Rephasing-invariant parametrizations of generalized Kobayashi-Maskawa matrices

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We discuss a simple rephase-invariant parametrization of the Kobayashi-Maskawa mixing matrix V which easily generalizes to more than three generations and which we believe to be suitable as a phenomenological standard. Our independent parameters are the magnitudes $|V_{i\alpha}|$ with $i < \alpha$ and the phase of plaquettes, $\arg(V_{i\alpha}V_{j\beta}V_{i\beta}^*V_{j\alpha}^*)$, where $j = i+1$, $\beta = \alpha + 1$, and $j < \beta$. The detailed discussion includes consequences of unitarity constraints, modifications in cases of degenerate quark masses, and the relation of Jarlskog's invariant functions of mass matrices. We reexpress the CP-violation phenomenology of the K-$\bar{K}$ and B-$\bar{B}$ systems in this rephase-invariant formalism. We exhibit a fourth-generation scenario where the top-quark mass need not be large even in the presence of large B_d-$\bar{B}_d$ mixing.

I. INTRODUCTION

In the standard model, CP violation is believed to be a consequence of complex values of elements of the 3×3 Kobayashi-Maskawa (KM) matrix[1] V which describe the couplings of the weak intermediate bosons $W^{\pm}$ to quarks. However, the phases of individual matrix elements of V are not themselves directly observable, because of arbitrariness in choice of phases of the quark fields. Therefore there is strong motivation to find a descriptive structure which is independent of such choices of phase. This problem has received a great deal of attention, and the 3×3 case is well understood.[2] We are motivated to address this issue again mainly by curiosity on how the three-generation description generalizes to n generations.[3] Here the situation is much less clear.

The description we offer does work in the $n \times n$ case, is reasonably simple and straightforward, and uses as raw material the quantities directly emergent from phenomenology. We believe it to be an especially suitable candidate for standardization of the phenomenology.

Our main suggestion is to replace the usual description of the Kobayashi-Maskawa matrix in terms of generalized Euler angles[4] by a description using moduli of matrix elements and plaquette phases, defined below. The name "plaquette" is motivated by a rough analogy to gauge theories; the rephasing transformations play a role analogous to gauge transformations. As the definition suggests, the plaquette phases are then analogous to the field strengths of gauge theories.

In the next section we present the general description. In Sec. III we present details of the argument. In Sec. IV we discuss the cases of three and four generations. A graphical method used to describe unitarity constraints is discussed in Sec. V. Section VI touches on Jarlskog invariants, and parametrizations of mass-degenerate cases are presented in Sec. VII. Rephasing-invariant phenomenology occupies Sec. VIII. Section IX concludes.

II. THE GENERAL PRESCRIPTION

We label the $n \times n$ Kobayashi-Maskawa matrix $V_{i\alpha}$ with latin indices for $Q = \frac{2}{3}$ quarks ($i = u, c, t, \ldots$) and greek indices for the $Q = -\frac{1}{3}$ quarks ($\alpha = d, s, b, \ldots$). The number of independent real parameters characterizing the (unitary) V is n^2. Of these, $n(n-1)/2$ are "angle" parameters (this being the number of independent parameters for $n \times n$ real rotations). Of the $2n$ possible rephasings of the quark fields, one (a common phase change of all $2n$ quark fields) leaves V invariant. Hence the number of independent "phase" variables is

$$n^2 - \frac{n(n-1)}{2} - (2n-1) = \frac{(n-1)(n-2)}{2} . \tag{1}$$

A typical observable (in particular anything obtainable from Feynman-diagram calculations) will be a polynomial in V's and V^*'s, with the restriction that in each term of the polynomial there be equal numbers of V's and V^*'s, and that in each term the set of indices $\{i\}$ in the product of V's be identical to the set $\{i\}$ in the V^*'s (this must of course also be true for the greek indices $\{\alpha\}$).

The simplest observable is the magnitude of each KM element: $(V_{i\alpha}V_{i\alpha}^*)^{1/2}$. The simplest which contains phase information is a product of four V's: $V_{i\alpha}V_{j\beta}V_{i\beta}^*V_{j\alpha}^*$. For the case $|i-j| = |\alpha - \beta| = 1$, we call this product a plaquette. The plaquettes, together with the $|V_{i\alpha}|^2$, will be our basic building blocks. We define plaquettes as

$$\square_{i\alpha} = V_{i\alpha}V_{i-1,\alpha-1}V_{i,\alpha-1}^*V_{i-1,\alpha}^* . \tag{2}$$

We furthermore define plaques as

$$^{j\beta}\square_{i\alpha} = V_{i\alpha}V_{j\beta}V_{i\beta}^*V_{j\alpha}^* . \tag{3}$$

As we demonstrate later, any observable consisting of a product of V's and V^*'s can be written as a product of plaquettes, possibly multiplied by a product of $|V_{i\alpha}|$, and possibly divided by another product of $|V_{i\alpha}|$. (We

assume, here and in what follows, except Sec. VII, that all elements of the KM matrix are nonvanishing.) It is therefore natural to associate the magnitudes of the $|V_{i\alpha}|$ with "angle" variables and the phases of the plaquettes (often just the imaginary part suffices) with the "phase" variables. In particular if we choose the $|V_{i\alpha}|$ with $\alpha > i$ as "angle" variables and also the $\arg\square_{i\alpha}$ with $\alpha > i$ as the "phase" variables, the counting comes out correctly: there are $n(n-1)/2$ independent $|V_{i\alpha}|$ and $(n-1)(n-2)/2$ $\arg\square_{i\alpha}$. [The top-most row with $i=1$ is unavailable, and one has $n(n-1)/2-(n-1)=(n-1)(n-2)/2$ elements remaining.]

