Quark-gluon plasma at finite baryon density: A large-N_c approach

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A simple model for the quark-gluon plasma with a nonzero baryon number B is developed in a microcanonical (i.e., fixed B) approach. The model exhibits features which one expects will emerge from a nonperturbative treatment of QCD. We show the existence of a critical chemical potential μ_c such that, for $T>0$, physical properties are unaffected by chemical potentials μ when $|\mu|<\mu_c$. μ_c therefore resembles a mass gap.

I. INTRODUCTION

Computer simulations of lattice quantum chromodynamics (QCD) as well as other arguments (for some recent reviews see, e.g., Refs. 1 and 2) suggest the existence of a phase transition from hadronic matter to a quark-gluon plasma phase. For temperatures much larger than the critical temperature, $T_c \approx 200$ MeV, the quark-gluon plasma is thought to behave as an ideal gas with a Stefan-Boltzmann energy-density distribution.

Model-dependent considerations (see, e.g., Ref. 1) suggest that the actual formation of the quark-gluon plasma in high-energy collisions may take place at a typical temperature $T\approx 200$ MeV and in a characteristic volume V of the order 1–10 fm^3. The parameter $\xi=(TL)^3$, where $L=V^{1/3}$, is then of the order one. For massless quarks ξ is a natural parameter of finite-volume corrections in the thermodynamics of confined particles (Refs. 3–9 and references cited therein). In such considerations the confinement mechanism of QCD forces one to impose the constraint that all physical states, which contribute to the partition function of the system, are color singlets.

In computer simulations of QCD one is, of course, forced to work on a finite lattice. The parameter ξ is then given by a purely "geometrical" factor $(N_\sigma/N_\beta)^3$ where N_σ (N_β) is the space (time) extent of the lattice. Finite-volume corrections to the energy density of quarks and gluons because of the global color-singlet constraint have been discussed in the literature and have been shown to be of the same order of magnitude as perturbative corrections on the lattice.[10,11]

Recently the quark-gluon plasma at high density, i.e., with a nonzero chemical potential μ, has been discussed in the literature from a general point of view.[12–15] Mean-field methods have been applied to study the chiral-symmetry restoration[16,17] and various Monte Carlo simulations have been carried out.[12,18–21] In this paper we consider the thermodynamical properties of an ideal quark-gluon gas at finite baryon density and at large N_c, where N_c is the number of colors. Since there are no bound states in an ideal gas, the baryon number simply stands for the quark number divided by N_c. We work in a finite volume V but use, for reasons of simplicity, a continuous momentum spectrum. This volume physically has the interpretation of either the actual size of the system under consideration or the size of "quark-gluon droplets," i.e., hadrons, which, in a baglike picture at sufficiently high temperatures when overlapping each other, form the quark-gluon plasma. In the large-N_c limit these droplets should be weakly interacting[22–24] and hence an ideal-gas approximation should be appropriate in the study of the thermodynamical properties of such a system. There is another reason why we believe that the large-N_c limit is appropriate for our considerations. In this limit "quenching" (see Refs. 25 and 26 and papers quoted therein) suggests that finite-volume considerations actually describe the infinite-volume-limit properties of the system under consideration.

For sufficiently small volumes the discrete character of the momentum spectrum becomes important suggesting that our analysis may be based on an invalid approximation using continuous-momentum degrees of freedom. Numerical investigations however show that the qualitative features remain roughly the same if we make use of a discrete momentum spectrum (see, e.g., Ref. 7). The continuous-momentum-spectrum approximation we used originally in order to simplify the calculations is, however, crucial for the existence of the phase structure of our model. In retrospect, we would like to turn this approximation into a virtue by suggesting that it accommodates effects of interaction.

The results of our paper show that our simple model of the quark-gluon plasma contains features of the phase transition which one expects will emerge in a nonperturbative treatment of QCD at finite temperature and density.

The paper is organized as follows. In Sec. II we summarize, for the convenience of the reader, the derivation of the color-singlet partition function for the ideal, massless quark-gluon gas at finite baryon density. It can be verified that the color-singlet condition corresponds to a nonperturbative integration over classical field configurations with zero action, provided the volume is sufficiently large. Elsewhere we will discuss this issue in detail. In Sec. III we consider the large-N_c limit and the phase structure of the system. Section IV contains a discussion on the thermodynamical properties of the system and a derivation of the phase diagram. Some of the technical aspects are discussed in the Appendix.

