. However, it is difficult to carry this out to higher orders. This Euclidean functional integral is manifestly diffeomorphism invariant and convergent. It is weighted by a nonlocal action different than that used in (2.23) and it is plausible, though not explicitly verified to all orders, that it equals the functional integral in the physical fields. We thus have obtained two prescriptions for Euclidean functional integrals with manifestly positive actions. This emphasizes that the form of the convergent parametrized Euclidean functional integral is not unique when its action is nonlocal in the original set of variables.

VI. CONCLUSION

When the Euclidean action for a theory with invariances is unbounded below, Euclidean functional integrals for its quantum states may or may not exist. Whether or not they do is determined by the action for the theory expressed in the physical degrees of freedom. In the case of linearized gravity, the physical degrees of freedom can be explicitly identified and the Hamiltonian in terms of them is positive definite. One can proceed to quantize the theory in its physical variables and indeed can form convergent Euclidean functional integrals in terms of the transverse traceless metric perturbations. Although the physical variables cannot be explicitly isolated for Einstein's theory in asymptotically flat spacetimes, the Hamiltonian is positive for metric configurations that satisfy the constraints by the positive-energy theorem. Again this means that convergent Euclidean functional integrals for the theory can be given in terms of the physical variables. However, in either theory, because of the unboundedness of the Euclidean action, one cannot find expressions equal to those given in the physical fields involving an action that are manifestly invariant and local in the full set of variables. Still, one can find useful parametrized forms of the Euclidean integrals that come close to achieving these goals.

For linearized gravity, we can recover functional integrals weighted by a manifestly gauge- and O(4)-invariant Euclidean action. It is even local in the set of variables $\phi_{\alpha\beta}$ and χ. However it is nonlocal when expressed in the original metric perturbation $\gamma_{\alpha\beta}$. The action weighting these convergent Euclidean functional integrals is the same as would be obtained by conformally rotating χ in the classical Euclidean action (3.1) for linearized gravity.

For the case of asymptotically flat spacetimes, the task of finding convergent parametrized Euclidean functional integrals is more difficult because the physical variables cannot be isolated explicitly. However, as we showed, one can proceed by using the manifestly invariant form of the Lorentzian transition functional as a guide. One looks for a parametrized transition functional weighted by an alternate nonlocal action that equals the first when both are evaluated in perturbation theory. There are many such nonlocal actions which will have this property. In addition we require that this action be chosen so that the Euclidean functional integrals corresponding to the alternate transition functional will be manifestly convergent. In Sec. V we demonstrated that one such choice resulted in manifestly convergent, O(4)-invariant Euclidean functional integrals. This action was not manifestly diffeomorphism invariant; however it was simply related to the manifestly invariant one of the full theory. This particular form of the Euclidean functional integral is local in the variables $\phi_{\alpha\beta}$ and χ of the linearized theory used to derive it; in this sense it is almost local in the metric perturbations $\gamma_{\alpha\beta}$. This property makes it an especially convenient form for calculation.

Can convergent Euclidean functional integrals for physical quantities be weighted by manifestly diffeomorphism invariant actions? The procedure by which (5.24) was derived suggests that such integrals correspond to another choice of a nonlocal action. However, there is a price to be paid; such integrals are highly nonlocal in the metric perturbations. This makes explicitly verifying their equivalence to those expressed in the physical degrees of freedom via perturbation theory difficult. However, as discussed at the end of Sec. V, the equivalence can be verified to hold through the first order in the interaction for the conformally rotated action of the full theory.