This is our main proposition: *use the* $|V_{i\alpha}|^2$ *and* $\arg\square_{i\alpha}$ *with* $\alpha > i$ *as the independent set of rephase-invariant variables.* We will show later that, given these parameters, the entire KM matrix is determined up to the $2n-1$ arbitrary quark-field phases, and up to a finite ambiguity which is no greater than 2^{n-2}-fold, coming from solving quadratic equations in determining the magnitude of unknown diagonal V's. In the 3×3 case, this implies that $|V_{us}|^2$, $|V_{ub}|^2$, $|V_{cb}|^2$, and $\arg\square_{cb}$ are the principal parameters. In the standard KM parametrization, it is the imaginary part of the plaquette:[5]

$$J = \mathrm{Im}\square_{cb} = \mathrm{Im}V_{cb}V_{us}V_{cs}^*V_{ub}^* = s_1^2 s_2 s_3 c_1 c_2 c_3 \sin\delta , \qquad (4)$$

which is the familiar and ubiquitous combination present in CP-violation phenomena. We note that

$$\mathrm{Im}\square_{cb} \leq |\square_{cb}| \leq (0.05)\times(0.2)\times1\times(0.01)$$
$$\sim 10^{-4} . \qquad (5)$$

We also note the important result that in the 3×3 case,[5] all plaques have the same imaginary part. In fact, for all $i\neq j$, $\alpha\neq\beta$,

$$\mathrm{Im}V_{i\alpha}V_{j\beta}V_{i\beta}^*V_{j\alpha}^* = \mathrm{const} = \pm J . \qquad (6)$$

This is a consequence of unitarity of the KM matrix, and is discussed further in Sec. IV.

In the 4×4 case, the parameters are supplemented, in an obvious notation, by five new quantities: namely, $|V_{uB}|^2$, $|V_{cB}|^2$, $|V_{tB}|^2$, $\arg\square_{cB}$, and $\arg\square_{tB}$. Thus, were new generations to emerge, the phenomenological structure need not undergo any major revision. New parameters become introduced and old unitarity constraints are modified. However the moduli and plaques below and on the diagonal will be complicated functions[6] of the above parameters.[7]

III. DETAILS

In order to substantiate the assertions of the previous section, it is necessary to first show that any (rephase-invariant) observable can be expressed in terms of $|V_{i\alpha}|$ and phases of plaquettes. Second, we have to show that given only the $|V_{i\alpha}|^2$ and $\arg\square_{i\alpha}$ with $\alpha > i$ all remaining parameters of the KM matrix are determined.

To demonstrate these assertions, it is useful to depict the observables, plaquettes, etc., which are products of V's and V^*'s graphically.[8] The procedure is as follows.

(i) If $V_{i\alpha}$ appears in the product, place an $\bigcirc$ in the $i\alpha$ entry of an originally empty $n\times n$ matrix. If $V_{i\alpha}^*$ appears, place an $\times$.

(ii) Then, from rephase invariance, each row (or column) must have equal numbers of $\times$'s and $\bigcirc$'s.

(iii) An $\otimes$ in a single given $i\alpha$ location is a factor $|V_{i\alpha}|^2$. These can be inserted or removed at will without changing the phase of the expression.

Now, given an arbitrary product $V_{i\alpha}\cdots V_{j\beta}^*$, which corresponds to a matrix with $\times$ and $\bigcirc$ entries, we may systematically eliminate the $\times$'s and the $\bigcirc$'s from the first column in terms of plaquettes, and then continue the procedure column by column. For example,[9]

$$\arg\begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ \times & \cdot & \bigcirc & \cdot \\ \cdot & \cdot & \times & \bigcirc \\ \bigcirc & \cdot & \cdot & \times \end{vmatrix} = \arg\begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ \times & \otimes & \bigcirc & \cdot \\ \otimes & \otimes & \times & \bigcirc \\ \bigcirc & \cdot & \cdot & \times \end{vmatrix} = -\arg\square_{32}+\arg\begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \times & \bigcirc & \cdot \\ \times & \bigcirc & \times & \bigcirc \\ \bigcirc & \cdot & \cdot & \times \end{vmatrix}$$

$$= -\arg\square_{32}-\arg\square_{33}+\arg\begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \times & \cdot & \cdot & \bigcirc \\ \bigcirc & \cdot & \cdot & \times \end{vmatrix}$$

$$= \cdots = -\arg\square_{32}-\arg\square_{33}-\arg\square_{42}-\arg\square_{43}-\arg\square_{44} . \qquad (7)$$

We trust the procedure is clear enough not to require the formal proof here. Therefore we argue it is possible to express all observables in terms of the magnitudes of KM matrix elements and the phases of plaquettes. What remains to be shown is that the limited set $\{|V_{i\alpha}|, \arg\square_{i\alpha}\}$ with $\alpha > i$ suffices as well. We do this in several steps, by construction.

(1) First choose the phases of $V_{1\alpha}$ and V_{in}. Since there are $2n-1$ such elements, this exhausts the arbitrariness associated with rephasing of quark fields. (We shall return later to a suggestion for how this phase choice might most conveniently be made.)

(2) Use unitarity to determine $|V_{11}|$ and $|V_{nn}|$:

$$|V_{11}|^2 = 1 - \sum_{\alpha>1}|V_{1\alpha}|^2 , \qquad (8a)$$

$$|V_{nn}|^2 = 1 - \sum_{i<n}|V_{in}|^2 . \qquad (8b)$$

(3) At this point all elements in the top row and right-hand column are fully determined. Thus the phase of $V_{2,n-1}$ can be determined from the phase of the plaquette $\square_{2n}$ in the upper-right-hand corner of the matrix

$$
\begin{bmatrix}
* & * & * & \boxed{* & *} \\
\cdot & \cdot & \cdot & \boxed{\cdot & *} \\
\cdot & \cdot & \cdot & \cdot & * \\
\cdot & \cdot & \cdot & \cdot & * \\
\cdot & \cdot & \cdot & \cdot & *
\end{bmatrix} . \tag{9}
$$

(4) In the same way, the phases of the remaining $V_{2\alpha}$ in the second row with $\alpha \geq 2$ may be determined iteratively in terms of plaquette phases: e.g.,

$$
\begin{bmatrix}
* & \boxed{* & *} & * & * \\
\cdot & \boxed{\cdot & *} & * & * \\
\cdot & \cdot & \cdot & \cdot & * \\
\cdot & \cdot & \cdot & \cdot & * \\
\cdot & \cdot & \cdot & \cdot & *
\end{bmatrix} . \tag{10}
$$

Note that the phase of V_{22} is determined at this stage, but not its magnitude.

