Gravitational radiation damping in systems with compact components

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The radiation reaction force and balance equations are derived for slow-motion, gravitationally bound systems with compact components such as neutron stars and black holes. To obtain these results, use is made of the Einstein-Infeld-Hoffmann (EIH) procedure. As a consequence, all quantities involved in the derivation are finite and hence no renormalization is required. Furthermore, no laws of motion for the components need be assumed. Approximate expressions for the fields required to evaluate the EIH surface integrals are obtained using the methods of matched asymptotic expansions and multiple time scales. The results obtained are the same as those derived previously for systems with noncompact components.

I. INTRODUCTION

Over the years numerous derivations of the gravitational radiation reaction force and balance equations have been given by various authors using many different approximation methods and assumptions.[1] The following question then arises: Are any of these derivations applicable to a system such as the binary pulsar PSR 1913+16? Since they all seem to lead to the same expressions for these quantities and since the prediction of the period change that follows from them agrees with observation to within observational error,[2] the answer appears to be yes. Nevertheless, an examination of these derivations shows that, as they stand, none of them can be applied to a gravitationally bound system with compact components such as neutron starts or black holes. The reasons for not being able to do so vary from derivation to derivation, but include one or more of the following.

(i) Source model not applicable. The binary pulsar PSR1913+16 consists of a neutron star and another compact object with a mass equal to 1.45 solar masses. While it is most likely that this other object is another neutron star, the possibility that it is a black hole cannot entirely be ruled out. Thus, source models capable of describing both types of objects are needed. Neither δ-function nor continuous matter sources can serve as models for black holes. δ-function sources bear no obvious relation to either neutron stars or black holes. Furthermore, the fields associated with such sources become arbitrarily large near the singularity so that no perturbation method can be applied in this region. Also, no perturbation method can hope to lead to the existence of an event horizon. While it is possible to model a neutron star in terms of a matter stress-energy tensor, one cannot use a model with weak internal gravity since the size of a neutron star may be only a few times its Schwarzschild radius.

(ii) Incomplete or inappropriate derivation. Any attempt to apply the results of the linearized theory to a gravitationally bound system must fail as Eddington[3] had long ago pointed out. Formal expansions can also give rise to misleading results. Thus, the first term in one formal fast-motion expansion predicted gravitational antidamping, while still other derivations predicted no damping at all. Derivations that resulted in such erroneous results in general neglected terms of the same order of magnitude as those retained.[4] On the other hand, a number of derivations made unnecessary or unjustified assumptions to obtain their final result. Several derivations, for example, take it for granted that one can equate the flux calculated by the use of the so-called quadrupole formula to the time rate of change of the Newtonian energy of the source. Other derivations assume that the first time-odd contribution to the time-time component of the gravitational field can be used as a gravitational potential in the Newtonian equations of motion. Finally, some derivations assume the validity of the geodesic equations of motion for the sources. Aside from the problem of what field to use in the case of compact sources, there is a much more serious objection to this procedure. In general relativity, the motion of the sources of the gravitational and electromagnetic fields follow from the field equations.[5,6] There is, thus, no need to make such an assumption, nor is it at all clear that it is consistent with these motions.

(iii) Mathematical inconsistencies. A number of approximation schemes lead to divergent integrals when carried to sufficiently high order and, hence, introduce uncontrolled errors even if the lower orders are finite and yield reasonable results.[7] Likewise, a number of derivations that make use of δ-function sources require the use of some form of regularization or infinite renormalization to obtain finite results. As we will see, such dubious procedures are completely unnecessary in general relativity. Finally, the radiation reaction force may lead to unphysical motions, e.g., runaway solutions, as it does in special relativistic derivations of the reaction force in electrodynamics.

It is not our intention to enter here into a detailed analysis of previous derivations, but rather to present an essentially new derivation of the effects of radiation reaction on gravitationally bound systems with compact components, which we believe avoids the difficulties outlined above. As such, it is the first derivation that can, with some confidence, be applied to the binary pulsar. We should perhaps add, to avoid unnecessary anxiety, that our results agree with those used to calculate the orbital period change in the binary pulsar.

Our basic procedure for deriving radiation reaction effects will be an application of the surface integral method first used by Einstein, Infeld, and Hoffmann[8] (EIH) to derive post-Newtonian corrections to the equations of motion of gravitationally bound systems. Although Goldberg[9] used the surface integral formalism to discuss the question of the existence of gravitational radiation, he argued that it could not be used for the detailed calculation of the effects of such radiation. Since then it has not been used for this purpose. This is especially strange since it is the only way we know of for dealing with compact sources without encountering one or more of the difficulties outlined above.

In order to apply the surface integral formalism it is necessary to construct solutions to the field equations corresponding to a number of compact objects in relative motion with respect to each other. Since no exact solutions of this kind exist, one must be content with approximate solutions. In their original papers, EIH employed what they called a slow-motion expansion to construct such solutions. This involved an expansion of the field variables in a small dimensionless parameter λ. In addition, they assumed that these fields depended on the time t through the combination λt; i.e., time derivatives were treated as higher order than spatial derivatives. EIH did not specifically identify λ, and later authors who employed this slow-motion approximation took it to be the reciprocal of the velocity of light. Since this is a dimensioned quantity, it can be given any value one desires by a change of units and is therefore inappropriate as an expansion parameter. Furthermore, such an expansion leaves open the question as to what type of system the approximation can be applied.

Since it is beyond our present powers to devise an approximation scheme that would be applicable to all types of systems, we shall be content to develop an approximation scheme that is applicable to slow-motion systems such as the binary pulsar. In such systems one can identify a slow-motion parameter ϵ, the ratio of the light travel time across the system to a characteristic time such as its orbital period. It is this parameter that we shall use in our approximation scheme. If its value is small compared to one, as it must be for slow motion (in the binary pulsar its value is approximately 10^{-4}), then the ratio of successive terms in an ϵ expansion will, in general, be small compared to 1.

It is unfortunately not possible to make a more precise statement in this regard since one cannot rule out entirely the possibility that the coefficients in an ϵ expansion will be such highly singular functions of their arguments that, for values of these arguments in the region of interest, the ratio of successive terms might not be small compared to one. In such a case, however, it is unlikely that any expansion scheme would lead to a useful approximation in this region. In what follows, we will assume that this is not the case, that the coefficients in our ϵ expansions are well-behaved functions of their arguments and that, therefore, the ratio of successive terms in these expansions are of the order of the ratio of the functions of ϵ that they multiply.

It would perhaps be naive to imagine that a single approximation scheme would suffice to construct a solution even for a slow-motion system. In fact, one needs to combine a number of such schemes for this purpose.[10] One of these, the method of matched asymptotic expansions (MAE) was first introduced into this subject by Burke.[11] The great virtue of this method is that it allows one to expand the fields in different ways in different regions of space-time. Thus, one can use the most efficient expansion in each region and then join them together by the matching procedure used in this method. Thus, in dealing with a slow-motion system one can employ a slow-motion expansion in the inner or induction zone and a fast-motion or weak-field expansion in the outer or wave zone. In order to obtain the needed accuracy in the inner zone using only a weak-field expansion one would need to go to third order in the weak-field expansion parameter, whereas using the MAE requires only first-order accuracy to calculate the lowest-order radiation reaction effects. The resultant saving in labor is enormous. Furthermore, it is the only consistent way known for incorporating radiation into a slow-motion approximation.

In addition to the MAE, one must employ several other approximation schemes to deal with the various nonuniformities that arise in the course of the construction. Unfortunately, there is no set prescription one can use for this purpose. Whenever one encounters a nonuniformity one must experiment with the various techniques that have been developed in the past to see which one works best. Unsatisfactory as this may appear, it is no worse and in many cases, considerably better, than the formal expansion schemes that have been used in the past. Of course, what one must require of any approximation scheme is that it does not lead to any obvious inconsistencies such as divergent integrals. The approximation schemes used here have at least that property.

Two approximation schemes in particular will be used here to deal with nonuniformities. One of them, the method of stretched coordinates (MSC) was used by us to deal with the $\ln r/r$ type nonuniformities that arise in the outer zone problem.[12] One finds that one can eliminate these terms by using the true retarded coordinate rather then the flat-space coordinate $t-r$ and, in addition, by modifying the standard deDonder coordinate condition.[13] Fortunately, these modifications have no effect on the lowest-order radiation reaction effects.

