Comparison between the coherent-pair approximation and projection from a hedgehog Fock state in chiral soliton models

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Comparisons are shown for approximations to the lowest-energy solution of a schematic Hamiltonian using either the coherent-pair approximation of Bolsterli or the hedgehog approximation with variation after projection as given by Fiolhais and Rosina.

I. INTRODUCTION

Effective mean-field models have been used to try to correlate hadronic properties without recourse to a full calculation within QCD. The model Lagrangians are constructed to have only those properties that are deemed important for low-energy phenomena. A typical chiral-invariant Lagrangian density[1–5] is given by

$$\mathcal{L}=\bar{q}[i\partial\!\!\!/-g(\sigma+i\gamma_5\tau\cdot\pi)]q+\tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma+\tfrac{1}{2}\partial_\mu\pi\cdot\partial^\mu\pi-U(\sigma,\pi)\ . \quad (1.1)$$

Here q, σ, and π represent, respectively, quark (fermion), σ (scalar, isoscalar), and pion (pseudoscalar, isovector) fields. The σ and pion fields are chosen as chiral partners to ensure the chiral symmetry. The Lagrangian density is chirally invariant if the nonlinear potential $U(\sigma,\pi)$ is only a function of $\sigma^2+\pi^2$. The form of $U(\sigma,\pi)$ is used to stabilize the system and includes (infinite) counterterms to ensure renormalizability.

Crucial to the application of the models, in the Hamiltonian approach, is the choice of a Fock state to define a physical system. Variation of the energy, with respect to field amplitudes defined by the Fock state, yields equations of motion for the amplitudes which will give a minimum in the energy.

Fock states have been chosen using the mean-field[2–5] (or coherent-state) approximation. In this paper we examine, for a schematic Hamiltonian, two alternative approaches to a coherent-state formalism in the case of the pion field where the relevant quanta of the fields are not S wave. We demonstrate, with the aid of a simple schematic model, that the hedgehog ansatz (as used by Fiolhais and Rosina[6]) is comparable to or better than practical applications of the coherent-pair approximation[7–9] for all values of the coupling constant. This dominance of the hedgehog ansatz (with some small extension) is valid after projection of relevant quantum numbers.

The simple schematic model Hamiltonian has the form

$$H=\xi\left[\sum_{tm}d^\dagger_{tm}d_{tm}-G(a^\dagger\times a)^{11}{:}(d^\dagger+\text{H.c.})\right]\ , \quad (1.2)$$

where $d^\dagger_{tm}$ creates an isovector, P-wave pion with projected quantum numbers t and m, and $a^\dagger_{tm}$ creates an S-wave fermion. G is a renormalized coupling constant which, in principle, includes an absolute coupling constant and space integrals over pion and fermion amplitudes arising, for example, from use of the Lagrangian in (1.1). The parameter ξ also involves, in principle, space integrals over amplitudes and is used here simply to set a scale ($\xi=1$). We use the notation $(a^\dagger\times a)^{11}{:}d^\dagger$ to indicate vector coupling of fermion operators in both isospin and spin spaces, and a scalar product in both spaces with the pion operator.

When operating within the space of S-wave quark states with N and Δ quantum numbers we may write[6]

$$G(a^\dagger\times a)^{11}_{tm}\equiv G'[\tau^{NN}_t\sigma^{NN}_m+(\tfrac{72}{25})^{1/2}(\tau^{N\Delta}_t\tau^{N\Delta}_m+\tau^{\Delta N}_t\sigma^{\Delta N}_m)+\tfrac{4}{5}\tau^{\Delta\Delta}_t\sigma^{\Delta\Delta}_m]\ ,$$

where τ^{NN} (σ^{NN}) are the Pauli spin matrices acting on the isospin (spin) of the bare (quark) nucleon; the operators $\tau^{\Delta\Delta}(\sigma^{\Delta\Delta}),\tau^{N\Delta}(\sigma^{N\Delta})$ have similar meanings for transformations between Δ states or between a Δ state and an N state.

II. COHERENT STATES

In most applications to date the σ field is approximated by a mean scalar classical field: i.e.,

$$\langle\sigma\rangle=\sigma(r)\ . \quad (2.1)$$

This mean-field approximation is equivalent to having a coherent state of S-wave σ quanta. Thus, if we consider the expansion of the σ field in terms of momentum amplitude states,

$$\sigma=\sum_{klm}f_l(kr)Y^{*l}_m[C^\dagger_{klm}+(-)^mC_{kl-m}]\ ,$$

the coherent state is defined by

$$|\mathcal{C}\rangle = N \exp\left[\int_0^\infty \lambda_k C_{k00}^\dagger dk\right]|0\rangle \tag{2.2}$$

with k, l, and m referring to the momentum, angular momentum, and projection of angular momentum, respectively, and N is a normalization coefficient. The coherent state is an eigenfunction of all destruction operators of σ quanta:

$$C_{k00}|\mathcal{C}\rangle = \lambda_k |\mathcal{C}\rangle \ . \tag{2.3}$$

It follows therefore that

$$\langle \mathcal{C}|\sigma|\mathcal{C}\rangle = \int_0^\infty \lambda_k j_0{}^2(kr)k^2 dk \equiv \sigma(r) \text{ by definition} \ . \tag{2.4}$$