II. THE COLOR-SINGLET PARTITION FUNCTION

General methods for extracting a contribution to a partition function Z, due to a partial summation of physical states transforming according to a definite irreducible representation of a group G, have been developed in the literature.[27–31] In these considerations the conserved generators of G are treated microcanonically. The partition function for a given baryon number B, Z_B, can be obtained, e.g., as

$$\begin{aligned} Z_B &= \mathrm{Tr}[\delta(N_c B - Q) e^{-\beta H}] \\ &= \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{-iN_c B\phi} \mathrm{Tr}(e^{-\beta H + i\phi Q}) \\ &= \frac{\beta}{2\pi i} \int_{-i\pi/\beta}^{i\pi/\beta} d\mu\, e^{-N_c B\beta\mu} Z(\mu) . \end{aligned} \tag{2.1}$$

In (2.1) H is the Hamiltonian of the system, β the inverse of the temperature T, and $Z(\mu)$ the grand canonical partition function: i.e.,

$$Z(\mu) = \mathrm{Tr}(e^{-\beta(H-\mu Q)}) , \tag{2.2}$$

where μ is the quark chemical potential, and

$$Q = \sum_{\mathbf{k},l} (n_{\mathbf{k},l} - \bar{n}_{\mathbf{k},l}) .$$

Here $n_{\mathbf{k},l}$ ($\bar{n}_{\mathbf{k},l}$) is the quark (antiquark) number operator, where l denotes a set of internal degrees of freedom.

For the singlet-SU(N_c) ideal and massless quark-gluon plasma with a finite baryon number B, (2.1) is generalized to (see, e.g., Refs. 31 and 32)

$$Z_B = \int_{\mathrm{SU}(N_c)} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \exp(-S[g,\phi]) . \tag{2.3}$$

Here

$$S[g,\phi] = iBN_c\phi + S_{\mathrm{gl}}[g] + S_{q,\bar{q}}(g,\phi) \tag{2.4}$$

contains a contribution from the gluons

$$S_{\mathrm{gl}}[g] = 2\sum_k \mathrm{Tr}[1 - \exp(-\beta\omega_k) R(g)] , \tag{2.5}$$

and a contribution from the fermions

$$\begin{aligned} S_{q,\bar{q}}[g,\phi] = -2n_F \sum_k \mathrm{Tr}[&\ln(1 + e^{-\beta\omega_k + i\phi} g) \\ &+ \ln(1 + e^{-\beta\omega_k - i\phi} g^+)] , \end{aligned} \tag{2.6}$$

where $\omega_k = |k|$. In these expressions $\sum_k$ denotes a momentum summation, Tr a trace over the group variables, $R(g)$ the adjoint representation of SU(N_c), and n_F the number of quark flavors. From now on we replace the momentum sum by an integral, i.e.,

$$\sum_k \to \frac{V}{(2\pi)^3} \int d^3k .$$

The "action" $S[g,\phi]$ depends only on the eigenvalues of g, i.e., $e^{i\alpha_k}$, $k = 1, \ldots, N_c$, $\sum_k \alpha_k = 0$, and hence we can make use of Weyl's parametrization of the reduced group measured.[33] Apart from an irrelevant multiplicative constant we therefore obtain, after an integration over the momentum degrees of freedom,

$$Z_B = \int_{\mathrm{U}(N_c)} \left[\prod_{i=1}^{N} d\alpha_i \right] \exp(-S[\alpha]) , \tag{2.7}$$

where

$$\begin{aligned} S[\alpha] = &-\sum_{i>j} \ln\left[4\sin^2\left[\frac{\alpha_i - \alpha_j}{2} \right]\right] + iB\sum_i \alpha_i \\ &- 2\xi_g \prod_{n=1}^{\infty} \sum_{i,j} \frac{1}{n^4} \cos[n(\alpha_i - \alpha_j)] \\ &+ 2N_c\xi_q \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \sum_i \cos(n\alpha_i) , \end{aligned} \tag{2.8}$$

where ξ_g (ξ_q) is given by the expression

$$2VT^3/\pi^2 \quad (2n_F VT^3/\pi^2 N_c) .$$

Even though $S[\alpha]$ is complex for $B \neq 0$ after integration, we will get a real-valued partition function. This must, of course, be so since the color-singlet constraint only limits the number of possible states which contribute to the partition function (cf. 34).