In addition to the MSC we have also employed the method of multiple-time scales (MTS) to deal with secular nonuniformities in time that arise in the course of the approximation. In addition to this function it also serves to obviate the need for the dipole terms EIH found necessary to introduce in order to satisfy the integrability conditions on their approximate field equations. In the end, they set the totality of these terms equal to zero, making the whole process seem somewhat arbitrary. In Sec. II we will give an example of the use of the multiple-time formalism in solving an equation of motion similar to the one encountered later in the paper for the motion of gravitationally bound compact sources with radiation reaction.

In order to construct approximate solutions by the approximation methods outlined above, it is necessary to characterize the kind of source with which we are dealing. Since the surface integral method only requires a knowledge of the field on and outside closed surfaces surrounding the individual components of the source, we will in fact characterize it by the types of fields these components produce. To do so, we will assume in what follows that the source consists of a number of compact components, each of which would, in the absence of the others, produce a spherically symmetric Schwarzschild field. (By compact we mean here that the largest distance scale associated with a given component, e.g., physical radius, Schwarzschild radius, etc., is small compared to the distance between components.) Furthermore, we will assume that these components possess no internal dynamics of their own and that the only distortion of their spherically symmetric fields is produced by their mutual interactions. These assumptions are equivalent to requiring that the components be either Schwarzschild black holes or else rigid, spherically symmetric mass distributions. (Within the context of a slow-motion approximation it makes sense to speak of rigid bodies: the ratio of the sound speed in such a body to its orbital speed is large compared to one.)

For a black-hole component these assumptions are clearly justified in all orders of approximation. For the neutron star(s) in PSR1913+16 they are justified to a high degree of accuracy. If we assume a star diameter of 10 km, the tidal distortion of its surfaces will be of the order of 10^{-9} cm and will therefore, have a completely negligible effect on the radiation emitted by the system. Furthermore, since a neutron star is essentially a superfluid, tidal friction will be nonexistent. The surface integral method is in fact quite capable of taking into account these distortions as well as the rapid rotation of the neutron stars. We shall not however consider these complications here since we are primarily concerned with radiative effects in such systems where their contributions will play no role.

In constructing approximate solutions to the field equations, these assumptions concerning the nature of our source are translated into the requirement, first employed by EIH, that the only allowed homogeneous solutions of the Poisson equations which arise in each order of the inner zone approximation, other than those determined by matching, be spherically symmetric. Higher-order multipole solutions would correspond to sources with internal structure of some kind and, hence, would require a knowledge of the internal dynamics of the source components for their determination.

In addition to characterizing the source structure in the inner zone it is necessary to restrict in some way the solutions in the outer or wave zone. In one way or another, such a restriction should reflect the fact that we are dealing with an isolated system, that is, that there is no coherent radiation incident upon it. There have been many attempts to deal with this problem, but none of them have proven to be completely satisfactory, especially when attempting to incorporate them into an approximation scheme. The simplest approach to this problem would be to assume that, in the lowest order of approximation, any radiation must be purely outgoing.

Schutz[14] has argued that one need make no such time-asymmetric assumption and that the time evolution of a source can be obtained by averaging over a statistical ensemble of slow-motion initial field configurations. He was able to show that such an approach yielded the expected radiation reaction force when applied to a weak-field, slow-motion source. On the other hand, Aichelburg and Beig[15] have shown, for the case of a model system, that its asymptotic behavior is independent of initial conditions, provided the energy of the initial fields is finite. Numerical studies by Anderson and Hobill[16] of both linear and nonlinear model systems bear out these conclusions. As suggestive as these results are, there seems to be at present no easy way to incorporate them into the EIH scheme for the simple reason that an initial-value problem by its very nature will violate the slow-motion assumption—in general, arbitrarily large frequencies will be present in the field components initially. It is of course possible to construct initial data that does not violate the slow-motion assumption[17] but they constitute only a small subset of all such data. What the EIH approach attempts to do is derive approximate equations of motion for a system far into the future of its initial startup time.

Our approach to the problem will be to assume that after some time sufficiently far in the past no incoming radiation was incident on the system. As a consequence we can require that, at least in the lowest order of approximation, the fields in the radiation zone will be pure outgoing fields. A problem arises however in constructing solutions of the inhomogeneous wave equations that one encounters in higher orders of approximation. Since the gravitational field acts like an inhomogeneous refractive medium, some part of the radiation emitted by a system will be back scattered, making it difficult to distinguish between this back-scattered radiation and incoming radiation. In order to overcome this difficulty we propose the use of the following causality condition: if the arbitrary functions of the retarded coordinate appearing in the lowest-order approximation of the wave zone fields are zero for all values of this coordinate less than some fixed value, then all of the higher-order solutions will have this property.[18]

The way in which equations of motion are obtained from the field equations in the EIH method is explained in Sec. III. Inner solutions of the field equations are also derived in this section and are used to obtain the Newtonian equations of motion for a gravitationally bound system as an illustration of the method. In Sec. IV we construct outer radiative solutions and use MAE to match them to the inner solutions found in Sec. III. Finally in Sec. V we will derive balance equations for energy and angular momentum that are in agreement with those found by Peters[19] using an invalid approximation scheme and later by this author[20] using approximation procedures of the type described above that avoided the problems of the Peters derivation. It is these equations that were used to compute the orbital period change in the binary pulsar and that give such good agreement with observation even though the nature of their derivation precluded their applicability to the binary pulsar. We will use the same methods to derive an expression for the radiation reaction force itself. It is interesting to note that the result obtained using deDonder coordinate conditions is not the standard expression obtained by a number of authors (see Ref. 1) although it is transformable into the standard expression by a coordinate transformation. It does, however, lead to the same balance equations as those obtained from the standard expression and those derived in Sec. V.[21]

II. MULTIPLE-TIME FORMALISM

The multiple-time formalism is a particularly effective way of dealing with secular nonuniformities in time. In using it, one assumes that the variables in the problem depend on time through a set of variables $t_n=\phi_n(\epsilon)t$, where the ϕ_n form an asymptotic sequence satisfying

$$\lim_{\epsilon\to 0}\phi_{n+1}(\epsilon)/\phi_n(\epsilon)=0 \ . \tag{2.1}$$

Just how many t_n's are required and what the ϕ_n will be depend upon the problem to which the method is being applied. As an example, consider the equation

$$x_{tt}+x=\epsilon x(x^2)_{ttttt} \ , \tag{2.2}$$

which is of the type we shall encounter later in the paper. We assume that $x=x(t,t_1,\epsilon)$ and that x is periodic in the fast time t. (t_1 is sometimes referred to as the slow time.) We next expand x according to

$$x=x_1+\epsilon x_2+\cdots \ . \tag{2.3}$$

Inserting this expansion into Eq. (2.2) and equating to zero the coefficients of the powers of ϵ we obtain, if we take $t_1=\epsilon t$, the first two equations

$$x_{1tt}+x_1=0 \tag{2.4}$$

and

$$x_{2tt}+x_2=-2x_{1tt_1}-x_1(x_1{}^2)_{ttttt} \ . \tag{2.5}$$

The solution of Eq. (2.2) is

$$x_1=A(t_1)e^{it}+\text{c.c.} \tag{2.6}$$

When this solution is substituted into Eq. (2.5), one obtains

$$x_{2tt}+x_2=(-2iA_{t_1}-32iA^2\overline{A})e^{it}-32iA^3e^{3it}+\text{c.c.} \tag{2.7}$$

The dependence of A on t_1 is determined by the requirement that the term in parentheses in this equation vanish. Note that if A had been assumed to be a constant, the first term on the right-hand side would give rise to a term in x_2 that would increase linearly with time and, hence, would represent a nonuniformity in the solution. As it is, we see that the expression in parentheses in Eq. (2.7) will vanish if we take

$$A=\tfrac{1}{2}[8t_1+1/(4A_0{}^2)]^{-1/2} \tag{2.8}$$

and that x_2 is then given by

$$x_2=4iA^3e^{3it} \ . \tag{2.9}$$

We can also derive a balance equation from Eqs. (2.4) and (2.5) by multiplying Eq. (2.4) by $(x_{1t_1}+x_{2t})$ and Eq. (2.5) by x_{1t} and adding to obtain

$$(x_{1t_1}+x_{2t})(x_{1tt}+x_1)+x_{1t}(x_{2tt}+x_2+2x_{1tt_1}) =-x_1(x_1{}^2)_{ttttt} \ , \tag{2.10}$$

which can be rewritten as

$$d_{t_1}[\tfrac{1}{2}(x_{1t}{}^2+x_1{}^2)]+d_t(x_{1t}x_{2t}+x_{1t_1}x_{1t}+x_1x_2) =-x_1(x_1{}^2)_{ttttt} \ . \tag{2.11}$$

Since x is assumed to be periodic in t (but not necessarily in t_1) we can average Eq. (2.11) over one period to obtain

$$d_{t_1}\langle\tfrac{1}{2}(x_{1t}{}^2+x_1{}^2)\rangle_t=-\tfrac{1}{2}\langle(x_1{}^2)_{ttt}{}^2\rangle_t \ , \tag{2.12}$$

where $\langle\ \rangle_t$ denotes an average over one period in the fast time t. This "balance" equation is similar in form to those we shall derive in the following sections.

