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Hamiltonian lattice gravity. Deformations of discrete manifolds

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The structure constants appearing in the Poisson-brackets relations among constraints in the Hamiltonian theory of gravity depend on the properties of deformations of a d-dimensional spatial manifold embedded in a $(d+1)$-dimensional continuum. The study of such deformations is extended to the case where the d-dimensional manifold is a piecewise flat Regge lattice. Although we cannot say whether the algebra of deformations does or does not close for the general lattice, should it close we provide a prescription for obtaining the structure constants. We discuss the case of small lattices where closure can be demonstrated explicitly.

I. INTRODUCTION

A discrete formulation of Lagrangian general relativity has existed[1,2] for over 25 years. A satisfactory Hamiltonian theory, one in which we keep "time" continuous while discretizing "space," is lacking. Hamiltonian gravity[3] is a theory of constraints; for each point in the d-dimensional spatial submanifold we have d "momentum constraints" and one Hamiltonian constraint. On the classical level the Poisson algebra and on the quantum level the commutator algebra of these constraints should close. In the continuum situation the classical algebra does close and the structure functions have a geometric interpretation[4] in terms of deformations of the spacelike manifold. Recently two groups[5] have proposed a transcription of the continuum constraints to a simplicial lattice. In these works the algebra of constraints does *not* close. As discussed in Ref. 4 the structure constants of these algebras depend only on properties of deformations of the spatial manifolds embedded in a space-time continuum. In this article we shall mimic this procedure for the spatially discrete case. We do not obtain explicitly the lattice momentum and Hamiltonian constraints, but find out, should these exist, what their commutation relations would be. We present a straightforward procedure for finding the momentum constraints; to find the Hamiltonian constraints is much more difficult, and it is not even obvious whether these will exist for all lattices.

One method of obtaining discrete Hamiltonian gravity is first to write a Lagrangian in d discretized dimensions while the time is taken to be continuous. One may then proceed with the usual canonical formalism. However, the relation between the canonical momenta and the time derivatives of the spatial metric tensor are not local and the solution for these derivatives in terms of the momenta involves the inverse of a nontrivial matrix. The resulting Hamiltonian is likewise nonlocal. The procedure followed in Ref. 5 was to start with the continuum Hamiltonian, which is local, and then attempt to discretize it. As mentioned earlier the resulting algebra of constraints does not close. In this article we show that should the algebra of deformations of a discretized submanifold close, it will do so in a nonlocal way. As in the continuum case only constraints at nearby points fail to commute, or their Poisson brackets[6] fail to vanish; however, whereas in the continuum the Poisson brackets is expressible as a sum of constraints at points in the vicinity of the points inside the brackets, in the discrete case, if the brackets is at all expressible as a sum of constraints, this sum will involve constraints at all points. Again, we are forced into a nonlocal situation. It is unclear whether the already nonlocal constraints obtained from the discretized Lagrangian would close. We intend to pursue this topic in subsequent works.

What are some of the interests in obtaining a Hamiltonian theory of discrete gravity? Although one has, in the Regge calculus, a fully discretized Lagrangian version, it is useful to have at ones disposal a canonical theory implied by a Hamiltonian formulation. This is especially true when one wants to quantize[7,8] such a theory. Naively one might argue that a Lagrangian is all one needs; a quantum theory is obtained upon the exponentiation of such a Lagrangian and subsequent path integration. However, we must still specify a measure for such a functional integration. Although we could transcribe a continuum measure to a lattice[9] it is desirable to have an *a priori* lattice measure. A Hamiltonian formulation would permit an unfolding,[10] at least on small lattices, of constraints enabling us to define a useful time variable as well as well as studying quantum cosmology on discrete manifolds.

In Sec. II we review the structure of deformations in the continuum situation where a d-dimensional space is embedded in a $(d+1)$-dimensional continuum. Emphasis is placed on the relation between the structure of the brackets relations for the deformations and the commutation relations for the generators of these deformations; these generators are just the dynamical variables that are constrained to be equal to zero in the Hamiltonian theory of gravity. Analogous deformations in the discrete situation are presented in Sec. III and their commutation relations studied in Sec. IV. We cannot state anything definite about the closure of the Poisson-brackets relations of the constraints of discrete Hamiltonian gravity for arbitrary lattices; positive statements about such a closure in the case of certain simple lattices are summarized in Sec. V.