(5) The same procedure may be followed to determine the phases of all $V_{i\alpha}$ with $\alpha \geq i$,

$$
\begin{bmatrix}
* & * & * & * & * \\
\cdot & * & * & * & * \\
\cdot & \cdot & * & * & * \\
\cdot & \cdot & \cdot & * & * \\
\cdot & \cdot & \cdot & \cdot & *
\end{bmatrix} , \tag{11}
$$

although it must again be remembered that $|V_{ii}|$ is not determined for $1 < i < n$.

(6) We now use unitarity to obtain the missing parameters in the second row. The orthogonality of the first and second rows gives a linear relation between the (complex) V_{21}, the real $|V_{22}|$, and previously determined quantities:

$$
V_{21}V_{11}^* + V_{22}V_{12}^* + \sum_{\alpha=3}^{n} V_{2\alpha}V_{1\alpha}^* = 0 . \tag{12}
$$

Now we may introduce the unitarity constraint of normalization of the second row:

$$
|V_{21}|^2 + |V_{22}|^2 = 1 - \sum_{\alpha=3}^{n} |V_{2\alpha}|^2 . \tag{13}
$$

The combination of Eqs. (12) and (13) is a quadratic equation in the unknown $|V_{22}|$. If the off-diagonal elements of the KM matrix are small (as in the case here), one root is positive (the physically correct solution). The other root will be negative, near -1, and thus physically unacceptable.[10]

(7) This procedure can be again iterated. In the third row there are two orthogonality equations which determine the (complex) V_{31} and V_{32} as linear functions of $|V_{33}|$ with coefficients determined in terms of known quantities (up to the remote possibility of a twofold ambiguity in determining $|V_{22}|$). Normalization of the third row leads to a quadratic equation for $|V_{33}|$ with

a remotely possible twofold ambiguity in its solution.

(8) When we reach the nth row, the same procedure again may be used to determine $|V_{nn}|$. However $|V_{nn}|$ was already determined in step 2 without ambiguity. Thus no additional ambiguity is introduced at this stage, and it can be expected that the overall degree of ambiguity will, if present at all, be reduced. A highly conservative statement is that there is at most a 2^{n-2}-fold ambiguity in reconstructing the KM matrix from the input data. However, as long as the off-diagonal elements are as small as those seen experimentally, there will in fact be no ambiguity at all.

This completes the general argument on reconstruction of all KM parameters from the input parameters. In the next section we will explicitly show how the procedure works for the three- and four-generation cases.

IV. PARAMETRIZATION

A. Three generations

The magnitudes of the KM elements which serve for us as inputs are[11]

$$
|V_{us}| = 0.220 \pm 0.002 , \tag{14a}
$$

$$
|V_{ub}| \leq 0.011 \ (90\% \ \text{C.L.}) , \tag{14b}
$$

$$
|V_{cb}| = 0.048 \pm 0.010 . \tag{14c}
$$

We suggest that for reconstruction purposes the five independent phase choices be made as follows. (1) V_{ud}, V_{us}, V_{cb}, and V_{tb} are chosen real and positive; (2) the phase of V_{ub}^* is chosen equal to the phase of the (only) input plaquette $\square_{cb}$:

$$
\arg V_{ub}^* = \arg \square_{cb} . \tag{15}
$$

This implies that V_{cs} is also real and positive. Then we may proceed to reconstruct the remaining V's.

B. Four generations

In the case for four generations we proceed in a similar way. Again it will be convenient to choose phases such that the phases of plaquettes of interest are directly related to phases of the KM matrix elements in the upper right-hand corner; i.e., V_{ub}, V_{uB}, and V_{cB}. We shall choose those such that their neighbors are real and positive. Specifically, the proposed generalization of the preceding section is as follows.

(1) Choose V_{ud}, V_{us}, V_{tB}, and V_{TB} real and positive.

(2) As before, choose the phase of V_{ub}^* equal to the phase of the plaquette $\square_{cb}$:

$$
\arg V_{ub} = -\arg \square_{cb} . \tag{16}
$$

(3) In the same way choose the phase of V_{cB}^* equal to the phase of the plaquette $\square_{tB}$:

$$
\arg V_{cB} = -\arg \square_{tB} . \tag{17}
$$

(4) Finally choose the phase of V_{uB} so that V_{cb} remains real and positive. This is accomplished by the choice

$$\arg V_{uB} = \arg V_{ub} + \arg V_{cB} - \arg\square_{cB}$$
$$= -\arg\square_{cb} - \arg\square_{tB} - \arg\square_{cB} \ . \qquad (18)$$

(5) From these definitions, it follows that, as in the 3×3 case, V_{cs}, V_{cb}, and V_{tb} remain real and positive. The situation is shown schematically as

$$V = \begin{vmatrix} R & R & * & * \\ \cdot & (R) & (R) & * \\ \cdot & \cdot & (R) & R \\ \cdot & \cdot & \cdot & R \end{vmatrix} , \qquad (19)$$

where R denotes real and positive by definition, an asterisk denotes complex, and (R) denotes real and positive as a consequence of the phase choices made for the starred elements.

It is clear there is a useful generalization here: for the $2n - 1$ phase choices, take the V_{ii} and $V_{i,i+1}$ to be real and positive. This puts the number of remaining phases for elements above the diagonal equal to the number of independent plaquette phases. The analysis of Sec. III remains valid with this convention. Indeed, we believe that if one insists on a standard phase convention, this one might be useful for phenomenology, since its connection to the rephase-invariant plaquettes is manifest.