III. THE EINSTEIN-INFELD-HOFFMANN METHOD

A. Equations of motion

The field equations of general relativity can be written in the form[22]

$$U^{\mu\nu\rho}{}_{,\rho}=\theta^{\mu\nu} \ , \tag{3.1}$$

where

$$U^{\mu\nu\rho}=-U^{\mu\rho\nu}=(1/16\pi)(\mathfrak{g}^{\mu\nu}\mathfrak{g}^{\rho\sigma}-\mathfrak{g}^{\mu\rho}\mathfrak{g}^{\nu\sigma})_{,\sigma} \ , \tag{3.2}$$

$$\theta^{\mu\nu}=(-g)(T^{\mu\nu}+t_{LL}{}^{\mu\nu}) \ , \tag{3.3}$$

$g_{\mu\nu}$ is the gravitational field variable with signature -2, $g=\det(g_{\mu\nu})$, $\mathfrak{g}^{\mu\nu}=\sqrt{-g}\,g^{\mu\nu}$, $T^{\mu\nu}$ is the stress-energy tensor of whatever else is interacting with the gravitational field and $t_{LL}{}^{\mu\nu}$ is the Landau-Lifshitz pseudotensor.[23] Because of the antisymmetry of $U^{\mu\nu\rho}$ in its last two indices it follows that $U^{\mu rs}{}_{,s}$ is a three-dimensional curl and therefore when Eq. (3.1) is integrated over a two-surface in a $t=$const hypersurface, we obtain

$$\oint(U^{\mu r0}{}_{,0}-\theta^{\mu r})n_r dS=0 \ , \tag{3.4}$$

where n_r is a unit surface normal. In a like manner we can obtain the result that

$$\oint[(x^\mu U^{\nu r0})_{,0}-(x^\nu U^{\mu r0})_{,0}-x^\mu\theta^{\nu r}+x^\nu\theta^{\mu r} -(\mathfrak{g}^{\nu r}\mathfrak{g}^{\mu 0}-\mathfrak{g}^{\mu r}\mathfrak{g}^{n0})_{,0}]n_r dS=0 \ . \tag{3.5}$$

It is these equations, (3.4) and (3.5), that we use to obtain equations of motion and balance equations.[24]

B. Inner solutions

In order to use Eqs. (3.4) and (3.5) it is necessary to obtain solutions of the field equations corresponding to the types of systems in which we are interested. Since no such exact solutions exist, we must employ approximate ones. Our system will be characterized by a small parameter ϵ, the ratio of the light travel time across our system to the shortest relevant time scale associated with its motion, in this case its orbital period.[25] This ratio is of course the same as the ratio of the size of the system to the wavelength of the radiation it emits. We also recognize two zones associated with our system, an inner zone which contains the sources and whose size is of the order of a few wavelengths and an outer zone that overlaps the inner zone and extends to infinity. Following Burke[11] we will employ different expansions in these two regions. The arbitrary functions that appear in these expansions are then determined by the requirement that they must agree in the overlap region between the two zones. We begin our expansion in the inner zone and use it to illustrate the EIH method of deriving equations of motion.

We will assume the existence of a coordinate system such that $g^{\mu\nu}$ can be expanded as an asymptotic series of the form

$$g^{\mu\nu} \sim \eta^{\mu\nu} + \sum \zeta_n(\epsilon) h_n{}^{\mu\nu}(t_1, t_2, \ldots, r) , \tag{3.6}$$

where $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, $t_1, t_2, \ldots$ are a set of multiple times and the $\zeta_n(\epsilon)$ form an asymptotic sequence satisfying a condition similar to Eq. (2.1). The reason that we do not simply use a power-series expansion in ϵ is that $g^{\mu\nu}$ is not analytic in ϵ, reflecting the fact that one encounters $\epsilon^n \ln\epsilon$ terms in higher orders of the approximation.[26] Fortunately, these terms will not concern us here. Therefore, up to the order of accuracy to which we are working we can take $\zeta_n = \epsilon^n$.

Since we are dealing with a source with compact components, the gravitational fields in their neighborhoods will not, in general, be small. Finding approximate solutions near the components is far from an easy task.[27] Fortunately, as long as the distances between components are large compared to their sizes, such solutions are not necessary for the application of EIH. The surface integrals in Eqs. (3.4) and (3.5) used to derive equations of motion and balance equations need never pass through regions where the fields are strong, as we shall see. Consequently the asymptotic expansions employed here will be adequate to the task at hand.

We begin our approximation by constructing the lowest-order contributions to h^{00} and h^{0r}. Our assumption that the source components are spherically symmetric and quasistatic will be satisfied if the lowest-order contribution of each component to h^{00} is the first term in the large-r expansion of a Schwarzschild field whose mass parameter is equal to the mass of the component. Furthermore, this contribution must be of order ϵ^2 since it appears as the Newtonian potential in the lowest-order equations of motion and, hence, by the virial theorem it must be of the same order of magnitude as the square of a typical orbital velocity which, by assumption, is ϵ^2. In what follows, it will prove convenient to use a length scale in which the distances between sources are of the order of unity in which case all masses must be of the order of ϵ^2. Finally we note that, if we impose the deDonder coordinate conditions

$$g^{\mu\nu}{}_{,\nu} = 0 , \tag{3.7}$$

it follows from the field equations (3.1) that $h_2{}^{00}$ must satisfy

$$\nabla^2 h_2{}^{00} = 0 . \tag{3.8}$$

The solution that satisfies all of our requirements is

$$h_2{}^{00} = 4 \sum m_{A2}/r_A , \tag{3.9}$$

where the sum is over all sources in the system, i.e., over $A=1$ to N, where $r_{Ai} = r_i - x_{Ai}$ and x_{Ai} is the coordinate of the Ath source. (In the case of a black hole, x_{Ai} is the effective coordinate of its center as determined by its Schwarzschild exterior solution.) For convenience we will take the center of mass of the system to be at the origin of coordinates so that $\sum m_{A2} x_{Ai} = 0$. As we will see, this condition is consistent with the lowest-order, Newtonian, equations of motion. Alternately, we can dispense with this condition and still obtain these equations. We can then impose this condition after having done so.

We are now in a position to determine the dependence on time of the m_{A2} which appear in Eq. (3.8) with the help of Eq. (3.4). Substitution of the expression (3.8) for $h_2{}^{00}$ into the expressions for U^{0r0} and θ^{0r}, yields the results that

$$16\pi U_2{}^{0r0} = h_2{}^{00}{}_{,r} \quad \text{and} \quad \theta_3{}^{0r} = 0 . \tag{3.10}$$

Since Eq. (3.4) is satisfied for any surface we choose, we will take it to be a sphere centered on the Ath source with a radius large compared to this source's size but small enough to exclude any other sources. In general, such integrals will contain terms that depend on the radius of the sphere and those that are independent of it. Since the overall integral must be independent of the radius of the sphere, it follows that the terms that depend on this radius must be identically zero and the sum of the terms that are radius independent must vanish as a consequence of the field equations. When one evaluates the integral (3.4) one finds that this will be the case provided that

$$\dot{m}_{A2} = 0 , \tag{3.11}$$

where an overdot denotes differentiation with respect to $t_1 = \epsilon t$. The quantity m_{A2} is just the mass that would appear in the Schwarzschild field of the Ath source if it were isolated from all the other sources, and we see that, in lowest order of approximation, it is time independent.