How are these Euclidean functional integrals for both linearized and perturbative gravity related to the conformally rotated ones of Gibbons, Hawking, and Perry? Their form suggests that these functional integrals could be obtained formally by an appropriate distortion of the contour of integration over χ to $i\chi$. However, this prescription of conformal rotation begins with an Euclidean functional integral that is manifestly divergent and consequently not well defined. In addition, it is hard to get the correct Jacobian factor in the measure by this method. Moreover, we showed that there are many more prescriptions for convergent Euclidean functional integrals than this one as there are many possible positive nonlocal actions. A more satisfactory way to view these integrals is the one presented in this paper; they arise naturally in the course of quantizing a theory with invariances. First one isolates the physical degrees of freedom. Then one constructs functional integrals for physical quantities in terms of them. Finally one adds in integrations over the redundant variables to recover manifest invariance. When doing this for quantities expressed as Euclidean functional integrals, there is an additional restriction on the parametrization process; one is only allowed to add back in manifestly convergent quantities. However, the result of this process is mostly determined by the form that is desired for the final answer; the content of the theory is contained in the functional integrals given in the physical variables.

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APPENDIX: THE MEASURE

To show that the functional integrals in (4.15) and (4.16) have a definite and concrete meaning, we shall evaluate the factors making up the measure $d\phi_{\alpha\beta}d\chi$, etc., in a particular set of "coordinates on the function spaces" with the specific gauge choice (4.11a). To make the mode sums discrete, we will take our spacetime to be a finite box of volume L^4 interior to the planes $\tau=0$, $\tau=-L$, and $x^i=\pm L/2$. We shall then use the coefficients of the Fourier expansion of the integration variables in this box as our coordinates on the function space. The expansion is

$$\begin{aligned} t_{\alpha\beta}(x) &= t^{\rm cl}_{\alpha\beta}(x)+\sum_{\nu=1}^{2}\sum_k{}' t^{(\nu)}(k)t^{(\nu)}_{\alpha\beta}(k,x) , \\ l_{\alpha\beta}(x) &= \sum_{\nu=1}^{3}\sum_k{}' l^{(\nu)}(k)l^{(\nu)}_{\alpha\beta}(k,x) , \\ \phi^L_{\alpha\beta}(x) &= \sum_{\nu=1}^{4}\sum_k{}' \phi^{L(\nu)}(k)\phi^{L(\nu)}_{\alpha\beta}(k,x) , \\ \phi^T_{\alpha\beta}(x) &= \sum_k{}' \phi^T(k)\phi^T_{\alpha\beta}(k,x) , \\ \chi(x) &= \sum_k{}' \chi(k)s(k,x) . \end{aligned} \qquad (A1)$$

This expansion is done by expanding around the classical solution of the linearized Einstein equations which satisfies the boundary conditions as fixed in Sec. IV. $t^{\rm cl}_{\alpha\beta}(x)$ is the classical solution which matches the argument of the wave function h^{TT}_{ij} on the $\tau=0$ boundary and vanishes on the other boundary surfaces; with our boundary conditions, the classical solutions of the other metric components vanish. (We assume that the finite volume box is chosen to be large enough so that the compact support of the initial data at $\tau=0$ is interior to it.)

The class of configurations that is summed over for the fluctuations around the classical solution is specified by the boundary conditions that the spatial components of the fields vanish at $\tau=0$ and $\tau=-L$ and are periodic in the spatial directions. The gauge choice we shall use is (4.11a). The modes on the right-hand side of (A1) will be constructed to satisfy these conditions. The notation $\sum_k'$ in (A1) means the sum over all $k^0>0$ and k such that $k^i\neq 0$. Modes with $k^i=0$ will have infinite action in the infinite-volume limit and thus will not contribute to the functional integrals; we omit them for convenience in defining the tensor modes. To explicitly construct these modes it is useful to first define, for a given k^α satisfying the above restrictions,

$$\begin{aligned} s(k,x) &= \frac{2}{L^2}\sin(k_0\tau)\sin(k_ix^i), \quad k^3>0 \\ &= \frac{2}{L^2}\sin(k_0\tau)\cos(k_ix^i), \quad k^3<0 , \\ p_\alpha(k,x) &= \frac{1}{kk_0}s^\gamma\partial_\gamma\dot{s}(k,x)n_\alpha+\frac{k_0}{k}s(k,x)s_\alpha , \\ p_{\alpha\beta}(k,x) &= \frac{1}{k^2}[(s^\gamma\partial_\gamma)^2 s(k,x)n_\alpha n_\beta \\ &\quad -2s^\gamma\partial_\gamma\dot{s}n_{(\alpha}s_{\beta)}+\ddot{s}s_\alpha s_\beta] , \end{aligned} \qquad (A2)$$