V. MODIFICATIONS TO THE RELATION $J = \pm \text{Im}\square$ IN FOUR GENERATIONS

We show some simple diagrammatics based on unitarity to prove the well-known result $\text{Im}\square = \pm J$ [Eq. (6)] in the three-generation case;[5] and then extend the diagrammatics to any number of generations. Define $\text{Im}\square_{cs} = J$; then unitarity of KM matrix gives

$$V_{11}V_{12}^* + V_{12}V_{22}^* + V_{13}V_{23}^* = 0 \ . \qquad (20)$$

One can write equivalently

$$\begin{vmatrix} \circ & \cdot & \cdot \\ \times & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \circ & \cdot \\ \cdot & \times & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & \circ \\ \cdot & \cdot & \times \\ \cdot & \cdot & \cdot \end{vmatrix} = 0 \ . \qquad (21)$$

Multiplying by $V_{12}^* V_{22}$ we obtain

$$\begin{vmatrix} \circ & \times & \cdot \\ \times & \circ & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \otimes & \cdot \\ \cdot & \otimes & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \times & \circ \\ \cdot & \circ & \times \\ \cdot & \cdot & \cdot \end{vmatrix} = 0 \ . \qquad (22)$$

Taking the imaginary part removes the second term and we get

$$\text{Im}\square_{22} = \text{Im}\square_{23} = J \ . \qquad (23)$$

In this fashion one easily sees that in three generations only one CP-sensitive parameter exists: $\text{Im}\square = \pm J$.

The generalization to 4×4 matrices or higher creates a very large number of such relations. It is interesting to see how far one can go with these. Already for four generations, the large number of linear unitarity constraints which one can write down contain many which are linearly dependent. After detailed examination it turns out that the nine $\text{Im}\square_{i\alpha}$ for plaquettes can be expressed linearly in terms of nine other quantities which are the imaginary parts of big plaques. By a big plaque we mean a quantity

$$\text{big plaque} = V_{i\alpha}V_{j\beta}V_{i\beta}^* V_{j\alpha}^* \ , \qquad (24)$$

with $|i - j| \geq 2$ and $|\alpha - \beta| \geq 2$.

There are four 2×2 big plaques, four 3×2 big plaques, and one 3×3 big plaque. This reexpression of plaquettes can be useful because, under the assumption that $|V|$ decreases the farther it is from the diagonal, one relates phases of plaquettes on the diagonal to phases of elements, the moduli of which are small. Note that there is only one big plaque which can reside in the three-generation submatrix. By repeated use of the unitarity condition,[12]

$$\text{Im}\square_{22} = \text{Im} \left[\begin{vmatrix} \circ & \cdot & \times & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \times & \cdot & \circ & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \circ & \cdot & \cdot & \times \\ \cdot & \cdot & \cdot & \cdot \\ \times & \cdot & \cdot & \circ \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \circ & \cdot & \times & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \times & \cdot & \circ & \cdot \end{vmatrix} + \begin{vmatrix} \circ & \cdot & \cdot & \times \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \times & \cdot & \cdot & \circ \end{vmatrix} \right] ,$$

$$\text{Im}\square_{23} = \text{Im}\square_{22} + \text{Im} \left[\begin{vmatrix} \cdot & \circ & \cdot & \times \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \times & \cdot & \circ \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \circ & \cdot & \times \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \times & \cdot & \circ \end{vmatrix} \right] ,$$

$$\text{Im}\square_{32} = \text{Im}\square_{22} + \text{Im} \left[\begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ \circ & \cdot & \times & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \times & \cdot & \circ & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ \circ & \cdot & \cdot & \times \\ \cdot & \cdot & \cdot & \cdot \\ \times & \cdot & \cdot & \circ \end{vmatrix} \right] ,$$

$$\text{Im}\square_{33} = \text{Im}\square_{23} + \text{Im}\square_{32} - \text{Im}\square_{22} + \text{Im} \left[\begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot & \times \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \times & \cdot & \circ \end{vmatrix} \right] . \qquad (25)$$

In the limit of a trivial fourth-generation contribution (i.e., no off-diagonal elements of V_{i4} or $V_{4\alpha}$), this reduces immediately to the three-generation case. To make good use of these relations, however, appears to require some knowledge of the fourth-generation KM matrix elements. One can say that sufficient conditions for the three-generation relations to survive are that $|V_{uB}|$, $|V_{cB}|$, $|V_{Td}|$, and $|V_{Ts}|$ all be small compared to 10^{-2}.

VI. JARLSKOG INVARIANTS

Jarlskog, in an interesting paper,[13] pointed out that all physical quantities must be independent of an arbitrary *unitary* transformation affecting simultaneously the up- and down-quark mass matrices, denoted, respectively, m and m'. One diagonalizes the "square" of the mass matrices via[13,14]

$$U(mm^\dagger)U^\dagger = D^2 , \qquad (26a)$$

$$U'(m'm'^\dagger)U'^\dagger = D'^2 , \qquad (26b)$$

where U and U' are unitary matrices. The KM matrix is defined as

$$V = UU'^\dagger . \qquad (27)$$

There are $n^2 + 1$ physical measurables, $2n$ quark masses, and $(n-1)^2$ physical parameters of the mixing

matrix. Jarlskog pointed out that physics does not change under the transformation

$$mm^\dagger \to Xmm^\dagger X^\dagger , \qquad (28a)$$

$$m'm'^\dagger \to Xm'm'^\dagger X^\dagger , \qquad (28b)$$

where X is an arbitrary unitary matrix. Under such a transformation, Eq. (28), the mass eigenvalues of the up and down quarks and even the mixing matrix