The next quantity to be determined is the lowest-order contribution to h^{0r}. We see that in order to satisfy the deDonder conditions it must be of order ϵ^3 since $h_2{}^{00}{}_{,0}$ is of this order and differentiation with respect to a spatial coordinate does not change the order of a quantity. It then follows from the field equations together with the deDonder conditions that it satisfies

$$\nabla^2 h_3{}^{0r} = 0 . \tag{3.12}$$

These conditions determine it to be

$$h_3{}^{0r} = 4 \sum m_A \dot{x}_{Ar}/r_A . \tag{3.13}$$

Equation (3.4) can now be used to obtain the equations of motion for the sources. For this purpose we need

$$16\pi U_3{}^{rs0} = h_3{}^{r0}{}_{,s} \tag{3.14}$$

and

$$16\pi\theta_4{}^{rs} = \tfrac{1}{4} h_2{}^{00}{}_{,r} h_2{}^{00}{}_{,s} - \tfrac{1}{8}\delta^{rs} h_2{}^{00}{}_{,m} h_2{}^{00}{}_{,m} \ . \tag{3.15}$$

When these expressions are substituted into Eq. (3.4) one finds that the surface-independent terms will vanish provided the x_{Ar} satisfy

$$m_{A2}\ddot{x}_{Ar} = m_{A2} \sum{}' m_{B2} x_{ABr}/x_{AB} \ , \tag{3.16}$$

where $x_{ABr} = x_{Ar} - x_{Br}$ and the prime on the sum indicates that it is over all $B \neq A$. It should be emphasized that the exclusion of A from the sum is not a matter of choice, but is a direct consequence of evaluating the integrals in Eq. (3.4). It of course could not be otherwise since including this term would yield a divergent result. Since Eq. (3.4) involves only surface integrals they must all be finite as long as the fields used in their evaluation are finite. Equations (3.16) are of course just Newton's equations of motion for gravitating bodies and were first obtained in this way by EIH.

Having constructed $h_2{}^{00}$ and $h_3{}^{0r}$ and having used them to derive the Newtonian equations of motion (3.16), we can proceed to the next level of approximation, the so-called post-Newtonian approximation, by first finding $h_4{}^{rs}$. It follows from the field equations that is satisfies an inhomogeneous Laplace equation of the form

$$\nabla^2 h_4 rs = \theta_4{}^{rs} \ , \tag{3.17}$$

where $\theta_4{}^{rs}$ is given by Eq. (3.15).

We see that $h_4{}^{rs}$ is determined modulo a homogeneous solution of Laplace's equation. In accordance with our discussion in the Introduction, we will require that all such homogeneous solutions be spherically symmetric so that our sources do not "grow wings" in the course of the approximation. The arbitrary functions appearing in this homogeneous solution are then determined by imposing the deDonder conditions (3.7). Because of the form of the source term in Eq. (3.17), we cannot give a closed-form expression for $h_4 rs$. It can, however, be determined both far from the sources and in the vicinity of each individual source. Fortunately, this is sufficient for our purposes and these expressions will be given later as needed.

One can continue in the above manner to construct post- and post-post-Newtonian corrections to the equations of motion. First the next correction to h^{00} is determined from the field equations, again modulo a homogeneous solution of Laplace's equation. The requirement of spherical symmetry implies that this homogeneous solution must be of the same form at that of $h_2{}^{00}$ given by Eq. (3.9) with new functions m_{A4} whose time dependence must again be determined through the use of Eq. (3.4). The equation for $h_4{}^{00}$ is

$$\nabla^2 h_4 00 = \tfrac{7}{8} h_2{}^{00}{}_{,r} h_2{}^{00}{}_{,r} + \ddot{h}_2{}^{00} \ . \tag{3.18}$$

The solution to this equation can be given in closed form and is

$$h_4{}^{00} = \tfrac{7}{16}(h_2{}^{00})^2 - 2\sum m_{A2}[(1/r_A) Y_{Amn}\dot{x}_{Am}\dot{x}_{An} + Y_{Am}\ddot{x}_{Am}] + 4\sum m_{A4}/r_A \ , \tag{3.19}$$

where

$$Y_{Amn} = n_{Am} n_{An} - \tfrac{1}{3}\delta_{mn} \quad \text{and} \quad Y_{Am} = n_{Am} \ , \qquad n_{Am} = r_{Am}/r_A \ , \tag{3.20}$$

are spherical harmonics of orders two and one, respectively.

In order to determine the m_{A4} appearing in the homogeneous term with the help of Eq. (3.4), we need the quantities

$$16\pi U_4{}^{0r0} = h_3{}^{0r} + h_4{}^{00}{}_{,r} \tag{3.21}$$

and

$$16\pi\theta_5{}^{0r} = h_2{}^{00}{}_{,m} h_3{}^{0m}{}_{,r} + \tfrac{3}{4} h_2{}^{00} h_2{}^{00}{}_{,r} - \delta^{mn} h_2{}^{00}{}_{,m} h_3{}^{0r}{}_{,m} \ . \tag{3.22}$$

When these expressions are inserted into Eq. (3.4) and the integrals evaluated, one finds that the surface-independent terms can be made to vanish if we take $\partial_{t_3} m_{A2} = 0$, where $t_3 = \epsilon^3 t$ and

$$m_{A4} = m_{A2}\left[\tfrac{5}{6}\dot{x}_A{}^2 - \tfrac{1}{2}\sum{}' m_{B2}/x_{AB}\right] \ . \tag{3.23}$$

The next step is to determine $h_5{}^{0r}$ from the field equations. The form of the homogeneous solution is again fixed by the deDonder conditions and the next-order equations of motion are again obtained from Eq. (3.4). In evaluating $U_5{}^{rs0}{}_{,0}$ for this purpose, there will appear derivatives of the x_{Ai} with respect to t_1 and t_3 (only derivatives with respect to t_1 appear in $\theta_6{}^{rs}$). The terms containing the t_3 derivatives will contribute a term $2m_{A2}\partial_{t_3}\dot{x}_{Ar}$ to the surface integral in Eq. (3.4), while the remaining terms will involve only derivatives with respect to t_1. This term is just the next one that one would obtain by expanding $m_{A2} d^2{}_t x_{Ar}$ using the MTS formalism. As a consequence, the equations that result from the vanishing of the surface integral in Eq. (3.4) will determine the dependence of x_{Ar} on t_3. Alternatively, we may add these equations to the Newtonian equations of motion (3.16) by, in effect resumming the series, to obtain the so-called post-Newtonian equations of motion involving derivatives with respect to t. Of course, when we come to solve the equations obtained in this way we will still have to proceed as we did in Sec. II.

EIH did not employ MTS in their approximation scheme. As a consequence they were forced to introduce fictitious dipole terms at each order of the approximation into their solutions of the field equations in order to ensure the latter's integrability. In the end, they required that the sum of all these dipoles vanish. The net effect of all this was to obtain again the post-Newtonian equations of motion. In the MTS method the terms involving derivatives with respect to t_3 replace the dipole terms of EIH, but now arise in a completely natural manner.

One could in principle continue on in this way indefinitely. In doing so, however, one would miss entirely the effects of radiation reaction since the fields that are so determined would correspond to a linear combination of half advanced, half retarded fields, that is, to a standing wave field. Among other things, such a field has an infinite-energy content and is therefore physically unacceptable. However, as emphasized by Burke,[11] the method employed to obtain these solutions is only valid in the near zone and does not lead to a unique result. To obtain a unique near-zone solution, we need to construct solutions in the outer zone and match them to inner zone solutions.

IV. RADIATION AND OUTER SOLUTIONS

The solutions we have so far constructed are near-zone or inner solutions. To obtain the effects of radiation, we must now construct an outer or far zone solution which will be matched to our inner-zone solution. This matching will determine both the arbitrary functions appearing in the outer solution and the radiative contributions to the inner solution. The method we shall employ for this purpose is essentially that devised by Burke[11] and uses the MAE. Not only is the MAE necessary in order to avoid nonuniformities, when dealing with equations in which the highest derivative is multiplied by a small parameter, such as is the case for slow-motion systems, but it is the only way we know of incorporating radiative effects in the framework of the EIH method. Fast-motion approximation schemes that attempt to solve the inhomogeneous wave equations that they encounter in terms of retarded integrals over the whole space must assume some form for the stress-energy tensor associated with the sources.[28] Whether the sources are represented by δ functions or continuous distributions, these methods cannot be applied to compact sources for reasons explained in the Introduction.