where $s_\alpha(k)$ is the unit vector in the direction of the projection of k^α onto the space orthogonal to n^α and $k=(k_\alpha k^\alpha)^{1/2}$. Using $\epsilon^{(\nu)}_\alpha$, two orthonormal vectors transverse to both k_α and s_α, we then construct the unit tensors

$$\begin{aligned} t^{(1)}_{\alpha\beta}(k) &= \sqrt{2}\,\epsilon^{(1)}_{(\alpha}\epsilon^{(2)}_{\beta)} , \\ t^{(2)}_{\alpha\beta}(k) &= \frac{1}{\sqrt{2}}(\epsilon^{(1)}_\alpha\epsilon^{(1)}_\beta-\epsilon^{(2)}_\alpha\epsilon^{(2)}_\beta) , \\ \phi^T_{\alpha\beta}(k) &= \frac{1}{\sqrt{2}}(\epsilon^{(1)}_\alpha\epsilon^{(1)}_\beta+\epsilon^{(2)}_\alpha\epsilon^{(2)}_\beta) . \end{aligned} \qquad (A3)$$

Then the tensor modes are

$$
\begin{aligned}
&t^{(v)}_{\alpha\beta}(k,x)=s(k,x)t^{(v)}_{\alpha\beta}(k)\ ,\\
&\phi^{T}_{\alpha\beta}(k,x)=s(k,x)\phi^{T}_{\alpha\beta}(k)\ ,\\
&\phi^{L(v)}_{\alpha\beta}(k,x)=\frac{\sqrt{2}}{k}\epsilon^{(v)}_{(\alpha}\partial_{\beta)}s(k,x),\quad v=1,2\ ,\\
&\phi^{L(3)}_{\alpha\beta}(k,x)=\frac{\sqrt{2}}{k}\partial_{(\alpha}p_{\beta)}(k,x)\ ,\\
&\phi^{L(4)}_{\alpha\beta}(k,x)=\frac{1}{k^2}\partial_\alpha\partial_\beta s(k,x)\ ,\\
&l^{(v)}_{\alpha\beta}(k,x)=\sqrt{2}\epsilon^{(v)}_{(\alpha}p_{\beta)}(k,x),\quad v=1,2\ ,\\
&l^{(3)}_{\alpha\beta}(k,x)=(\tfrac{1}{3})^{1/2}\phi^{T}_{\alpha\beta}(k,x)-(\tfrac{2}{3})^{1/2}p_{\alpha\beta}(k,x)\ .
\end{aligned}
\tag{A4}
$$

These tensor modes are real and normalized to 1 on the finite volume. The real functions $t^{(v)}(k)$, $\phi^{T(v)}(k)$, $\phi^{L(v)}(k)$, and $\chi(k)$ become the coordinates on the space of functions over which we integrate and the actions may be expressed in terms of them. For example, the action I^g_2 (4.12) is

$$
\begin{aligned}
&I^g_2=\sum_k{}' I^g_2(k)\ ,\\
&I^g_2(k)=\frac{k^2}{4}\sum_{v=1}^{3}\{[l^{(v)}(k)]^2+a[\chi(k)]^2\}\ ,
\end{aligned}
\tag{A5}
$$

and the action i_2 is

$$
\begin{aligned}
&i_2=i^{\rm cl}_2[h^{TT}_{ij}]+\sum_k{}' i_2(k)\ ,\\
&i_2(k)=\frac{k^2}{4}\sum_{v=1}^{2}[t^{(v)}(k)]^2\ ,
\end{aligned}
\tag{A6}
$$

and $i^{\rm cl}_2[h^{TT}_{ij}]$ is the classical action evaluated for the appropriate classical solution. The functional δ functions and Faddeev-Popov determinants are, in the gauge (4.11a),