$$V \to V \qquad (29)$$

stay invariant. As is well known, not all the mixing matrix elements are physical quantities. The transformation discussed in previous sections which leaves physics invariant, but *changes* the phases of KM elements is

$$U \to TU , \qquad (30a)$$

$$U' \to B^\dagger U' , \qquad (30b)$$

where T, B are arbitrary diagonal unitary matrices

$$V \to TVB . \qquad (31)$$

It appears that Jarlskog's approach includes all the physics, since one can not only express any $|V|$,

$$|V_{i\alpha}|^2 = \text{tr}[v_i(S)v'_\alpha(S')]/(\det v \det v') , \qquad (32a)$$

but also any plaque as an invariant function of mass matrices, Eq. (28). In particular

$$^{k\beta}\square_{i\alpha} = V_{i\alpha}V_{k\beta}V^*_{i\beta}V^*_{k\alpha} = V_{i\alpha}V^*_{k\alpha}V_{k\beta}V^*_{i\beta} = V_{i\alpha}V^\dagger_{\alpha k}V_{k\beta}V^\dagger_{\beta i} = \text{tr}(E_i VE_\alpha V^\dagger E_k VE_\beta V^\dagger)$$

$$= \text{tr}[v_i(S)v'_\alpha(S')v_k(S)v'_\beta(S')]/(\det v \det v')^2 . \qquad (32b)$$

Here $S = mm^\dagger$, $S' = m'm'^\dagger$; E_i and E_α are the elementary matrices; $v_i(S)$ and $v'_\alpha(S')$ are the Vandermonde-type matrices for the up and down sectors, respectively (consult Ref. 13 for details[15]).

It appears that invariance under Eq. (28) can be likened to redefinition of fields and not to an underlying internal symmetry. A lucid example is the case of multiple scalar fields $\phi^t = (\phi^1, \ldots, \phi^n)$ with a ϕ^4 interaction. The Lagrangian density reads

$$L = \frac{1}{2}[\partial^\mu \phi^\dagger \partial_\mu \phi - \phi^\dagger m^2 \phi + \lambda(\phi^\dagger \phi)^2] . \qquad (33)$$

The mass matrix can be diagonalized by a unitary transformation U, i.e.,

$$U^\dagger m^2 U = D^2 . \qquad (34)$$

Then under the redefinition of fields

$$\phi \to U\phi \qquad (35)$$

we will have a diagonalized Lagrangian. Obtaining the "same physics" does not require identical actions $S = \int d^4x \, L$, but rather "same physics" falls into equivalence classes, definable by all Lagrangians having identical m^2 eigenvalues (not necessarily equal m^2 matrices).

VII. DEGENERATE MASSES

A. Mass degeneracies

It is interesting to consider cases of d-fold degeneracies[16] in the up- or down-quark masses. For instance, as is well known, in the three-generation case existence of one twofold degeneracy implies the nonexistence of a CP-violating phase. For four generations and two twofold degeneracies, there remain four angle parameters and one phase, and not five angles and no phases.

Consider a d-fold mass degeneracy, and for concreteness take the first d up quarks to be degenerate. Under a $d \times d$ unitary reshuffling U of the first d rows of the KM matrix physics cannot change:

$$V \to \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} V . \qquad (36)$$

Call the KM matrix restricted to the first d rows $\bar{V}$. We observe that $\bar{V}^\dagger \bar{V}$ is invariant under U transformations Eq. (36) (this is trivial for $V^\dagger V$). The invariants under Eq. (36) are

$$(\bar{V}^\dagger \bar{V})_{\alpha\beta} = \sum_{i=1}^d \bar{V}^*_{i\alpha}\bar{V}_{i\beta} = \sum_{i=1}^d V^*_{i\alpha}V_{i\beta} . \qquad (37)$$

The summation extends only over the degenerate mass rows. Physically invariant quantities are obtained when we create rephasing-invariant combinations in the down sector of $V^\dagger V$, Eq. (37); for example,

$$\sum_{i=1}^{d} V_{i\alpha}^* V_{i\alpha} \, , \tag{38a}$$

$$\sum_{i=1}^{d} V_{i\alpha}^* V_{i\beta} (V_{j\beta}^* V_{j\alpha}) = \sum_{i=1}^{d} {}^{i\beta}\square_{j\alpha} \quad \text{for } j > d \, , \tag{38b}$$

$$| V_{j\alpha} | , {}^{j\alpha}\square_{k\beta} \quad \text{for } j,k > d \, . \tag{38c}$$

It seems possible to express any other physical quantity in terms of Eqs. (38a)–(38c). For instance,

$$\sum_{i=1}^{d} V_{i\alpha}^* V_{i\beta} \sum_{j=1}^{d} V_{j\beta}^* V_{j\alpha} = \sum_{i,j=1}^{d} {}^{j\alpha}\square_{i\beta}$$

$$= \left[\sum_{i,j=1}^{d} {}^{k\alpha}\square_{i\beta} \, {}^{j\alpha}\square_{r\beta} \right] \bigg/ {}^{r\beta}\square_{k\alpha}$$

$$\text{for } k,r > d \tag{39}$$

or

$$\sum_{i=1}^{d} V_{i\alpha}^* V_{i\beta} V_{j\beta}^* V_{k\alpha} V_{k\gamma}^* V_{j\gamma} = {}^{k\alpha}\square_{j\gamma} \sum_{i=1}^{d} {}^{j\alpha}\square_{i\beta} \bigg/ | V_{j\alpha} |^2$$

$$\text{for } k,j > d \, . \tag{40}$$

As a physical parametrization we could choose the angle and phase parameters from the region bounded above by the dth row and bounded to the right by the diagonal (the dth row and the diagonal are not included):

$$d \left\{ \begin{bmatrix} \diagdown & \cdot & \cdot & \cdot & \cdot \\ \cdot & \diagdown & \cdot & \cdot & \cdot \\ \cdot & \cdot & \diagdown & \cdot & \cdot \\ \diamond & \diamond & \diamond & \diagdown & \cdot \\ \diamond & \diamond & \diamond & \diamond & \diagdown \end{bmatrix} \right. \, . \tag{41}$$

The angle parameters, denoted by diamonds, are taken as the magnitudes of KM elements in the above region. The phase parameters are taken as the arguments of all those plaquettes that involve at least three KM elements from the above region. Indeed any plaque with two elements from the first d rows can always be rotated to 0; it has no physical significance. By a unitary transformation on the first d rows, Eq. (36), one can rotate all the first d rows below the diagonal simultaneously to 0. Therefore no angle or phase content is neglected in the parametrization, Eq. (41).