In Burke's original work he imposed deDonder coordinate conditions and used flat-space null cones to construct his approximate solutions. While such a procedure causes no trouble in the lowest order of approximation it leads to $\ln r/r$-type nonuniformities in higher orders. This difficulty can be avoided if one makes use of MSC (Ref. 12). In this method one introduces new coordinates as functions of the old coordinates in such a way as to eliminate the nonuniformities. This means that the functional form of the relation between the new and the old coordinates cannot be specified *ab initio*, but must be determined as part of the problem. In the present case this amounts to using the true null cones rather than the flat-space ones and in modifying the deDonder conditions so that the effective source terms in the inhomogeneous wave equations that arise from the nonlinearities of the field equations are of order r^{-3} rather than r^{-2}. Fortunately, these modifications only affect higher-order approximations and can be safely ignored if one wishes to find only the lowest-order radiative effects.

In order to obtain expressions for the field variables in the outer or wave zone we follow Burke[11] and introduce the outer coordinates $\hat{x}^{\mu}=(\epsilon t,\epsilon r)$. The quantities $\hat{h}^{\mu\nu}(\hat{x})=h^{\mu\nu}(x)$ are again expanded in an asymptotic series similar to the one employed in Eq. (3.6) except that now the coefficients in the expansion are functions of the outer coordinates. If one now imposes the deDonder conditions it then follows from the field equations that the lowest-order contribution to $\hat{h}^{\mu\nu}$ satisfies the flat-space homogeneous wave equation

$$\hat{\Box}\hat{h}_n{}^{\mu\nu}=0 \ . \tag{4.1}$$

Since the solution of Eq. (4.1) is to be matched to our inner solution we must construct the latter's outer expansion so as to determine the form of this outer solution. The outer expansion is obtained by letting $r\to\infty$ while holding ϵ fixed. If we keep only the first nontrivial terms in this limit we obtain the results

$$h_2{}^{00}\to 4M/r+6I_{rs}Y_{rs}/r^3 \ , \tag{4.2a}$$

$$I_{rs}=\sum m_{A2}x_{Ar}x_{As}, \quad M=\sum m_{A2} \ ,$$

$$h_4{}^{00}\to -2\ddot{I}_{mn}Y_{mn}/r+4\sum E_A/r \ , \tag{4.2b}$$

where

$$I=I_{mm} \quad \text{and} \quad E_A=\tfrac{1}{2}m_{A2}\left[\dot{x}_A{}^2-\sum{}' m_{B2}/x_{AB}\right] ,$$

$$h_3{}^{0r}\to 4\sum m_{A2}\dot{x}_{Ar}x_{As}Y_s/r^2 \ , \tag{4.3}$$

and

$$h_4{}^{rs}\to 4\sum m_{A2}(\dot{x}_{Ar}\dot{x}_{As}+\ddot{x}_{Ar}x_{As})/r \ . \tag{4.4}$$

We see from these outer expansions that the lowest-order contributions to $\hat{h}^{00}$ must contain terms with zero- and second-order spherical harmonics, those to $\hat{h}^{0r}$ first-order harmonics and those to $\hat{h}^{rs}$ zero-order harmonics.

A knowledge of the harmonic dependence of a solution of Eq. (4.1) is of course not sufficient for its determination; we need in addition some kind of radiation condition. As indicated in the Introduction, we will require that, in the lowest order of approximation, any radiation present must be purely outgoing. As a consequence, the arbitrary functions appearing in the solutions of Eq. (4.1) will be functions only of the retarded null coordinate $\hat{u}_0=\hat{t}-\hat{r}$.

Wave-zone solutions that match on to our inner-zone solutions and that contain only outgoing waves are given by

$$\hat{h}_3{}^{00}=4M/\hat{r} \ , \tag{4.5a}$$

$$\hat{h}_5{}^{00}=a/\hat{r}+(a_{rs}''/\hat{r}+3a_{rs}'/\hat{r}^2+3a_{rs}/\hat{r}^3)Y_{rs} \ , \tag{4.5b}$$

$$\hat{h}_5{}^{0r}=(b_{rs}'/\hat{r}+b_{rs}/\hat{r}^2)Y_s \ , \tag{4.6}$$

$$\hat{h}_5{}^{rs}=c_{rs}/\hat{r} \ , \tag{4.7}$$

where the quantities a, a_{rs}, b_{rs}, and c_{rs} are functions of the flat-space retarded coordinate $\hat{u}_0$ and a prime denotes differentiation with respect to this variable.

These functions are restricted by the requirement that the $h_5^{\mu\nu}$ satisfy the deDonder conditions. This will be the case provided

$$a_{rs}=\tfrac{1}{2}(b_{rs}+b_{sr})-\tfrac{1}{3}\delta_{rs}b_{mn}\ ,\qquad a=\tfrac{1}{3}b_{mm},\quad b_{rs}=c_{rs}\ . \tag{4.8}$$

The functions of $\hat{u}_0$ appearing in Eqs. (4.5)–(4.7) can be determined by matching the outer field found above to the inner field found previously. For this purpose it is necessary to construct the inner expansion of this outer field and equate it to the outer expansion of the inner field given in Eqs. (4.2)–(4.4). The inner expansion of the outer field is obtained by taking the limit $\hat{r}\to 0$ holding ϵ fixed. One finds in this way that

$$\hat{h}_5{}^{00}\to[3a_{rs}/(\epsilon r)^3-\tfrac{1}{2}\ddot{a}_{rs}/(\epsilon r)+\tfrac{1}{8}(\epsilon r)a_{rs}^{(4)}-\tfrac{1}{15}(\epsilon r)^2a_{rs}^{(5)}]Y_{rs}+a/(\epsilon r)-\dot{a}\ , \tag{4.9}$$

$$\hat{h}_5{}^{0r}\to[b_{rs}/(\epsilon r)^2-\ddot{b}_{rs}+\tfrac{1}{3}(\epsilon r)b_{rs}^{(3)}]Y_s\ , \tag{4.10}$$

$$\hat{h}_5{}^{rs}\to c_{rs}/(\epsilon r)-\dot{c}_{rs}\ , \tag{4.11}$$

where a number in parentheses denotes the number of derivatives to be taken with respect to t_1. In each case we have carried out the inner expansion to the point where the first time-odd terms appear.

By comparing Eqs. (4.2) and (4.9) we see that

$$a_{rs}=2\epsilon^5 I_{rs}\quad\text{and}\quad a=4\epsilon^3 M+\tfrac{2}{3}\epsilon^5 I,\quad I=I_{mm}\ . \tag{4.12}$$

Likewise, comparing Eqs. (4.3) and (4.10) gives

$$b_{rs}=4\epsilon^5\sum m_{A2}\dot{x}_{Ar}x_{As} \tag{4.13}$$

and comparing Eqs. (4.4) and (4.11) gives

$$c_{rs}=4\epsilon^5\sum m_{A2}(\ddot{x}_{Ar}x_{As}+\dot{x}_{Ar}\dot{x}_{As})\ . \tag{4.14}$$

At this point we have all of the information needed to evaluate the surface integrals required for the derivation of the balance equations using Eqs. (3.4) and (3.5).

To derive an expression for the lowest-order radiation reaction force is at least an order of magnitude more difficult than deriving the balance equations. In addition to the field components already obtained in the inner zone we need $h_5{}^{00}$, $h_7{}^{00}$, $h_6{}^{0r}$, $h_8{}^{0r}$, $h_5{}^{rs}$, and $h_7{}^{mm}$. The components $h_5{}^{00}$ and $h_5{}^{rs}$ can be read off directly from Eqs. (4.9) and (4.11). They are

$$h_5{}^{00}=-\tfrac{2}{3}I^{(3)} \tag{4.15}$$

and

$$h_5{}^{rs}=-2I_{rs}^{(3)}\ . \tag{4.16}$$

These components are seen to satisfy Laplace's equation which, as follows from the field equations, they must. Furthermore, they are valid everywhere in the inner region. On the other hand, Eq. (4.10) only determines the lowest-order contribution to the outer expansion of $h_6{}^{0r}$ which is, by itself, insufficient to determine $h_6{}^{0r}$ everywhere in the inner region. In particular, it is insufficient to determine it in the region occupied by the sources where it is needed for the determination of the reaction force.