$$
\delta[\partial_\alpha\phi^{\alpha\beta}]\left|\frac{\delta G^\alpha}{\delta f^\beta}\right|=\prod_k{}' D_1\left[\prod_{v=1}^{4}\delta[k\phi^{L(v)}(k)]\right][k^2]^4\ ,
\tag{A7a}
$$

$$
\delta[R^{(1)}(\phi^T)]\left|\frac{\delta R^{(1)}}{\delta\omega}\right|=\prod_k{}' D_2\delta[k^2\phi^T(k)][k^2]\ ,
\tag{A7b}
$$

where D_1 and D_2 are numerical constants determined by the normalization of the fields and implementation of the δ functions and determinants. The products of factors $[k^2]^4$ and $[k^2]$ in Eqs. (A7) are the Faddeev-Popov determinants represented as modes. Defining the measure (4.13) as

$$
dl_{\alpha\beta}d\phi^L_{\alpha\beta}d\chi=\prod_k{}'\frac{N}{D_1}\left[\prod_{v=1}^{3}dl^{(v)}(k)\right]\left[\prod_{\mu=1}^{4}d\phi^{L(\mu)}(k)\right]d\chi(k)\ ,
\tag{A8a}
$$

$$
d\phi^T_{\alpha\beta}=\prod_k{}'\frac{1}{D_2}d\phi^T(k)\ ,
\tag{A8b}
$$

and using (A5) and (A7) Eqs. (4.15) become (suppressing the label k on the mode amplitudes)

$$
\begin{aligned}
&1=\int\prod_k{}' N\left[\prod_{v=1}^{3}dl^{(v)}\right]d\chi\left[\prod_{\mu=1}^{4}d\phi^{L(\mu)}\delta[k\phi^{L(\mu)}]\right][k^2]^4\exp[-I^g_2(k)]\ ,\\
&1=\int\prod_k{}' d\phi^T\delta[k^2\phi^T][k^2]\ ,
\end{aligned}
\tag{A9}
$$

where N is a constant needed to set (A9) to unity including a factor of $a^{1/2}$ as well as other numerical constants.

Using the Fourier modes (A1) the path integral over the physical degrees of freedom can also be made concrete. Defining the measure over the fluctuations around the classical solution to be

$$
dt_{\alpha\beta}=\prod_k{}'\frac{\pi}{4}\left[\prod_{v=1}^{2}dt^{(v)}(k)\right]
\tag{A10}
$$

and using (A6), the wave functional is

$$
\Psi_0[h^{TT}_{ij}]=\mathcal{N}\exp(-i^{\rm cl}_2[h^{TT}_{ij}])\int\prod_k{}'\frac{\pi}{4}\left[\prod_{v=1}^{2}dt^{(v)}(k)\right]\exp[-i_2(k)]\ ,
\tag{A11}
$$

where $\mathcal{N}$ is a normalization parameter that can be computed explicitly in the following way. First evaluate the path integral (A11) over the fluctuations in the finite volume L^4. Then fix $\mathcal{N}$ by requiring that the resulting wave functional be the normalized product of the ground-state harmonic-oscillator wave functions whose arguments are the amplitudes of the Fourier transform of $h^{TT}_{ij}(x)$ in the appropriate measure as $L\to\infty$. One finds that $\mathcal{N}$ is the properly normalized ground-state wave functional for 0 initial data on the boundary surface at $\tau=-L$.

Finally for completeness we give the form of the parametrized wave functional (4.18) in the Fourier coordinates after integration over the Grassmann fields:

$$\Psi_0[h_{ij}^{TT}]=\mathcal{N}\int \prod_k{}' \frac{\pi N}{4} d^{10}\phi\, d\chi \left[\prod_{\mu=1}^{4} \delta[k\phi^{L(\mu)}]\right] \delta[k^2\phi^T][k^2]^5 \exp(-\hat{I}_2[h_{ij}^{TT},t^{(\nu)},l^{(\nu)},\chi]) \, , \tag{A12}$$

where $\prod_k' [\pi N/(4D_1D_2)]d^{10}\phi\, d\chi$ is the product of (A8a), (A8b), and (A10) and $\hat{I}_2$ is the sum of (A5) and (A6).

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