In the next section we are going to consider the action $S[\alpha]$ in the large-N_c limit. In order to obtain tractable analytical results, we will approximate $S[\alpha]$ by keeping only the $n=1$ term in (2.8). (This means that we are considering, instead of Bose-Einstein and Fermi-Dirac, only Boltzmann statistics.) This approximation does not change the qualitative functional character of $S[\alpha]$ but only the quantitative features of our results. For a pure massless Fermi gas with a zero baryon number, it was shown in Ref. 5 that there is a third-order phase transition for $\xi_q = 0.5$ ($\xi_g = 0$) under such an approximation. This is the same kind of phase transition as found by Gross and Witten[35] in the $N_c = \infty$ limit of pure Yang-Mills lattice gauge theory in two dimensions at zero temperature. If we keep all terms in the sum over n in (2.8) we would instead get a critical value $\xi_q = 0.5/\zeta(3)$, where ζ is the Riemann zeta function. For a pure gluon gas ($\xi_q = 0$) it was also shown in the same way[5] that there is a first-order phase transition at $\xi_g = 1.0$ such that the energy density is zero if ξ_g is less then 1.0. It is straightforward to show that this phase transition remains if one keeps all terms in the sum over n in (2.8) and that one obtains the same critical value for ξ_g.

III. THE LARGE-N_c LIMIT

In the large-N_c limit we replace the discrete sums over eigenvalues in (2.8) by integrals, i.e.,

$$\sum_{i=1}^{N_i} f(\alpha_i) \to N_c \int_0^1 dt\, f(\alpha(t)) . \tag{3.1}$$

In order to make all terms in (2.8) of order N_c^2 we assume (i) that there are xN_c flavors, so that $\xi_q = 2xVT^3/\pi^2$ is independent of N_c, and (ii) that $B = \sigma N_c$.

The expressions (2.7) and (2.8) for the singlet-projected fixed-baryon-charge partition function then become a functional integral. Keeping only the $n=1$ term in (2.8) we get

$$Z_B=\int_{0\le t\le 1} D\alpha(t)^{-N_c^2 S[\alpha(t)]} , \tag{3.2}$$

where

$$\begin{aligned} S[\alpha(t)]=&-\tfrac{1}{2}\int_0^1 dt\int_0^1 dt' \ln\left[4\sin^2\left[\frac{\alpha(t)-\alpha(t')}{2}\right]\right] \\ &+i\sigma\int_0^1 dt\,\alpha(t) \\ &-2\xi_g\int_0^1 dt\int_0^1 dt'\cos[\alpha(t)-\alpha(t')] \\ &-2\xi_q\int_0^1 dt\,\cos\alpha(t) . \end{aligned} \tag{3.3}$$

To obtain the leading term in the large-N_c expansion for the partition function, we make a stationary phase approximation of the α integrals. Thus their integration paths are deformed into the complex plane, and the integral is approximated by the value of the integrand at the stationary point:

$$\ln(Z_B)\approx -N_c^2 S_{\text{eff}} , \tag{3.4}$$

where S_{eff} is the value of $S[\alpha]$, as given by (3.3), at the stationary point $\{\alpha(t), t=[0,1]\}$ determined by

$$\frac{\delta S[\alpha(t)]}{\delta\alpha(t)]}=0 , \tag{3.5}$$

i.e.,

$$-⨍_0^1 dt'\coth\left[\frac{\alpha(t)-\alpha(t')}{2}\right]+i\sigma+2\xi_g\int_0^1 dt'\sin[\alpha(t)-\alpha(t')]+2\xi_q\sin\alpha(t)=0 . \tag{3.6}$$