To determine the remaining piece of $h_6{}^{0r}$ one needs terms in $\hat{h}_6{}^{0r}$ which in turn are expanded to produce a match in the inner zone.[29] We shall leave the details of this determination as well as those for obtaining the other required components of $h^{\mu\nu}$ to the Appendix since they follow along similar lines to those used above and give here only the final results. They are

$$h_6{}^{0r}=\tfrac{2}{3}d_{t_1}{}^3\sum m_{A2}(2\dot{x}_{Ar}x_{As}r_s-\dot{x}_{Ar}x_A{}^2)-4d_{t_1}{}^2\sum E_A x_{Ar}\ , \tag{4.17}$$

$$h_7{}^{00}=-\tfrac{1}{15}d_{t_1}{}^5\sum m_{A2}(2x_{Am}x_{An}r_m r_n+x_A{}^2r^2+2x_{As}x_A{}^2r_s)+\tfrac{4}{3}d_{t_1}{}^3\sum E_A x_{As}r_s+4I_{rs}^{(3)}\sum m_{A2}(r_{Ar}r_{As}/r_A{}^2-\delta_{rs})/r_A+\cdots\ , \tag{4.18}$$

$$h_7{}^{rs}=-\tfrac{2}{3}d_{t_1}{}^4\sum m_{A2}\dot{x}_{Ar}x_{As}r^2+\tfrac{1}{4}(\phi_r r_s+\phi_s r_r)+\cdots\ ,\qquad \phi_r=-\tfrac{2}{3}d_{t_1}{}^4\sum m_{A2}\dot{x}_{Ar}x_A{}^2+4d_{t_1}{}^3\sum E_A x_r\ , \tag{4.19}$$

and

$$h_8{}^{0r}=\tfrac{1}{15}I_{rs}^{(6)}r^2r_s+4I_{rs}^{(4)}\sum m_{A2}r_{As}/r_A+4I_{mn}^{(3)}\sum m_{A2}(r_{Am}r_{An}/r_A{}^2-\delta_{mn})\dot{x}_{Ar}/r_A+\cdots\ , \tag{4.20}$$

where the ellipses denote terms that are not needed for the calculation of the reactive force in lowest order.

V. CALCULATION OF DAMPING EFFECTS

A. Balance equations

Balance equations for "energy" and "angular momentum" can be obtained from Eqs. (3.4) and (3.5) by taking the surface of integration to be that of a large sphere centered at the center of mass of the sources and lying in the radiation zone. (We use quotes here to indicate that the terms energy and angular momentum have only heuristic significance. We emphasize that nothing in the derivation depends on this terminology.) Since the Newtonian energy and its post- and post-post-Newtonian corrections are conserved up to the order where radiative effects enter, that is, the derivative of their sum with respect to t_1 is zero,[30] it follows that the first nontrivial contributions to the integrals will be of order ϵ^{10}. In order to evaluate these integrals then we need $U_4{}^{0r0}$ and $\theta_{10}{}^{0r}$. After averaging with respect to t_1 one finds

$$dt_6\langle E_{\text{Newton}}\rangle_{t_1}=-\tfrac{1}{5}\langle \mathcal{I}_{rs}^{(3)}\mathcal{I}_{rs}^{(3)}\rangle_{t_1}\ , \tag{5.1}$$

where $t_6=\epsilon^6 t$,

$$E_{\text{Newton}}=\sum E_A\ , \tag{5.2}$$

and

$$\not{I}_{rs}=I_{rs}-\tfrac{1}{3}\delta_{rs}I \ . \tag{5.3}$$

This is the "standard" expression for the energy loss due to the emission of gravitational waves and has been derived by many authors in many different ways for many different types of systems. It has been remarked upon before that it is quite remarkable that one always obtains the same result regardless of what one does.

One can also derive a similar balance equation for angular momentum from Eq. (3.5), although with considerably more effort. After a straightforward but lengthy calculation (details upon request) one again finds the standard result that

$$dt_6\langle L_{rs\ \mathrm{Newton}}\rangle_{t_1}=-\tfrac{2}{5}\langle \not{I}^{(2)}_{rm}\not{I}^{(3)}_{sm}-\not{I}^{(2)}_{sm}\not{I}^{(3)}_{rm}\rangle_{t_1} \ , \tag{5.4}$$

where

$$L_{rs\ \mathrm{Newton}}=m_{A2}(x_{Ar}\dot{x}_{As}-x_{As}\dot{x}_{Ar}) \ . \tag{5.5}$$

Equations (5.1) and (5.5) can now be used to determine the time rate of change of the Newtonian orbit parameters. In particular, Eq. (5.1) can be used to determine the period change as a function of these parameters.

Before we leave the subject of balance equations we should perhaps comment briefly on the famous (or infamous, depending on one's point of view) "quadrupole formula" of general relativity.[31] Many authors have presented derivations of this formula in the past and have interpreted it as an expression for the flux of gravitational energy radiated by a system. This formula is usually derived by integrating the time-space component of one or another of the energy-momentum pseudotensors over a large sphere surrounding the sources. And indeed, if one integrates θ^{0r} over such a sphere one obtains, in the limit that the radius of the sphere is allowed to approach infinity, the negative of the right-hand side of Eq. (5.1).

As it stands, the quadrupole formula is at best of academic interest, however. It is not something that one would ever measure directly and its interpretation as an energy flux is just that, an interpretation. Furthermore, setting its negative equal to the rate of change of the Newtonian energy of the system on the basis of this interpretation can hardly be considered a derivation of the energy balance equation. For these reasons we eschew any attempt to derive quadrupole formulas in this paper.

B. Radiation Reaction Force

Although the balance equations derived above are sufficient for comparing the predictions of general relativity with present day observations, it is instructive to derive an expression for the radiation reaction force. For this purpose we will again use as a surface in Eq. (3.4) a sphere surrounding the Ath source. To perform the integrations we need to know $U_8{}^{rs0}$ and $\theta_9{}^{rs}$. The evaluation is again straightforward (after you find all of the terms that contribute to the integrals), but exceedingly lengthy. In all, there are some 88 terms that must be collected to give the final result. One finds that

$$\begin{aligned}F^r_{A\ \mathrm{react}}&\equiv \oint_A (U_8{}^{rs0}-\theta_9{}^{rs})n_{As}dS_A\\ &=m_{A2}\left[\tfrac{3}{5}x_{As}I^{(5)}_{rs}+2\dot{x}_{As}I^{(4)}_{rs}+\tfrac{2}{3}\dot{x}_{Ar}I^{(4)}+\tfrac{4}{3}\ddot{x}_{Ar}I^{(3)}-3\sum{}' m_{B2}\frac{x_{ABr}x_{ABm}x_{ABn}}{x_{AB}{}^5}I^{(3)}_{mn}+Q_r\right] \ ,\end{aligned} \tag{5.6}$$

where

$$Q_r=-\left[\tfrac{1}{3}d_{t_1}{}^4\sum m_{A2}\dot{x}_{Ar}x_A{}^2-\tfrac{1}{30}d_{t_1}{}^5\sum m_{A2}x_{Ar}x_A{}^2+\tfrac{5}{3}d_{t_1}{}^3\sum E_A x_{Ar}\right] \ . \tag{5.7}$$

This form of the radiation reaction force is quite different from the so-called standard form. Nevertheless it yields the same balance equations as does the latter. This agreement between two such different calculations serves to enhance our confidence in the final results. It is also in agreement with a general result, of which this is a special case, proved by Schutz.[32]

The result given above for the radiation reaction force is in deDonder coordinates. Since this expression is not coordinate invariant, it should be possible to transform it to the standard form since both forms yield the same balance equations. However, one cannot transform it directly since there is no reason to believe that it is a geometric object. Rather, one must add it to the Newtonian equations of motion (3.16) and seek a transformation such that the transform of a solution of these equations satisfies the Newtonian equations with the standard form of the radiation reaction added. Indeed, if one takes

$$\begin{aligned}x'^r&=x^r-\epsilon^5 I_{rs}x_s+\epsilon b_r \ ,\\ x'^0&=x^0-\tfrac{2}{3}\epsilon^4 I \ ,\end{aligned} \tag{5.8}$$

where

$$d_{t_1}{}^2 b_r=Q_r \ , \tag{5.9}$$

one finds that the transformed solution does satisfy the Newtonian equations with the standard form of the reaction force

$$F'^{\,r}_{A\ \mathrm{react}}=-\tfrac{2}{5}m_{A2}x_{As}I^{(5)}_{rs} \tag{5.10}$$

added. It should be emphasized that this result cannot be obtained without the inclusion of the gravitational interaction tern in the Newtonian equations of motion.

