Perturbative QCD and electromagnetic form factors

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We calculate nucleon magnetic form factors using perturbative QCD for several distribution amplitudes including a general one given in terms of Appell polynomials. We find that the magnitude and sign of both nucleon magnetic form factors can be explained within perturbative QCD. The observed normalization of G_{Mp} requires that the distribution amplitude be broader than its superhigh-momentum-transfer limit, and the G_{Mn}/G_{Mp} data may require the distribution amplitude to be asymmetric, in accordance with distribution amplitudes derived from QCD sum rules. Some speculation as to how an asymmetric distribution amplitude can come about is offered. Finally, we show that the soft contributions corresponding to the particular distribution amplitudes we use need not be bigger than the data.

I. INTRODUCTION

There has been much discussion about the validity of using perturbation theory with QCD (PQCD) to make predictions for exclusive processes at experimentally feasible energies.[1–4] It is generally granted that the predicted scaling behavior works, by luck or otherwise, at reasonably low Q^2. Absolute normalizations then become the next testing ground for PQCD. Unfortunately the normalization, unlike the energy or momentum-transfer scaling behavior, is dependent upon unknown and/or perhaps not well understood wave functions of the quarks in a hadron.[1,2,5,6] Still, the question remains whether wave functions can be found for which the calculated normalizations are in agreement with the data. In this paper it is shown that such wave functions can be found, and the nature of these wave functions is examined, but no firm stand can be taken on whether or not these wave functions are the correct ones. *Ab initio* calculations of the correct nuclear wave functions requires the use of nonperturbative techniques beyond the scope of this paper.

In order to clarify the discussion, several categories of predictions of PQCD for exclusive processes can be distinguished.

(1) Scaling behavior. High-energy or high-momentum-transfer scaling behavior of form factors or differential cross sections can be obtained.[7] Taking electromagnetic form factors as an example, the helicity-conserving one is always the biggest and goes like

$$F(Q^2)=\frac{A}{(Q^2)^{N-1}}\left[1+\frac{b}{Q^2}+\cdots\right], \tag{1}$$

for a system of N constituents. Predictions of power-law behavior tend to work well. Figure 1 shows one example: the proton magnetic form factor. The currently published data[8] for Q^4G_{Mp} is shown as a function of Q^2 and the PQCD scaling behavior appears substantially right for $Q^2\geq 5\ \text{GeV}^2$.

(2) Normalization. The normalization of the form factors [the coefficient A in Eq. (1)] or scattering amplitudes could be obtained. These calculations depend on the quark wave functions, and in this paper explicit calculations for the nucleon magnetic form factors are shown. (There has been a claim[3,4] that no reasonable wave function can give a PQCD calculated form factor as large as the data, so that the agreement seen with the PQCD scaling behavior in Fig. 1 is just luck.)

(3) Logarithmic corrections. The logarithmic corrections to the power-law behavior can be calculated.[1,2] Like the normalization, these calculations are, in general, wave-function dependent. For example, the leading term in a form factor is more completely given as

$$F(Q^2)=\frac{1}{(Q^2)^{N-1}}[\alpha_s(Q^2)]^{N-1}\times\sum_{ij}d_{ij}(\ln Q^2/\Lambda^2)^{-\gamma_i-\gamma_j}\ . \tag{2}$$

The γ_j are calculable, positive, and monotonically increasing with j, but the d_{ij} are wave-function dependent. Only one prediction is wave-function independent, and that is the $\ln Q^2$ behavior at sufficient Q^2 that only one term in the above sum survives. This requires extremely high Q^2. In contrast with category (1), "logarithmic asymptopia" is now needed rather than "power-law asymptopia"; that is to say, $\ln Q^2$ must be large rather than just Q^2 being large.

(4) Polarization. Quantities specifically involving polarization can be calculated.[9]

In this paper the normalization of the nucleon magnetic form factor, category (2) above, will be studied, using plausible wave functions or distribution amplitudes including a flexible class of distribution amplitudes that can be expressed in terms of the first six Appell polynomials.[1,2,6] (These are eigensolutions of the evolution