II. REVIEW OF CONTINUUM DEFORMATIONS

The analysis we wish to apply to the study of deformations of a discrete lattice is the analogue of the one applied in Ref. 4 for continuum deformations. Mapping a d-dimensional Euclidean space into a $(d+1)$-dimensional Minkowski space generates a d-dimensional spatial manifold. The coordinates of this manifold will be denoted by $y^A(x)$, where $A=1,\ldots,d+1$ and x is in the underlying Euclidean space. Instead of using the coordinates y^A, it is convenient to use coordinates intrinsic to the embedded manifold. We introduce the tangent vectors

$$y^A_{,r}(x)=\frac{\partial y^A(x)}{\partial x^r} \tag{2.1}$$

and the vector $n^A(x)$ normal to the manifold. An arbitrary deformation δy^A can be decomposed into a normal and a tangent part

$$\delta y^A(x)=\eta^{\perp}(x)n^A(x)+\eta^r(x)y_{,r}(x) \ . \tag{2.2}$$

In Ref. 4 the commutation relations between two such transformations were obtained. The difference in the product of two transformations specified by the infinitesimal parameters $\eta(x)$ and $\xi(x)$ yields a transformation whose parameters are $\zeta(x)$. The relations among these are

$$(\xi^s\eta^r_{,s}-\eta^s\xi^r_{,s})=\zeta^r \ , \tag{2.3a}$$

$$(\eta^r\xi^{\perp}_{,r}-\xi^r\eta^{\perp}_{,r})=-\zeta^{\perp} \ , \tag{2.3b}$$

$$(\eta^{\perp}\xi^{\perp}_{,s}-\xi^{\perp}\eta^{\perp}_{,s})=-g_{rs}\zeta^r \ . \tag{2.3c}$$

The metric g_{rs} is related to the tangent vectors

$$g_{rs}(x)=g_{AB}y^A_{,r}y^B_{,s} \ . \tag{2.4}$$

In the above g_{AB} is the metric of the $(d+1)$-dimensional embedding Minkowski space. The Hamiltonian and momentum constraints of general relativity are the generators of the normal and tangent deformations respectively. Let these be denoted by $\mathcal{H}_{\perp}(x)$ and $\mathcal{H}_r(x)$. The Poisson brackets or commutators of these generators may immediately be inferred from (2.3):

$$[\mathcal{H}_r(x),\mathcal{H}_s(y)]=\mathcal{H}_r(y)\delta_{,s}(x-y)+\mathcal{H}_s(x)\delta_{,r}(x-y) \ , \tag{2.5a}$$

$$[\mathcal{H}_r(x),\mathcal{H}_{\perp}(y)]=\mathcal{H}_{\perp}(x)\delta_{,r}(x-y) \ , \tag{2.5b}$$

$$[\mathcal{H}_{\perp}(x),\mathcal{H}_{\perp}(y)]=[\mathcal{H}^r(x)+\mathcal{H}^r(y)]\delta_{,r}(x-y) \ . \tag{2.5c}$$

Equation (2.5a) ensures that the $\mathcal{H}_r$'s generate the diffeomorphism group of the spatial manifold, (2.5b) indicates that $\mathcal{H}_{\perp}$ transforms as a scalar density under this diffeomorphism. All the dynamics are in Eq. (2.5c). It is the purpose of this paper to obtain analogous relations for the situation where the spatial manifold is discretized into a Regge lattice.

III. LATTICE DEFORMATIONS

The d-dimensional space will be approximated by a collection of d-dimensional simplices, with common $(d-1)$-dimensional subsimplices, embedded in a $(d+1)$-dimensional Minkowski space. We let y_i^A denote the coordinates of vertex i and (ij) represent the directed link joining vertex i to vertex j; $A=1,\ldots,d+1$. We shall study deformations of this simplicial manifold by considering displacements of the vertices, y_i^A, in the $(d+1)$-dimensional embedding space. As in the continuum case, we shall decompose these displacements into "tangent" and "normal" parts. We write these terms in quotes in order to emphasize that these directions are not tangent or normal to the discrete manifold but have roles similar to those in the continuum situation.

On a lattice scalars reside on vertices and vectors on links. It is straightforward to obtain the lattice analogue of the tangent vector $y^A_{,r}$. For the link (ij) the tangent vector is

$$y_{ij}^A=y_i^A-y_j^A \ . \tag{3.1}$$

The definition of the normal vector at point i is not unique. We cannot obtain a vector orthogonal to all the y_{ij}^A's connected to the vertex i. A natural choice for a normal vector at vertex i is obtained by taking the average of the unit vectors normal to simplices emerging from this vertex and weighted by the volume of the corresponding simplex; this vector is then normalized to unity. For example consider the three-dimensional simplex, S, with vertices at y_i, y_j, y_k, and y_l. The normal to S at i is proportional to

$$N_i^A(S)=\epsilon^{ABCD}(y_i^B-y_j^B)(y_i^C-y_k^C)(y_i^D-y_l^D) \ . \tag{3.2}$$

The unit normal at i is defined as

$$n_i^A=\sum_{S(i)}N_i^A(S)\Big/\left|\sum_{S(i)}N_i(S)\right| \ . \tag{3.3}$$

The sums in the above equation range over all simplices having i as a vertex. Using continuum notation

$$N_i^A(S)=\int d^dx\sqrt{g(x)}n^A(x) \ ; \tag{3.4}$$

the integration is over the d-dimensional volume dual to the vertex i.