As usual, in the large-N_c limit approach[35] it is convenient to introduce the spectral density $\rho(\alpha)$ given by

$$\rho(\alpha)=\left[\frac{d\alpha}{dt}\right]^{-1} . \tag{3.7}$$

In terms of ρ the action assumes the form

$$\begin{aligned} S[\alpha]=&-\tfrac{1}{2}\int d\alpha\int d\alpha'\rho(\alpha)\rho(\alpha')\ln\left[4\sin^2\left[\frac{\alpha-\alpha'}{2}\right]\right] \\ &-i\sigma\int d\alpha\,\alpha\rho(\alpha) \\ &-\xi_g\int d\alpha\int d\alpha'\rho(\alpha)\rho(\alpha')\cos(\alpha-\alpha') \\ &-2\xi_q\int d\alpha\,\rho(\alpha)\cos(\alpha)-\lambda\left[\int d\alpha\,\rho(\alpha)-1\right] . \end{aligned} \tag{3.8}$$

We have conveniently added a Lagrange multiplier term in order to enforce the normalization constraint on ρ. Eq. (3.6) expressed in terms of the spectral density can be obtained from (3.8) by differentiation, first with respect to $\rho(\alpha)$ and then with respect to α. It reads

$$⨍ d\alpha'\rho(\alpha')\left[\coth\left[\frac{\alpha-\alpha'}{2}\right]-2\xi_g\sin(\alpha-\alpha')\right] =i\sigma+2\xi_q\sin\alpha . \tag{3.9}$$

Introducing the variables $z=e^{i\alpha}$ and $z'=e^{i\alpha'}$ we obtain

$$\begin{aligned} ⨍_L \frac{dz'}{z'-z}\rho(z')=&\frac{i}{2}(1-\sigma) \\ &+\frac{1}{2}\left[\xi_g\int_L\rho(z')\frac{dz'}{z'^2}+i\xi_q\right]z \\ &-\frac{1}{2}\left[\xi_g\int_L\rho(z')dz'+i\xi_q\right]\frac{1}{z} , \end{aligned} \tag{3.10}$$

where $\rho(z)$ is subjected to the normalization condition

$$\int_L \frac{\rho(z')}{iz'}dz'=1 . \tag{3.11}$$

The integration path L should in principle be determined by the variational equations. It must be such that $\rho(z)dz/iz$ is real, since this quantity measures a number of eigenvalues. It turns out, however, that this is an analytic differential, so that when we are only interested in the values of the integrals, we may choose the path L with great freedom. Techniques for solving singular integral equations of the form (3.10) are well known in mathematics,[36,37] and have been applied in various physical problems. In the Appendix we summarize some relevant expressions. In our case we need the following identifications:

$$F_0=\frac{i}{2}(1-\sigma) , \tag{3.12}$$

$$F_1=\frac{1}{2}\left[\xi_g\int_L\rho(z')\frac{dz'}{z'^2}+i\xi_q\right] , \tag{3.13}$$

$$F_{-1}=-\frac{1}{2}\left[\xi_g\int_L\rho(z')dz'+i\xi_q\right] . \tag{3.14}$$

We see from (3.12) and (A4) that when $\sigma\neq 0$, there is no solution with a closed contour since the normalization condition (3.11) is then not satisfied. A solution therefore exists only for an open contour L (whose end points we denote A and B). Then F_1 and F_{-1} must satisfy the following system of equations:

$$\begin{aligned} F_1&=\frac{\xi_g}{2}\left[F_1I_2-\frac{1}{\sqrt{AB}}F_{-1}I_3\right]+i\frac{\xi_q}{2} , \\ F_{-1}&=-\frac{\xi_g}{2}\left[F_1I_0-\frac{1}{\sqrt{AB}}F_{-1}I_1\right]-i\frac{\xi_q}{2} . \end{aligned} \tag{3.15}$$

Expressions for I_i $(i=0,1,2,3)$ are given in the Appendix.