Moreover,

$$\langle \mathcal{C}|:\sigma^n:|\mathcal{C}\rangle \equiv \sigma^n(r) \tag{2.5}$$

with the colons denoting normal ordering.

We note the obvious fact that the coherent state is a spherical scalar (S wave). This will be a valid approximation as long as the quarks are themselves in S states. Should the quarks be in nonspherically symmetric states (e.g., when the system is deformed) then other quanta than S wave should be considered for the σ field.

In the case of the pion, the pseudoscalar coupling with S-wave fermions implies that the relevant multipole is P wave. And of course the pion is isovector. A coherent state of pions of the relevant quanta, constructed in analogy to Eq. (2.2), would take the form

$$|\mathcal{C}_\pi\rangle = N \exp\left[\int_0^\infty \sum_{mt} \lambda_{kmt} d_{k1mt}^\dagger dk\right]|0\rangle \tag{2.6}$$

(where we now add an isospin index to the pion field quanta description). Although the state $\mathcal{C}_\pi$ is an eigenfunction of all pion destruction operators, it does not have definite spherical tensor characteristics. When such a state is included in a Fock state, therefore, a projection over the whole Fock state is necessary to recapture the observed quantum numbers of a hadron. The "hedgehog" form of the coherent state, for which $\lambda_{kmt} = \lambda_k \delta_{t,-m}$, has particular relevance since this form has been shown[10] to minimize the energy before projection if restriction is made to the chiral circle ($\sigma^2 + \pi^2 = \text{const}$).

In the application of the hedgehog ansatz with the schematic Hamiltonian in Eq. (1.2), Fiolhais and Rosina[6] chose the full fermion + pion state in the form

$$\chi = [\cos\delta(\tfrac{1}{2})^{1/2}(N_{1-1} - N_{-11}) + \sin\delta(\tfrac{1}{4})^{1/2}(\Delta_{3-3} - \Delta_{1-1} + \Delta_{-11} - \Delta_{-33})]\mathcal{C}_\pi(\text{hedgehog}) \ . \tag{2.7}$$

Here $N_{2t,2m}$ and $\Delta_{2t,2m}$ are the "bare" fermion states which can be considered as having three S-wave quarks coupled to the quantum numbers of the N and Δ, respectively, with projection t and m in the standard way. It is from the full state χ however that "physical" states with the quantum numbers of the N and Δ are projected. The physical state of the N is thus seen to have some bare components with the Δ quantum numbers which, when coupled to the pion field, yields the observed quantum numbers of the N.

In conventional applications of the hedgehog ansatz[1-5] a fermion hedgehog is defined at the quark level, i.e.,

$$h = (u\downarrow + d\uparrow)^3 \ , \tag{2.8}$$

where $u\downarrow$ denotes an up-quark with spin-down, etc., and we suppress an implied antisymmetric color state. The state h, when expanded, yields the state in Eq. (2.7) with $\delta = \pi/4$. The state in Eq. (2.7) is thus seen to have a greater degree of freedom than Eq. (2.8)—a fact that is essential to getting a variational minimum for weak coupling.

Fiolhais and Rosina[6] have computed the energy with projected states from χ in various situations.

(i) VBP. Variation of parameters $\lambda_{mt=-m}$, in hedgehog form of Eq. (2.6) ($\delta = \pi/4$), to yield an energy minimum before projection.

(ii) VAP. Variation of $\lambda_{mt=-m}$ parameters ($\delta = \pi/4$) to yield an energy minimum after projection.

(iii) VAP + δ. As in VAP but with additional variation of δ in Eq. (2.7). Results of their VAP and VAP + δ calculations are given in our Fig. 1.