What do we mean by tangent and normal deformations? A tangent deformation along the link (ij) displaces the vertices i and j,

$$\delta y_i^A=\eta^{ij}y_{ij}^A \ , \quad \delta y_j^A=\eta^{ij}y_{ij}^A \ , \tag{3.5}$$

while the normal displacement along n_i^A is

$$\delta y_i^A=\eta^{\perp,i}n_i^A \ . \tag{3.6}$$

In analogy with the continuum case, we denote the generators of these transformations be $\mathcal{H}_{ij}$ and $\mathcal{H}_{\perp,i}$, respectively. The first question we may ask is whether an arbitrary deformation of the vertex i may be written as a sum of these deformations. Namely, can any δy_k^A be expressed as

$$\delta y_k^A=\sum_{ij}\xi^{ij}y_{ij}^A+\sum_i\xi^{\perp,i}n_i^A \ . \tag{3.7}$$

The second question we can ask is whether or not this decomposition is unique. For an arbitrary lattice, the answer to both questions is no. This may easily be seen in the case of $d=2$. The simplest two-dimensional surface is the boundary of a tetrahedron; this surface has four vertices and six links. There are thus a total of ten parameters available for specifying the deformations of (3.7). However, in three dimensions, the four vertices have 12 independent displacements. The nonuniqueness may be ascertained by considering the displacements of a plane triangulated surface, with all the displacements being in the plane. For a sufficiently large number N_0 of vertices, Euler's theorem tells us that the number of links N_1 is equal to $3N_0$. Thus we have $3N_0$ parameters in (3.7), but only $2N_0$ independent displacements. *The question we wish to address is whether the commutator of two deformations of the type of (3.7) is again a deformation of this type and what are the relations of the parameters of the commutator with those of the original deformations.*

IV. COMMUTATIONS OF DEFORMATIONS

In this section we shall systematically study the three types of commutation relations among the displacements of the lattice vertices: namely, "tangent-tangent," "tangent-normal," and "normal-normal." As mentioned earlier it is not obvious whether the difference of the product of such transformations, once taken in one order and then in reverse order, is again expressible as a sum of such transformations. At best, in the general case, this difference will involve displacements at all points of the lattice. In the last section we shall make some comments for the case of simple lattices.

A. Tangent-tangent commutator

It is evident that the commutator of two tangent deformations, one at link (ij) and the other at (kl) fails to vanish provided the two links have one and only one vertex in common. Consider transformations, as specified in (3.5), with η^{ij} for link (ij) and then with ξ^{jk} for link (jk). It is straightforward to show that the commutator of these transformations yields the following displacements of the vertices:

$$\begin{aligned} \delta y_i{}^A &= \eta^{ij}\xi^{jk} y_{jk}^A \ , \\ \delta y_j{}^A &= \eta^{ij}\xi^{jk}(y_{ij}^A - y_{jk}^A) \ , \\ \delta y_k{}^A &= \eta^{ij}\xi^{jk} y_{ij}^A \ . \end{aligned} \tag{4.1}$$

For an arbitrary lattice, we have not found a way of obtaining this set of displacements as a sum of tangent and normal deformations as specified in (3.7). This question has to be investigated lattice by lattice. Should these displacements be expressible in this form, then we have determined the brackets relations for the generators of the tangent transformations:

$$\eta^{ij}\xi^{jk}[\mathcal{H}_{ij},\mathcal{H}_{jk}] = \sum_{pq}\zeta^{pq}\mathcal{H}_{pq} + \sum_{p}\zeta^{\perp,p}\mathcal{H}_{\perp,p} \ . \tag{4.2}$$

In the above the ζ's are the parameters of deformations that generate the displacements in (4.1).