Because of the above-mentioned analyticity, we may, without loss of generality, choose as path L an arc with radius l and end points $A=le^{-i\phi_c}$ and $B=le^{+i\phi_c}$. Then the solution of (3.15) can be written

$$F_1(l)=\frac{i\xi_q[1-\xi_g\gamma+l^{-2}\xi_g\gamma(1-\gamma)]}{2[(1-\xi_g\gamma)^2-\xi_g{}^2\gamma^2(1-\gamma)^2]}\ ,$$
$$F_{-1}(l)=-F_1(1/l)\ , \tag{3.16}$$

where $\gamma=\sin^2(\phi_c/2)$. From the normalization condition (3.11) one gets

$$F_1 l - F_{-1}l^{-1}=i(2\gamma)^{-1}\ , \tag{3.17}$$

or, with the help of (3.16),

$$\frac{\xi_q(l+l^{-1})}{1+\xi_g\gamma^2-2\xi_g\gamma}=\frac{1}{\gamma}\ . \tag{3.18}$$

This is the first of two equations from which we must find γ (or ϕ_c) and l as functions of ξ_q and ξ_g. The second one can be obtained from the condition that the solution should be bounded at the ends (A4), i.e.,

$$\sigma=\frac{(1-\gamma)(l-l^{-1})}{1-\xi_g\gamma^2}\xi_q\ . \tag{3.19}$$

Since $0<\gamma\leq 1$ one must require that

$$\xi_g+\xi_q(l+l^{-1})\geq 1\ . \tag{3.20}$$

In the case when $\sigma=0$, $l=1$ and the solution of (3.10) with the contour L open, exists only if $\xi_g+2\xi_q\geq 1$ (Ref. 5). When $\xi_g+2\xi_q\leq 1$ we have the solution where L is a closed contour. Since $\sigma=0$, the normalization condition is then automatically satisfied and one obtains a third-order phase transition at $\xi_g+2\xi_q=1$. If $\sigma\neq 0$ (3.18) and (3.19) have a solution for all values of ξ_g, ξ_q, and σ; i.e., there is no phase transition in this case.

IV. THERMODYNAMICAL QUANTITIES

By using (3.5) and (3.9) we can express the effective action (free energy) in the following form:

$$-S_{\text{eff}}=\frac{i\sigma}{2}\int_L \rho(z)\ln z\frac{dz}{z}+\frac{\xi_q}{2}\int_L\rho(z)\left[z+\frac{1}{z}\right]\frac{dz}{iz}-\frac{\lambda}{2}\ . \tag{4.1}$$

Some relevant integrals are calculated in the Appendix. We get the following explicit form for the free energy:

$$-S_{\text{eff}}=\frac{\sigma^2}{2}\left[\ln(1-\gamma)+\frac{\gamma}{1-\gamma}\right]-\sigma\ln l+\frac{1}{2\gamma}+\tfrac{1}{2}\ln\gamma-\tfrac{1}{2}+\xi_q\nu\ , \tag{4.2}$$

where

$$\nu=\left[\xi_q\gamma^2+\frac{1}{2}\left[1+\frac{1}{l}\right](1-\gamma)\right]\Big/(1-\xi_g\gamma^2)\ . \tag{4.3}$$

Thermodynamical quantities can be found as derivatives of the free energy over different parameters, such as T, V, σ, etc. It is rather difficult to calculate these derivatives from the expression (4.2) directly, and it is therefore more convenient to use (3.8). In this case, because of (3.5) one needs only to differentiate with respect to the parameters which enter explicitly into this expression. Thus for the energy density divided by $N_c{}^2$, ϵ_{df}, one gets

$$\epsilon_{df}=-\frac{1}{N_c{}^2V}\frac{\partial}{\partial\beta}\ln Z_B=\frac{3T}{V}\left\{\xi_g\left[\left[\int\rho(\alpha)\cos\alpha\,d\alpha\right]^2+\left[\int\rho(\alpha)\sin\alpha\,d\alpha\right]^2\right]+2\xi_q\int\rho(\alpha)\cos\alpha\,d\alpha\right\}\ , \tag{4.4}$$

where

$$\int\rho(\alpha)\cos\alpha\,d\alpha=\nu \tag{4.5}$$

and

$$\int\rho(\alpha)\sin\alpha\,d\alpha=\frac{-i}{2}\frac{l-l^{-1}}{l-\xi_g\gamma^2}(1-\gamma)\ . \tag{4.6}$$

In the limit when $\sigma\to 0$ can one verify that (4.4) reduces to the corresponding expression in Ref. 5.