It is also possible to analyze cases for which there are two or more sets of degenerate quarks. For example, look at a four-generation model where $m_u = m_c$ and $m_d = m_s$. Then just choose $| V_{33} |$, $| V_{34} |$, $| V_{43} |$, $| V_{44} |$, and $\arg\square_{44}$ as our parameters. Indeed the submatrix

$$\begin{bmatrix} V_{33} & V_{34} \\ V_{43} & V_{44} \end{bmatrix} \tag{42}$$

is not necessarily unitary and contains phase information.

B. Different number of up and down generations

Inspired by E_6 models[17] we consider the following. If we were to have an unequal number (n) of "up generations" and (m) of "down generations," then the mixing matrix could satisfy only one of the two equations:

$$VV^\dagger = \mathbb{1}_{n \times n} \, , \tag{43a}$$

$$V^\dagger V = \mathbb{1}_{m \times m} \, . \tag{43b}$$

Both unitarity conditions cannot be met simultaneously since the combined number of constraints $n^2 + m^2$ exceeds the initial number of real parameters $2nm$ characterizing an arbitrary complex $n \times m$ V matrix. To be definite take $n < m$, and assume Eq. (43a) holds, as would be the case if all up quarks were members of electroweak doublets. Then the number of physical angle parameters is

$$n[(m-1)+(m-n)]/2 \tag{44a}$$

and phase parameters is

$$[(n-2)(m-1)+n(m-n)]/2 \, , \tag{44b}$$

where we assume a nondegenerate up mass and down mass spectrum. A physical parametrization can proceed as follows: take the region bounded to the left by the diagonal, which is not included:

$$n \left\{ \begin{bmatrix} \diagdown & \diamond & \diamond & \diamond \\ \cdot & \diagdown & \diamond & \diamond \\ \cdot & \cdot & \diagdown & \diamond \end{bmatrix} \right. \, . \tag{45}$$

$$\underbrace{}_{m}$$

The angle parameters are the magnitudes of the KM elements in the bounded region, the phases are the arguments of the plaques constructed in that same region.

VIII. REPHASE-INVARIANT PHENOMENOLOGY

Here we display in rephasing-invariant form[18] the kaon parameters Δm, $\Delta\Gamma$, ϵ, ϵ', and look at the KM constraints from $K_L \to \mu^+ \mu^-$ and from B^0-$\bar{B}^0$ and D^0-$\bar{D}^0$ mixings. In a later study we will include constraints coming from $K^+ \to \pi^+ \nu\bar{\nu}$ and from the electric dipole moment of the neutron.

A. Kaon system

We define the short- and long-lived species, assuming CPT invariance, as

$$| K_S \rangle = p | K^0 \rangle + q | \bar{K}^0 \rangle \, , \tag{46a}$$

$$| K_L \rangle = p | K^0 \rangle - q | \bar{K}^0 \rangle \, . \tag{46b}$$

The parameters p and q are not rephase invariant. In the absence of CP violation the ratio p/q is of modulus unity. Define

$$e^{i\omega_0} \equiv \frac{\langle \pi\pi, I=0 | H' | K^0 \rangle}{\langle \pi\pi, I=0 | H' | \bar{K}^0 \rangle} \, . \tag{47}$$

The Wu-Yang phase convention, together with $CP |K^0\rangle = + |\bar{K}^0\rangle$, implies $\omega_0 = 0$. Here we leave this phase, Eq. (47), arbitrary. From their definitions it is clear that the following combinations are rephase-invariant:

$$e^{i\omega_0} M_{12} = e^{i\omega_0} \sum_n P \frac{\langle K^0 | H' | n \rangle \langle n | H' | \bar{K}^0 \rangle}{m_K - E_n} , \quad (48a)$$

$$e^{i\omega_0} \Gamma_{12} = e^{i\omega_0} 2\pi \sum_n \rho_n \langle K^0 | H' | n \rangle \langle n | H' | \bar{K}^0 \rangle$$

$$\times \delta(m_K - E_n) , \quad (48b)$$

$$e^{-i\omega_0} \frac{q}{p} = e^{-i(\omega_0 + \arg\Gamma_{12})} \left[\frac{M_{12}^*/\Gamma_{12}^* - (i/2)}{M_{12}/\Gamma_{12} - (i/2)} \right]^{1/2}$$

$$= e^{-i\omega_0} \frac{2[M_{12}^* - (i/2)\Gamma_{12}^*]}{-\Delta\lambda} , \quad (49)$$

where P stands for principal value and ρ_n is the density of states. Here $\Delta\lambda = \lambda_L - \lambda_S = \Delta m - i\Delta\Gamma/2$ and λ_L and λ_S are the eigenvalues of the 2×2 mass matrix $M - i\Gamma/2$. It is useful to recognize that M_{12}/Γ_{12} is rephase invariant. In the limit of CP conservation it is real. We now may express the (rephase-invariant) mass and lifetime difference of the kaons as[19]

$$\Delta m \approx -2 \operatorname{Re}(M_{12} e^{i\omega_0}) , \quad (50a)$$

$$\Delta\Gamma \approx -2 \operatorname{Re}(\Gamma_{12} e^{i\omega_0}) . \quad (50b)$$