VI. DISCUSSION OF RESULTS

We have, we believe, been able to overcome the basic objections which we raised in the Introduction to the application of previous derivations of the balance equations and the radiation reaction force to a system such as the binary pulsar PSR1913+16. The source models are consistent with what we know about the components of this system. The mathematical operations are all internally consistent and, in particular, no infinite renormalizations are necessary. Furthermore, no unnecessary assumptions had to be made—the equations of motion followed directly from the field equations.

Finally, we should point out that these equations do not have runaway solutions. At first sight it would appear that they do admit such solutions since the radiation reaction force appears to contain third and even higher time derivatives of the source coordinates. However, this appearance is deceptive since the whole derivation rested on the assumption of slow motion and employed the MTS formalism as an integral part of this derivation. As we saw in the example in Sec. II, when one attempts to solve these equations of motion one must substitute solutions of the Newtonian equations of motion without the radiation reaction force into this force in the full equations as well as on the right-hand sides of the balance equations. The motions will then decay with time as they do in the example and as they do in the binary pulsar.

While we have overcome the above-mentioned objections, there are still, however, in our opinion, two problems which must be solved before one can say that the derivation is complete. We have had to assume that our outer solutions were causal. While justified on physical grounds, in principle such an assumption should be unnecessary since we are dealing with a Cauchy system. It should be possible to show that for all physically reasonable initial data and for times sufficiently far in the future of the initial-data hypersurface the solutions become causal in the sense used here. The model calculation of Aichelberg and Beig[15] and the numerical studies of Anderson and Hobill[16] lend support to this conjecture, but so far we do not have a proof of it.

The other problem concerns our assumption that our expansions are well behaved and that, as a consequence, the ratio of successive terms is of the order of the ratio of the functions of the slowness parameter ϵ, which they contain as factors. Such behavior is of course necessary if one is to justify using only the first few terms in an expansion to approximate an exact result. It is, however, not sufficient since the sum of all the remaining terms might be larger than those retained. What is needed to overcome this objection is a proof that the expansions are asymptotic. Such a proof will, in all likelihood, be exceedingly difficult to come by because of the many different expansions needed to obtain our final results.

We did not include these problems in the list we gave in the Introduction because it was not these that rendered earlier derivations inapplicable to the binary pulsar. Indeed, similar problems exist with all extant derivations. Furthermore, the assumptions made here to deal with them are not inconsistent with what we know about the binary pulsar. And finally, because our derivation is, we believe, free of the objections raised in the Introduction to earlier derivations and because of the very good agreement between the calculated and observed period change in the binary pulsar we would argue that the assumptions we have made appear to be justified.

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APPENDIX

In this appendix we will outline the steps for determining the field components needed to calculate the lowest-order contribution to the reaction force. To save writing we will drop the caret over the outer zone coordinates where it will not cause confusion.

1. $h_6{}^{0r}$

In order to determine $h_6{}^{0r}$ we must first determine $h_5{}^{0r}$, which in turn will determine a piece of $\hat{h}_6{}^{0r}$ through matching. This piece will then determine, again through matching, the additional piece of $h_6{}^{0r}$ required for the calculation of the reaction force. From the field equations it follows that $h_5{}^{0r}$ satisfies

$$\nabla^2 h_5{}^{0r} = \ddot{h}_3{}^{0r} - h_2{}^{00}{}_{,m} h_3{}^{0m}{}_{,r} - h_2{}^{00}{}_{,m} h_3{}^{0r}{}_{,m} + \tfrac{3}{4} h_2{}^{00}{}_{,0} h_2{}^{00}{}_{,r} \;. \tag{A1}$$

We now set $h_5{}^{0r} = h_{5\mathrm{I}}{}^{0r} + h_{5\mathrm{II}}{}^{0r}$, where $\nabla^2 h_{5\mathrm{I}}{}^{0r}$ equals the first term on the right-hand side of the above equation while $\nabla^2 h_{5\mathrm{II}}{}^{0r}$ equals the remainder. One then finds that

$$h_{5\mathrm{I}}{}^{0r} = 2\sum m_{A2}[r_A x^{(3)}_{Ar} - (Y_{Amn}/r_A)\dot{x}_{Am}\dot{x}_{An}\dot{x}_{Ar} - Y_{Am}(\ddot{x}_{Am}\dot{x}_{Ar} + 2\dot{x}_{Am}\ddot{x}_{Ar}) + f_{Ar}/r_A] \;, \tag{A2}$$

where f_{Ar} is to be determined.

It is not possible to find $h_{5\mathrm{II}}{}^{0r}$ in closed form. However, for our purposes, it is sufficient to find $h_{5\mathrm{II}}{}^{0r}{}_{,r}$, which can be obtained in closed form and is given by

$$h_{5\mathrm{II}}{}^{0r}{}_{,r} = -\tfrac{7}{16}[(h_2{}^{00})^2]_{,0} + \sum g_A/r_A \;. \tag{A3}$$

The unknown quantities f_{Ar} and g_A appearing in Eqs. (A2) and (A3) are fixed by the imposition of the deDonder conditions:

$$h_4{}^{00}{}_{,0} + h_5{}^{0r}{}_{,r} = 0 \;. \tag{A4}$$

One finds in this way that

$$f_{Ar}=\tfrac{4}{3}m_{A2}\dot{x}_A{}^2\dot{x}_{Ar}+4E_A\dot{x}_{Ar} \tag{A5}$$

and

$$g_A=-4\dot{E}_A\ . \tag{A6}$$

We can now determine $\hat{h}_6{}^{0r}$ to the necessary accuracy in order to find $h_6{}^{0r}$ in the inner zone. We do this by constructing the outer expansion of $h_5{}^{0r}$. Some of the terms in this expansion will match to terms already present in $\hat{h}_5{}^{0r}$. The lowest-order unmatched terms in the outer expansion of $h_{5\mathrm{I}}{}^{0r}$ are of the form p_r/r, where

$$p_r=\tfrac{2}{3}d_{t_1}{}^2\sum m_{A2}x_A{}^2\dot{x}_{Ar}+4\sum E_A\dot{x}_{Ar}\ . \tag{A7}$$

They match to an outer solution of the homogeneous wave equation given by

$$\hat{h}_{6\mathrm{I}}{}^{0r}=p_r/r\ , \tag{A8}$$

which has an inner expansion

$$\hat{h}_{6\mathrm{I}}{}^{0r}\rightarrow p_r/\epsilon r-\dot{p}_r+\cdots\ . \tag{A9}$$

In matching $h_{5\mathrm{II}}{}^{0r}$ we need only concern ourselves with the homogeneous piece $\sum g_A/r_A$ since the quadratic terms will match to a solution of an inhomogeneous wave equation whose source term $\hat{\theta}_8{}^{0r}$ is $O(\epsilon^8)$. When these terms are in turn matched back into the inner zone, they will therefore match to terms of at least $O(\epsilon^{10})$ and, hence, will not contribute to the reaction force in lowest order. The outer expansion of the homogeneous term in $h_{5\mathrm{II}}{}^{0r}{}_{,r}$ has the form

$$-4\sum\dot{E}_A/r_A\rightarrow-4\sum\dot{E}_A[(1/r)+(Y_m/r^2)x_{Am}+\cdots]\ , \tag{A10}$$

where the terms represented by the ellipsis match to terms which also do not contribute to the reaction force in lowest order. Furthermore, the first term on the right-hand side of this equation is zero since $\sum\dot{E}_A=O(\epsilon^2)$ as follows from the Newtonian equations of motion (3.16). The second term will match to the homogeneous outer solution

$$\hat{h}_7{}^{0r}{}_{,r}=(f_r'/r+f_r/r^2)Y_r,\quad f_r=-4\sum\dot{E}_Ax_{Ar}\ , \tag{A11}$$

which leads to the conclusion that

$$\hat{h}_6{}^{0r}=-f_r/r+\cdots\rightarrow-f_r/\epsilon r+\dot{f}_r+\cdots\ . \tag{A12}$$

Collecting contributions from Eqs. (4.10), (A9), and (A12) leads finally to the result (4.17).