In Fig. 1 we compare ϵ_{df}, with $x=0.1$, and the energy density ϵ_c is calculated without the color-singlet constraint and with chemical potential μ, i.e.,

$$\epsilon_c=\frac{3T}{V}[\xi_g+\xi_q(e^{\beta\mu}+e^{-\beta\mu})]\ . \tag{4.7}$$

The chemical potential μ is related to the parameter σ by

$$\sigma=\xi_q(e^{\beta\mu}-e^{-\beta\mu})\ . \tag{4.8}$$

We have analytically verified that ϵ_{df}/ϵ_c approaches one for $\xi_g\to\infty$ as well as for $\xi_g\to 0$. If $\sigma\neq 0$ one finds that

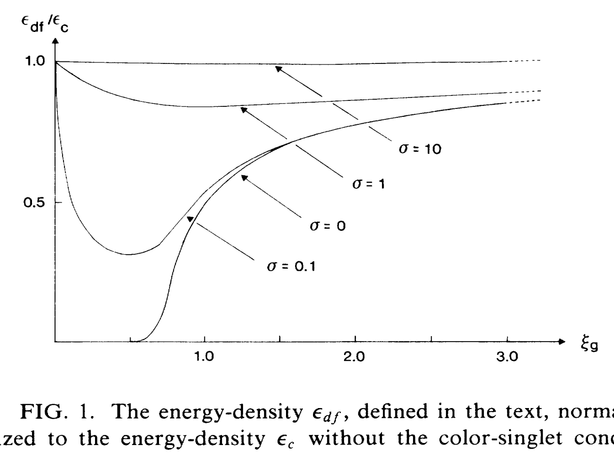

FIG. 1. The energy-density ϵ_{df}, defined in the text, normalized to the energy-density ϵ_c without the color-singlet condition, for $x=n_F/N_c=0.1$ and for various valus of $\sigma=B/N_c$. (B is the baryon number.) For $\sigma=0$ there is a third-order phase transition at $\xi_g\approx 0.833$.

$$\epsilon_{df} \approx \frac{3T}{V}\sigma \tag{4.9}$$

for sufficiently small ξ_g and fixed x.

So far we have focused our attention on the microcanonical partition function corresponding to a fixed baryon number $B=\sigma N_c$, and its effective action $S_{\rm eff}(\sigma)$. Since we find it illuminating, we now discuss the alternative grand-canonical description of the same system.

The chemical potential μ, conjugate to the baryon number, and the grand-canonical partition function were introduced in (2.1) and (2.2). According to (2.1) the microcanonical partition function is the Fourier transform of the grand-canonical partition function. We are interested in the large-N_c approximation in which all angular integrals, including the Fourier transform, are evaluated by saddle-point approximation. Since (by charge symmetry) the saddle point occurs at a real μ, we obtain the grand-canonical effective action from the microcanonical effective action in the large-N_c approximation (where the mean value of σ is σ) by a Legendre transformation

$$\begin{gathered} S_{\rm eff}(\sigma)=S_{\rm eff}(\mu)+\sigma\mu\beta\ , \\ \frac{\partial S_{\rm eff}(\mu)}{\partial\mu}+\sigma\beta=0,\quad \mu\beta=\frac{\partial S_{\rm eff}(\sigma)}{\partial\sigma}\ . \end{gathered} \tag{4.10}$$

We obtain

$$\beta\mu=\int\frac{dz}{iz}\rho(z)\ln(z)=\ln l-\sigma\ln(1-\gamma)-\frac{\sigma\gamma}{1-\gamma}\ . \tag{4.11}$$

One can explicitly check, using (3.18) and (3.19), that μ is an increasing function of σ, tending to infinity when σ tends to infinity, just as one might have expected. However, μ does not necessarily approach zero in the limit $\sigma\to 0$.

From (3.18) and (3.19) we see that there are two different possibilities at $\sigma=0$.

(i) $l=1$. Then $\xi_g+2\xi_q\geq 1$ and $\mu=0$.