An alternative to using a single exponential coherent state [as in Eq. (2.6)] has been proposed in the "coherent-pair" approximation by Bolsterli.[7,8] Here coherent states are defined which are eigenfunctions of scalar *pairs* of boson destruction operators. In the case of isovector, P-wave pion operators this takes the form

$$d:d\,|ny\rangle = y\,|ny\rangle \ , \tag{2.9}$$

where we use the colon notation to indicate a scalar product in both J and T spaces. Here the states $|ny\rangle$ have the form

$$|ny\rangle = \sum_{m=0}^{\infty} C_m (y d^\dagger : d^\dagger)^m |n\rangle \ , \tag{2.10}$$

where $|n\rangle$ is a state of n unpaired creation operators on the absolute vacuum. Clearly the J and T characteristics of the state $|ny\rangle$ are carried by the state $|n\rangle$. The coefficients in Eq. (2.10) have been given by Bolsterli in the form

$$C_m = \frac{(2n+\nu-2)!!}{2^m m!(2n+\nu+2m-2)!!} \ , \tag{2.11}$$

where in our case $\nu = 3\times 3 = 9$ is the dimension of the isovector ($2T+1=3$) P-wave ($2J+1=3$) bosons. Bolsterli[7] has given a complete J and T classification of all the (32 in number) $|nyJT\rangle$ states up to four unpaired pions and, in a later paper,[8] up to six unpaired pions. With such states one can now write down multicomponent expression for N and Δ Fock states which au-

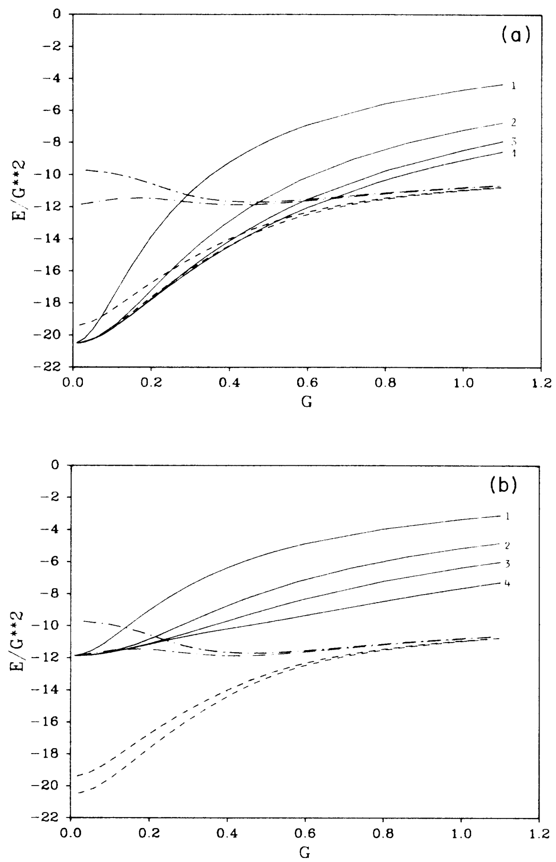

FIG. 1. Approximations to E/G^2 as a function of G. The four solid curves refer to the coherent-pair approximation with indicated maximum number of unpaired pions for the N (a) and for the Δ (b). The dashed and dot-dash curves in each part refer to N's and Δ's, respectively; in each case the upper member of a pair gives VAP and the lower gives VAP + δ.

tomatically have the correct quantum numbers. Thus

$$\tilde{B}=\sum_{nJ_\pi T_\pi}\alpha_{nJ_\pi T_\pi}\{N\mid nyJ_\pi T_\pi\rangle\}+\beta_{nJ_\pi T_\pi}\{\Delta\mid nyJ_\pi T_\pi\rangle\}\ . \tag{2.12}$$

Here N and Δ refer to the standard bare (three-quark) states with N and Δ quantum number, curly brackets denote vector coupling to the correct JT quantum numbers of the baryon state $\tilde{B}$, and α and β are expansion coefficients. Thus with $J=T=\frac{1}{2}$ or $\frac{3}{2}$, $\tilde{B}$ refers to a "physical" N or Δ, respectively. On truncating the sum in Eq. (2.12), the lowest energy of the state $\tilde{B}$ is found by diagonalizing the matrix of the Hamiltonian in the basis $\{B\mid nyJ_\pi T_\pi\rangle\}$ (where B is either N and Δ) and simultaneously varying y to reach a minimum. (In principle one also has the freedom to vary y for each component separately[7] but we do not examine this additional freedom here.)

III. RESULTS AND CONCLUSIONS

The structure of the coherent-pair approximation ensures that the perturbation limit is reached for very small values of the effective coupling constant G. Figure 1 shows how E/G^2 varies with G as the number of unpaired pions is increased (from 1 to 4). In Table I we show actual values for various small values of G. (In the solution of the Hedgehog approximation it was convenient to find the coupling constant corresponding to a given coherence parameter.[6] In order to make a direct comparison therefore these same coupling constants were used in the coherent-pair approximation.) For the nucleon we see that the coherent-pair approximation is better than the hedgehog ansatz up to about $G=0.5$, though it agrees with the VAP + δ to within about 1%. For $G>0.5$ the VBP, VAP, and VAP + δ approximations rapidly become

TABLE I. Explicit values for E/G^2 for small values of G in the VAP + δ approximation (column 2) and the coherent-pair approximation with the 1, 2, 3, or 4 maximum number of unpaired pion components in columns 3, 4, 5, and 6. Shown for the N's (a) and Δ's (b).