Rephase-invariant definitions of the CP-violating parameters are

$$\epsilon = \frac{\langle \pi\pi, I=0 | H' | K_L \rangle}{\langle \pi\pi, I=0 | H' | K_S \rangle} \quad (51a)$$

and

$$\epsilon' = (1/\sqrt{2}) i e^{i(\delta_2 - \delta_0)} \operatorname{Im}(a_2/a_0) . \quad (51b)$$

Here we have defined

$$\langle \pi\pi, I | H' | K^0 \rangle = a_I e^{i\delta_I} . \quad (52)$$

It follows that

$$\epsilon \approx \frac{-i \operatorname{Im}(M_{12} e^{i\omega_0}) - \operatorname{Im}(\Gamma_{12} e^{i\omega_0})/2}{\Delta\lambda}$$

$$\approx \frac{-i \operatorname{Im}(M_{12} e^{i\omega_0})}{\Delta\lambda} . \quad (51a')$$

Equations (50) and (51) are all physical quantities and shown in a rephasing-invariant way; under the rephasings

$$|K^0\rangle \rightarrow e^{i\theta} |K^0\rangle , \quad (53a)$$

$$|\bar{K}^0\rangle \rightarrow e^{i\bar{\theta}} |\bar{K}^0\rangle , \quad (53b)$$

physics does not change.

We remark that in the standard model the $I=2$ amplitude arises only[20] from the spectator diagram and hence[21]

$$a_2 = b V_{us}^* V_{ud} , \quad (54a)$$

b being a real constant. On the other hand,

$$a_0 = \sum_{q=u,c,t,\ldots} c_q V_{qs}^* V_{qd} , \quad (54b)$$

where c_q are coefficients whose short-distance contributions have been calculated.[22] We therefore obtain

$$\epsilon' \approx \frac{1}{\sqrt{2}} i e^{i(\delta_2 - \delta_0)} |a_2/a_0|^2 \frac{1}{|V_{us} V_{ud}|^2} (-)\operatorname{Im} \sum_{q=u,c,t,\ldots} (c_q/b)^{qd}\square_{us} . \quad (55)$$

Utilizing unitarity and assuming (c_q/b) to be real we obtain, in the four-generation case,

$$\epsilon' \approx \frac{1}{\sqrt{2}} i e^{i(\delta_2 - \delta_0)} |a_2/a_0|^2 \frac{1}{|V_{us} V_{ud}|^2 b} [-(c_T - c_c)\operatorname{Im}\square_{cs} + (c_T - c_t)\operatorname{Im}{}^{us}\square_{td}] . \quad (56)$$

One sees that ϵ'/ϵ can be positive, negative, or even 0 and that ϵ'/ϵ does not depend on the long-distance c_u coefficient, which might harbor the $\Delta I = \frac{1}{2}$ rule explanation. Note that in the three-generation case the square bracket in Eq. (56) reduces to $(c_c - c_t)J$.

To calculate ϵ, one realizes that to a good approximation one can neglect the phase difference between a_0 and a_2. Then, exploiting the simple relation[20] [Eq. (54a)] we obtain[23]

$$M_{12} e^{i\omega_0} \sim \sum_{i,k=u,c,t,\ldots} {}^{ud}\square_{is} {}^{ud}\square_{ks} S\left(\frac{m_i^2}{M_W^2}, \frac{m_k^2}{M_W^2} \right) . \quad (57)$$

In the simple limit $x_i \ll x_k \ll 1$, $[x_i \equiv (m_i/M_W)^2]$ we have

$$S(x_i, x_k) \approx x_i \ln(x_k/x_i) , \quad (58a)$$

$$S(x_k) \equiv S(x_k, x_k) \approx x_k . \quad (58b)$$

For arbitrary quark masses exact expressions could be used.[24] ϵ follows as[23]

$$\epsilon \sim \frac{-i}{\Delta\lambda |V_{us} V_{ud}|^2} \sum_{i,k=u,c,t,\ldots} S(x_i, x_k) \operatorname{Im}({}^{ud}\square_{is} {}^{ud}\square_{ks}) . \quad (59)$$

The K_L-K_S mass difference is believed to arise mainly from long-distance effects $K^0 \leftrightarrow 2\pi \leftrightarrow \bar{K}^0$. However, Gaillard and Lee[25] predicted the charm mass within the two-generation Glashow-Iliopoulos-Maiani (GIM) model:

$$\Delta m \mid_{\text{expt}} \sim m_c{}^2 \, \text{Re}(^{ud}\square_{cs})^2 \approx m_c{}^2 \theta^4 \; . \tag{60}$$

As a rough upper bound, we note that higher-generation contributions must not exceed Δm and so must not exceed the charm contribution.

While the ϵ' parameter involves the imaginary parts of the $^{qd}\square_{us}$ plaques [Eq. (55)]; the short-distance contributions[26] of $K_L \to \mu^+\mu^-$ contains information about their real parts. An estimate of the modulus of the short-distance amplitude leads to

$$\left| \text{Re} \left[\sum_{q=u,c,t,\ldots} {}^{qd}\square_{us} m_q{}^2 \right] \right| \leq \mid V_{us}{}^2 V_{ud} \mid 55 \text{ GeV}^2 \; . \tag{61}$$

Utilizing unitarity we may eliminate any one plaque; this yields, for the four-generation case,

$$\mid m_c{}^2 \, \text{Re}(^{cd}\square_{us}) + m_t{}^2 \, \text{Re}(^{td}\square_{us}) + m_T{}^2 \, \text{Re}(^{Td}\square_{us}) \mid$$
$$\leq \mid V_{us}{}^2 V_{ud} \mid 55 \text{ GeV}^2 \; . \tag{62}$$