(ii) $l>1$, $\gamma=1$. Then $\xi_g+2\xi_q<1$ and

$$\mu=\mu_c=T[\ln l-(l^2-1)/(l^2+1)]\ , \tag{4.12}$$

separated by a Gross-Witten third-order phase transition.[35] These correspond to different phases of the system. An example of the behavior of μ_c/T is shown in Fig. 2. Since μ jumps discontinuously when σ is varied, there is a first-order phase transition on the half-line $\sigma=0$, $\xi_g+2\xi_q<1$, in the $(\sigma,\xi_g+2\xi_q)$ plane. This phase transition is somewhat analogous to the liquid-gas transition in condensed-matter physics, σ and μ playing the roles of volume and pressure. It is not necessarily associated with any symmetry breaking and we have not found any order parameter.

The effective action $S_{\rm eff}(\mu)$ for $|\mu|<\mu_c$ remains to be found. It is evident that such an action, and a corresponding partition function $Z(\mu)$ (2.2) and its derivative, are well defined for all real μ. The Boltzmann statistics and large-N_c approximations do not change this fact. To find it, we recall the discussion after (3.11). The integration path L can be determined from the reality requirement on $\rho(z)dz/iz$ once one point on it is known. This happens in case (i) when $\gamma<1$; L:s end points are unique. In case (ii), however, L is closed and has no *a priori* fixed point. In this case, the expression for ρ is very simple:

$$\rho(z)=\frac{1}{2\pi}\left[1+\left[z+\frac{1}{z}\right]\Big/\left[l+\frac{1}{l}\right]\right]\ . \tag{4.13}$$

The reality condition determining the curve is

$$\mathrm{Re}\left[\ln z+\left[z-\frac{1}{z}\right]\Big/\left[l+\frac{1}{l}\right]\right]=\text{const}\ . \tag{4.14}$$

It is not difficult to see that this gives an acceptable curve L starting with any point in the internal $(-l,-1/l)$. These curves correspond exactly to the

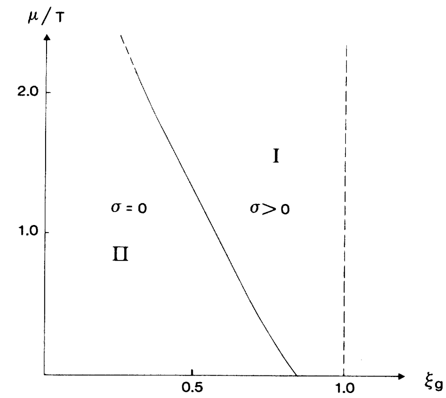

FIG. 2. The curve μ/T at $\sigma=0$ and $x=n_F/N_c=0.1$ divides the μ/T-ξ_g plane into two regions: I and II. Only positive values of μ/T are exhibited. The negative values can be obtained by a reflection in the ξ_g axis. For $\sigma=0$ there is a third-order phase transition at $\xi_g\approx 0.833$ when moving along the ξ_g axis. Any (positive) σ can be obtained in region I. In region II $\sigma=0$ and thermodynamical quantities are independent of μ/T for a fixed value of ξ_g. The dashed curve at $\xi_g=1.0$ corresponds to the limit $x=0$.

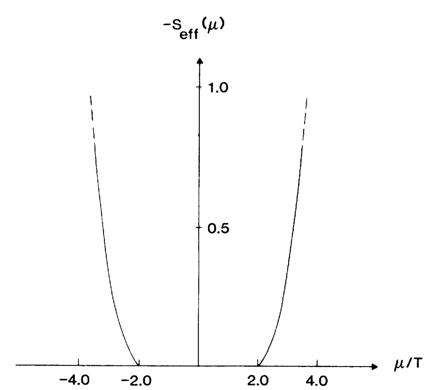

FIG. 3. The grand-canonical effective action, $S_{\rm eff}(\mu)=S_{\rm eff}(-\mu)$, for $\xi_g=0.5$ and $\xi_q=0.05$. For $|\mu|\leq\mu_c\approx 0.31$, $S_{\rm eff}(\mu)=\xi_q{}^2/(1-\xi_g)$.

missing chemical potentials. Since most observables are integrals of analytic functions over the closed contour L, they do not depend on μ in this phase. In particular $\mu=0$ and $L=\{\,|z|=1\}$, describe the same physics at $\mu=\mu_c$. At $T\neq0$, μ_c therefore resembles a mass gap. (At $T=0$ we find, however, that $\mu_c=0$.) This is how Gross and Witten's strong-coupling phase[35] appears in our calculation. Figure 3 shows an example of the behavior of $S_{\text{eff}}(\mu)=S_{\text{eff}}(-\mu)$ for $\xi_g=0.5$ and $x=0.05$ in which case we find that $\mu_c\approx1.313$. For $|\mu|\leq\mu_c$ it follows that $S_{\text{eff}}(\mu)=S_{\text{eff}}(-\mu)=\xi_q{}^2/(1-\xi_g)$.