G	Hedgehog VAP + δ	Coherent pair 1	2	3	4
			(a)		
0.019 27	−20.479 37	−20.382 87	−20.481 35	−20.481 52	−20.481 52
0.098 38	−19.568 11	−17.802 63	−19.554 84	−19.635 32	−19.637 20
0.191 13	−17.807 13	−14.135 09	−17.429 04	−17.923 84	−17.965 44
0.297 12	−15.928 47	−11.138 47	−14.882 63	−15.875 86	−16.050 53
0.404 19	−14.341 32	−9.130 95	−12.785 10	−14.051 96	−14.382 66
0.518 89	−13.076 62	−7.044 45	−10.313 07	−11.705 87	−12.207 88
			(b)		
0.019 40	−11.855 76	−11.833 13	−11.870 78	−11.870 83	−11.870 83
0.111 56	−11.507 26	−10.632 32	−11.557 12	−11.602 73	−11.604 41
0.220 78	−11.526 39	−8.665 64	−10.624 54	−10.975 87	−11.031 46
0.329 22	−11.800 94	−7.123 13	−9.458 14	−10.211 72	−10.487 90
0.442 64	−11.880 88	−5.963 49	−8.334 04	−9.377 62	−10.015 59

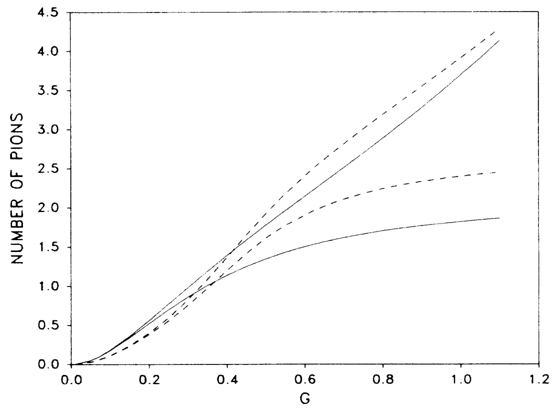

FIG. 2. The average number of pions in N (solid curves) or Δ (dashed curves). The lower of each pair of curves shows the number of pions not paired while the upper shows the total of unpaired and paired pions.

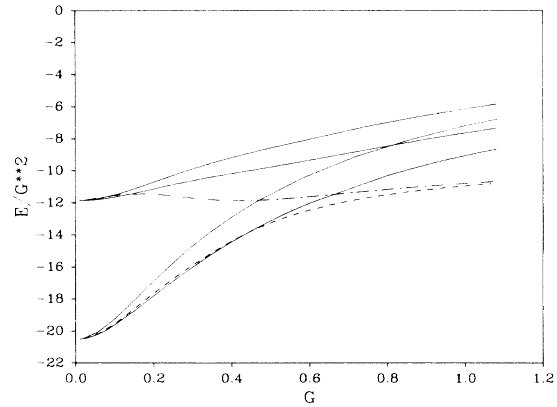

FIG. 3. Showing the effect of including (lower solid curve in each set) or not including (upper solid curve in each set) pairs of pions in the coherent-pair approximation including a maximum of four unpaired pions. The upper (lower) set of curves refers to the Δ (N). Shown also for comparison is the hedgehog in the VAP + δ approximation for both the Δ (dot-dash) and N (dashed).

degenerate leading to the classical field solution as $G \to \infty$. All yield an energy lower than the coherent-pair approximation up to four unpaired pions. The range of applicability of the coherent-pair approximation with up to four unpaired pions for the Δ is limited to $G < 0.2$. For $G > 0.2$ the coherent-pair approximation completely fails to reproduce the characteristic form of the VAP + δ approximation, at least up to four unpaired pions.

Figure 2 shows some of the properties of the coherent-pair solution. Here it is seen that the mean number of unpaired pions in the expansion up to four unpaired pions is always greater than the mean number of pions in pairs. Nevertheless, as Fig. 3 shows, the paired pions play an important role in lowering the energy of the coherent-pair state: at $G=0.5$ the value of E/G^2 is decreased by about 20% by the introduction of coherent pairs. It is also interesting to note that for small G there are, on average, more pions in the N Fock state than in the Δ.

Despite the advantage of the coherent-pair approximation in generating basis states with the correct quantum numbers of a given hadron, the generalized hedgehog ansatz, with variation after projection, is seen to be as good an approximation for small coupling constants and far superior for large coupling constants. Although this analysis was carried out for the schematic Hamiltonian, the general conclusions are expected to be valid for more general Hamiltonians.

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