B. B-$\bar{B}$ mixing

The large B_d mixing observed by the ARGUS Collaboration,[27]

$$(\Delta m / \gamma)_d \approx 0.7 \; , \tag{63}$$

implies, in the case of three generations,[28]

$$m_t \gtrsim 60 \text{ GeV} \; . \tag{64}$$

We review the reasoning as follows. To good approximation[28–31]

$$(\Delta m / \gamma)_d$$
$$\approx 0.8 B_d \left| \frac{f_{B_d}}{100 \text{ MeV}} \right|^2 \frac{S(x_t)}{S(m_t = 40 \text{ GeV})} \left| \frac{V_{td}}{V_{cb}} \right|^2 , \tag{65}$$

where B_d is the "bag constant" and f_{Bd} is the decay constant. From unitarity of the KM matrix and the experimental data one obtains that

$$\mid V_{td} / V_{cb} \mid \approx \theta \; . \tag{66}$$

The easiest way to reconcile the experimental B_d-mixing result, Eq. (63), is to choose a larger top-quark mass. This may not be obligatory given theoretical and experimental uncertainties. However, in the four-generation case, unitarity conditions are much relaxed. We know[11] from the B lifetime that

$$\mid V_{cb} \mid \approx \theta^2 \tag{67}$$

and from indirect unitarity bounds

$$\mid V_{td} \mid \leq 0.17 \; (95\% \text{ C.L.}), \quad 0 \leq \mid V_{tb} \mid < 1 \; . \tag{68}$$

Assume, for the sake of an argument, that the top contribution is dominant even in the four-generation scenario. Then one can easily fit the large B_d mixing with small top-quark masses (say 40 GeV), by choosing $\mid V_{td} \mid \sim \mid V_{cb} \mid \sim \theta^2$.

Furthermore, were it to happen that B_s-$\bar{B}_s$ mixing is

less than maximal, one would then have to look outside the standard three-generation model.[30] One possible explanation could be found with four generations.[32]

C. Remarks and speculations

Large fourth-generation mixing: An example

To get a feel about the mixing magnitudes of a fourth generation consider

$$\mid V_{\text{big}} \mid \sim \begin{bmatrix} 1 & \theta & \theta^3 & \theta^2 \\ \theta & 1 & \theta^2 & \theta \\ \theta^3 & \theta^2 & 1 & \theta \\ \theta^2 & \theta & \theta & 1 \end{bmatrix} . \tag{69}$$

We list a few consequences:

$$m_T \lesssim 30 \text{ GeV} \quad (K_L\text{-}K_S \text{ mass difference}) \; , \tag{70a}$$

$$m_T \lesssim 34 \text{ GeV} \quad (K_L \to \mu^+\mu^-) \; , \tag{70b}$$

$$m_B \lesssim 100 \text{ GeV} \quad (D^0\text{-}\bar{D}^0 \text{ mixing}) \; , \tag{70c}$$

$$m_T \lesssim 170 \text{ GeV} \quad (B_d\text{-}\bar{B}_d \text{ mixing}) \; . \tag{70d}$$

We make the following remarks.

(a) In order that higher-generation contributions not exceed the charm contribution to the K_L-K_S mass difference, one must have

$$m_c{}^2 \theta^4 \gtrsim m_T{}^2 \theta^8 \tag{71}$$

leading to $m_T \lesssim 30$ GeV.

(b) The $K_L \to \mu^+\mu^-$ analysis [Eq. (62)] leads to Eq. (70b).

(c) In the two-generation case[33]

$$\frac{\Delta m_D}{\Delta m_K} = \left[\frac{f_D}{f_K} \right]^2 \left[\frac{m_D}{m_K} \right] \left[\frac{m_s}{m_c} \right]^2 . \tag{72}$$

With the experimental D lifetime and $m_S = 500$ MeV one gets

$$(\Delta m / \gamma)_D = 10^{-3} \; . \tag{73}$$

The existence of an ultraheavy fourth-generation B quark leads to

$$\Delta m_D \sim [\theta^2 m_s{}^2 + m_B{}^2 (V_{cB} V_{uB}^*)^2] \; , \tag{74}$$

$$(\Delta m / \gamma)_D \approx 10^{-3}[1 + \theta^4 (m_B / m_s)^2] \; . \tag{75}$$

From experiment[34] $(\Delta m / \gamma)_D \lesssim 10^{-1}$ hence Eq. (70c).

(d) By demanding that the fourth-generation contribution to B_d-$\bar{B}_d$ mixing not exceed the ARGUS observation Eq. (63), and by analogous reasoning to (c), we obtain[35] the bound (70d).

It appears that V_{big}, Eq. (69), is experimentally marginal.

IX. CONCLUSIONS

The main purpose of this paper is the proposed parametrization of the KM matrix. It is quite directly related to phenomenology, since the parameters consist of

moduli of matrix elements and plaquette phases [defined in Eq. (2)]; these are manifestly rephase invariant. In the three-generation case, the question of how to parametrize the KM matrix is not too important. However if a generalization to a higher number of generations turns out to be necessary, the problem is less trivial.

If one does insist on a phase-dependent convention, we believe that choosing diagonal elements and those next-to diagonal elements which are above the diagonal to be real and positive guarantees a simple relationship to plaquette phases. However one suffers from increasing complexity in the lower diagonal half. It may also be that experiment may dictate other choices; if a given set of $|V_{i\alpha}|$ are measured especially accurately, it makes sense to include them in the set of independent parameters. Likewise one might consider to use phases of those plaques directly related to the observed CP violation. Our basic point is to highlight the importance of rephase invariance of any future parametrization, because only then is the physics manifest and not obscured by arbitrary phase conventions.

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$$
\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \times & \cdot & \bigcirc & \cdot \\ \cdot & \cdot & \times & \bigcirc \\ \bigcirc & \cdot & \cdot & \times \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \times & \cdot & \bigcirc & \cdot \\ \otimes & \cdot & \times & \bigcirc \\ \bigcirc & \cdot & \cdot & \times \end{bmatrix} \Big/ |V_{td}|^2
$$

$$
= ({}^{21}\square_{33}\,{}^{31}\square_{44})^* / |V_{td}|^2 \ .
$$

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