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APPENDIX: SOLUTION OF THE SINGULAR INTEGRAL EQUATION (3.10), AND SOME RELEVANT INTEGRALS

A method to solve the integral equation

$$⨍_L \frac{dz'}{z'-z}\rho(z')=F_0+F_1z+F_{-1}/z \tag{A1}$$

is given in Refs. 36 and 37. In our case, the function ρ must be normalized, i.e., satisfy (3.11). Unlike the standard situation[36,37] the contour L is not given and different possibilities must be considered.

(i) The contour L is closed. Then the solution of (A1) has the form

$$\rho(z)=\frac{1}{\pi i}(F_0+F_1z-F_{-1}/Z)\ , \tag{A2}$$

and from the normalization condition

$$F_0=i/2\ . \tag{A3}$$

(ii) The contour L is open and consists of an arc with end points A and B. In this case one must require that solutions are bounded at the ends. This is the case if[36,37]

$$F_0=F_1\left[-\frac{A+B}{2}\right]-F_{-1}/\sqrt{AB}\ . \tag{A4}$$

The solution then has the form

$$\rho(z)=\frac{1}{\pi i}\left[F_1-\frac{F_{-1}}{z\sqrt{AB}}\right]\sqrt{(z-A)(z-B)}\ , \tag{A5}$$

and from the normalization condition it follows that

$$iF_1\left[\frac{A+B}{2}-\sqrt{AB}\right]+iF_{-1}\left[\frac{1}{\sqrt{AB}}-\frac{1}{2A}-\frac{1}{2B}\right]=1\,. \tag{A6}$$

We also present some useful formulas in the case when the curve L lies on the circumference of a circle of radius l, and has end points $A=le^{-i\phi_c}$, $B=le^{i\phi_c}$. Let

$$I_n=\frac{1}{\pi i}\int_L\frac{dz}{z^n}\sqrt{(z-A)(z-B)}dz$$

and

$$L_n=\frac{1}{\pi i}\int_L\frac{\ln z}{z^n}\sqrt{(z-A)(z-B)}dz$$

then

$$\begin{aligned}&I_0=l^3I_3=2l^2\gamma(1-\gamma),\quad I_1=lI_2=2l\gamma\ ,\\&I_3=\frac{2\gamma(1-\gamma)}{l}\ ,\end{aligned} \tag{A7}$$

and

$$L_1=-2l(1-\gamma)\ln(1-\gamma)+2l\gamma\ln l-2l\gamma\ , \tag{A8}$$

$$L_2=2(1-\gamma)\ln(1-\gamma)+2\gamma\ln l+2\gamma\ , \tag{A9}$$

where $\gamma=\sin^2(\phi_c/2)$.

To obtain the explicit expression (4.2) for the effective action, we must also calculate the Lagrange multiplier λ. Differentiation of (3.8) with respect to $\rho(\alpha)$ and transformation to z variables leads to the following expression:

$$\begin{aligned}\lambda=&-\int_L\frac{dz}{iz'}\rho(z')\ln\left[\frac{-(z-z')^2}{z'-z}\right]+\sigma\ln z\\&-\xi_q\left[z+\frac{1}{z}\right]-\xi_g\int_L\rho(z')\frac{z^2+z'^2}{zz'}\frac{dz'}{iz'}\ .\end{aligned} \tag{A10}$$

The most deceptive term here is the first one. The branch of the logarithm should be chosen such that it is real on the arc L. In this integral, as in all others in this appendix, it is possible to replace the integration along L by integration around a closed contour, and use analytic function methods. A lengthy but straightforward calculation gives

$$\lambda=1-\frac{1}{\lambda}-\ln\gamma+\sigma\ln l\ . \tag{A11}$$

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