2. $h_7{}^{00}$

From the field equations it follows that $h_7{}^{00}$ satisfies

$$\nabla^2h_7{}^{00}=-\tfrac{2}{3}I^{(5)}-I_{rs}^{(3)}h_2{}^{00}{}_{,rs}\ , \tag{A13}$$

which has, as a solution,

$$h_7{}^{00}=-\tfrac{1}{9}I^{(5)}r^2+4I_{rs}^{(3)}\sum m_{A2}(r_{Ar}r_{As}/r_A{}^2-\tfrac{1}{3}\delta_{rs})/r_A+\psi_{\mathrm{hom}}\ . \tag{A14}$$

The contributions to ψ_{hom}, which is a solution of Laplace's equation, are to be determined by matching and by coordinate conditions. One such contribution,

$$\psi_{I\ \mathrm{hom}}=-\tfrac{2}{15}r^2I_{rs}^{(5)}Y_{rs}\ , \tag{A15}$$

comes from the first time-odd term in Eq. (4.9). Additional contributions are obtained by matching parts of $h_4{}^{00}$ to an outer solution. In addition to pieces that match to $\hat{h}_5{}^{00}$, $h_4{}^{00}$ contains terms in its outer expansion of the form $(Y_s/r^2)T_s$, where

$$T_s=\tfrac{2}{5}d_{t_1}{}^2\sum m_{A2}x_{As}x_A{}^2+4\sum E_Ax_{As}\ , \tag{A16}$$

which match to a term

$$\hat{h}_6{}^{00}=(T_s'/r+T_s/r^2)Y_s+\cdots \tag{A17}$$

in the outer solution. The inner expansion of this term in turn yields a contribution

$$\psi_{\mathrm{II\ hom}}=\tfrac{1}{3}rT_s^{(3)}\ . \tag{A18}$$

The final contribution is obtained from the deDonder condition

$$h_7{}^{00}{}_{,}+h_8{}^{0r}{}_{,}=0 \tag{A19}$$

and is

$$\psi_{\mathrm{III\ hom}}=-\tfrac{8}{3}I^{(3)}\sum m_{A2}/r_A\ . \tag{A20}$$

When all of these contributions are added up, we obtain the result given by Eq. (4.18).

Although it is not necessary for the determination of the reaction force, it is perhaps instructive to see how the nonlinear term $\frac{7}{16}(h_2{}^{00})^2$ appearing in $h_4{}^{00}$ matches to an outer solution. The lowest-order nonlinear contribution to the outer solution satisfies

$$\Box\hat{h}_{6\mathrm{NL}}{}^{00}=\hat{\theta}_6{}^{00}=-14M^2[\nabla(1/r)]^2\ , \tag{A21}$$

which has as a solution

$$\hat{h}_{6\mathrm{NL}}{}^{00}=7M^2/r^2\ . \tag{A22}$$

Its inner expansion is seen to be just equal to the nonlinear term in $h_4{}^{00}$.

3. $h_7{}^{rs}$

The field equation for $h_7{}^{rs}$ is

$$\nabla^2h_7{}^{rs}=-2I_{rs}^{(5)}\ , \tag{A23}$$

which has as a solution

$$h_7{}^{rs}=-\tfrac{2}{3}I_{rs}^{(5)}r^2+\psi_{\mathrm{hom}}{}^{rs}\ . \tag{A24}$$

The part of the $\psi_{\mathrm{hom}}{}^{rs}$ that contributes to the reaction force can most easily be gotten by imposing the deDonder condition

$$h_6{}^{r0}{}_{,0}+h_7{}^{rs}{}_{,s}=0\ . \tag{A25}$$

In this way one finds that

$$\psi_{\mathrm{hom}}{}^{rs}=-\tfrac{1}{4}(\phi_rr_s+\phi_sr_r)\ , \tag{A26}$$

where ϕ_r is defined in Eq. (4.19). The resulting expression for $h_7{}^{rs}$ is then given by Eq. (4.19).

4. $h_8{}^{0r}$

The field equation for $h_8{}^{0r}$ is

$$\nabla^2 h_8{}^{0r} = 2I^{(3)}_{mn} h_3{}^{0r}{}_{,mn} - 2I^{(4)}_{rs} h_2{}^{00}{}_{,s} - \tfrac{2}{3} I^{(6)}_{rs} r_s + S^r = 0 \; , \tag{A27}$$

where S^r is the second t_1 derivative of the part of $h_6{}^{0r}$ that depends only on t_1. Its exact form is immaterial since it does not lead to a contribution to the reaction force. The parts of the solution that contribute to this force are given by

$$h_8{}^{0r} = 4I^{(3)}_{mn} \sum m_{A2}(r_{Ar} r_{As}/r_A{}^2 - \tfrac{1}{3}\delta_{rs})\dot{x}_{Ar}/r_A + 4I^{(4)}_{rs} \sum m_{A2} r_{As}/r_A + \psi_{\text{hom}}{}^r + \cdots \; , \tag{A28}$$

where the homogeneous piece is again obtained from the deDonder conditions (A19). The resultant expression for $h_8{}^{0r}$ is given by Eq. (4.20).

5. $\theta_9{}^{rs}$ and $U_8{}^{rs0}$

The parts of θ^{rs} and U^{rs0} needed to calculate the reaction force by means of the surface integral appearing in Eq. (5.6) are

$$16\pi U_8{}^{rs0} = h_5{}^{rs}{}_{,0} h_2{}^{00} - h_3{}^{r0}{}_{,m} h_5{}^{sm} + h_8{}^{r0}{}_{,s} \tag{A29}$$

and

$$\begin{aligned}16\pi\theta_9{}^{rs} = {} & \delta^{rs}(-\tfrac{3}{4}h_2{}^{00}{}_{,0}h_5{}^{00}{}_{,0} - h_3{}^{0k}{}_{,m}h_6{}^{0m}{}_{,k} - h_2{}^{00}{}_{,k}h_6{}^{0k}{}_{,0} + h_3{}^{m0}{}_{,k}h_5{}^{mk}{}_{,0} + h_3{}^{0n}{}_{,m}h_6{}^{0n}{}_{,m} - \tfrac{1}{4}h_2{}^{00}{}_{,m}h_3{}^{00}{}_{,m}) \\ & + h_6{}^{s0}{}_{,0}h_2{}^{00}{}_{,r} + h_6{}^{r0}{}_{,0}h_2{}^{00}{}_{,s} + h_3{}^{s0}{}_{,m}h_6{}^{0m}{}_{,r} + h_3{}^{r0}{}_{,m}h_6{}^{0m}{}_{,s} + h_6{}^{s0}{}_{,m}h_3{}^{0m}{}_{,r} + h_6{}^{r0}{}_{,m}h_3{}^{0m}{}_{,s} - h_5{}^{sm}{}_{,0}h_3{}^{m0}{}_{,r} \\ & - h_5{}^{rm}{}_{,0}h_3{}^{m0}{}_{,s} - h_6{}^{r0}{}_{,m}h_3{}^{s0}{}_{,r} - h_6{}^{r0}{}_{,m}h_3{}^{s0}{}_{,s} - h_3{}^{0m}{}_{,r}h_6{}^{0m}{}_{,s} - h_6{}^{0m}{}_{,r}h_3{}^{0m}{}_{,s} \\ & + \tfrac{1}{4}(h_2{}^{00}{}_{,r}h_7{}^{00}{}_{,s} + h_7{}^{00}{}_{,r}h_2{}^{00}{}_{,s} + \delta^{rs}h_2{}^{00}{}_{,0}h_5{}^{mm}{}_{,0}h_2{}^{00}{}_{,r}h_7{}^{mm}{}_{,s} + h_2{}^{00}{}_{,s}h_7{}^{mm}{}_{,r} - \delta^{rs}h_2{}^{00}{}_{,m}h_7{}^{nn}{}_{,m}) \; . \end{aligned} \tag{A30}$$

When the fields calculated above and in the body of the text are inserted into these expressions and the surface integrals calculated, the result is the reaction force given by Eq. (5.6).

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