B. Tangent-normal commutator

The commutator of a tangent deformation on link (ij), Eq. (3.5), and a normal one at vertex k, Eq. (3.6), will fail to vanish whenever the vertex k coincides either with i or j or it is connected by a link to either of these vertices. These commutation relations depend on how the various unit normal vectors change as we perform the deformation along (ij):

$$n_k{}^A \to n_k{}^A + \eta^{ij}\delta_{ij}(n_k{}^A) \ , \tag{4.3}$$

for k either coincident with i or j or linked to either of them; all other normal vectors are unaffected. The detailed form of $\delta_{ij}(n_k{}^A)$ is lattice dependent. Suffice it that for a given lattice it may be determined by straightforward geometric manipulations. We now consider the commutator of a transformation of the form specified by (3.5), with parameter η^{ij} along link (ij), and one specified by (3.6) with parameter $\xi^{\perp,k}$ for vertex k. For $k=i$ we find that the commutator yields the deformations

$$\begin{aligned} \delta y_i{}^A &= \eta^{ij}\xi^{\perp,i}[\delta_{ij}(n_i{}^A) - n_i{}^A] \ , \\ \delta y_j{}^A &= -\eta^{ij}\xi^{\perp,i} n_i{}^A \ , \end{aligned} \tag{4.4}$$

while for k equal to neither i nor j but connected by a link to one of them we obtain

$$\begin{aligned} \delta y_i{}^A &= 0 \ , \\ \delta y_j{}^A &= 0 \ , \\ \delta y_k{}^A &= \eta^{ij}\xi^{\perp,k}\delta_{ij}(n_k{}^A) \ . \end{aligned} \tag{4.5}$$

The brackets of the generators of these transformations is

$$\eta^{ij}\xi^{\perp,k}[\mathcal{H}_{ij},\mathcal{H}_{\perp,k}] = \text{RHS of (4.2)} \ , \tag{4.6}$$

where this time the parameters ζ^{pq} and $\zeta^{\perp,p}$ correspond to the deformations indicated in (4.5).

C. Normal-normal commutator

The procedure should now be clear. The commutator of two normal deformations at vertices i and j fails to vanish only when these vertices are connected by a common link. The geometric quantity we need is the change in $n_i{}^A$ due to a deformation along the normal at vertex j with magnitude $\eta^{\perp,j}$:

$$n_i{}^A \to n_i{}^A + \eta^{\perp,j}\delta_j(n_i{}^A) \ . \tag{4.7}$$

The detailed form of $\delta_j(n_i{}^A)$ is lattice dependent. The commutator of two such transformations, one located at i with magnitude $\eta^{\perp,i}$ and the other one at j with magnitude $\xi^{\perp,j}$ gives displacements only at i and j:

$$\begin{aligned} \delta y_i{}^A &= -\eta^{\perp,i}\xi^{\perp,j}\delta_j(n_i{}^A) \ , \\ \delta y_j{}^A &= \eta^{\perp,i}\xi^{\perp,j}\delta_i(n_j{}^A) \ . \end{aligned} \tag{4.8}$$

From these we find the brackets of the generators of two normal transformations:

$$\eta^{\perp,i}\xi^{\perp,j}[\mathcal{H}_{\perp,i},\mathcal{H}_{\perp,j}] = \text{RHS of (4.2)} \ , \tag{4.9}$$

where this time the parameters in (4.2) are determined by the transformations in (4.8).

V. CONCLUSION

Although we can obtain the form of the deformation induced by the commutator of two displacements we labeled as tangent or normal, it is unclear whether this resultant deformation can be expressed as a sum of the standard displacements. In the continuum case any deformation at a point will be expressible as a sum of tangent and normal displacements restricted to this point. As discussed in Sec. III, this is not true for the general lattice. Even if the commutator turns out to be expressible as a sum of deformations it will involve these at all vertices and links of the lattice, not just the ones in the immediate vicinity of the two deformations being commuted.

For simple lattices the answers to the above questions are positive. Detailed studies were made for the case of a two-dimensional manifold that is the boundary of a three-dimensional tetrahedron and for the case of a three-dimensional manifold that is the boundary of a four-dimensional simplex. *In both cases the commutations relations did close.* Details will be presented in a subsequent article. It is likewise easy to show that we can express the lattice version of the momentum constraints in terms of canonical lattice variables as for example the link lengths and their canonical momenta. $\mathcal{H}_{ij}$ involves changes in the lengths of links having i or j as a boundary. A detailed expression will depend, as it should, on the intrinsic lattice geometry and not on the particular embedding. An expression for the Hamiltonian constraint $\mathcal{H}^{1,i}$ is *not* obtainable in such a straightforward manner; as in the continuum case[11] the commutation relations will determine its general structure. The expression for this quantity is likely to be nonlocal. It should be possible to find a closed form for these Hamiltonian constraints in the case of the simple lattices described above.

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