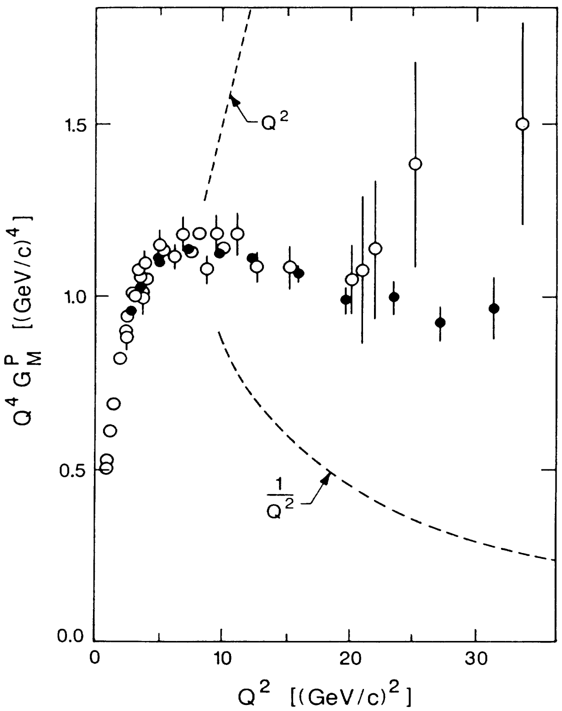

FIG. 1. Data for $Q^4 G_{Mp}$ plotted vs Q^2. (Taken from Ref. 8.) The two dashed lines indicate how the data would behave if $G_{Mp} \sim 1/Q^2$ or $G_{Mp} \sim 1/Q^6$.

equation for the distribution amplitude. For this paper any basis set would do as well, but the Appell polynomials are very convenient for any studies of the logarithmic Q^2 dependence of the form factor.) It will be shown that it is possible to match at high Q^2 the observed normalization of G_{Mp}, without running afoul of wave function normalization conditions. The simplest wave functions, however, are not the best ones to use. The observed normalization of G_{Mp} requires a broad distribution amplitude, and moreover the observed value of the ratio G_{Mp}/G_{Mn} may not be readily explained without asymmetric distribution amplitudes.[6] Indeed, a distribution amplitude based on QCD sum rules which shows an asymmetry in the quark spatial wave function has already been suggested by Chernyak and Zhitnitsky.[5] The foregoing is discussed in Sec. II. Section IV includes some speculation about how it is possible to have an asymmetric distribution amplitude even though SU(6) results, which are based on completely symmetrical spatial wave functions, work fairly well for quantities measured at low Q^2.

Section III contains a study of the "soft contributions."[4] In QCD the impulse approximation gives the correct leading-order (in $1/Q^2$) form factors at high Q^2 and the result comes from the tail, meaning the high-transverse-momentum part, of the quark wave function. High enough transverse momenta allow the use of perturbation theory with QCD and indeed the PQCD calculation can be seen as a way of generating the correct tail of the wave function and immediately using it in an impulse-approximation calculation of the form factor. If we do an impulse-approximation calculation of the form factor keeping only the low-transverse-momentum part of the wave function, for example, by purposely using a wave function such as a Gaussian which is plausible at low momenta falls much too quickly at high momenta, then we get the "soft contributions." At high Q^2 the soft contributions fall faster than the PQCD or "hard" contributions so the PQCD result must eventually dominate, but at any Q^2 the size of the contributions is a test of the validity of the approximations that go into the PQCD result. The question is whether the soft contributions are significant or even dominant at present experimental Q^2. It is found for some wave functions of interest that the soft contributions may be important but are not necessarily larger than the PQCD result. As discussed in Sec. III, it is important to note the effects of the tail of the wave function on wave-function parameters.

A summary and some speculation are given in Sec. IV.

II. NUCLEON FORM FACTORS IN PQCD

One can show that in QCD the impulse approximation contains the leading contribution to the form factor at high Q^2 and further that the leading contribution to either F_1 or G_M (these two are equivalent to leading order since F_2 falls faster by one power of Q^2) is (Fig. 2)

$$G_M = \int [dx][dy]\phi^*(x,Q)T_H(x,y,Q)\phi(y,Q) , \tag{3}$$

where ϕ, the three-quark distribution amplitude, and the other quantities in Eq. (3) are defined below. To obtain the leading contribution it is sufficient to consider the three-quark part of the wave function; Fock components with more constituents require more gluon exchanges to give all the constituents parallel momenta in the final state and their contribution to the form factor fall faster with Q^2. We work in an infinite-momentum frame where the entering proton is moving along the z axis. The transverse-momentum components of the ith quark are

$$\mathbf{k}_{iT} = (k_i^1, k_i^2) , \tag{4}$$

and the momentum fractions are

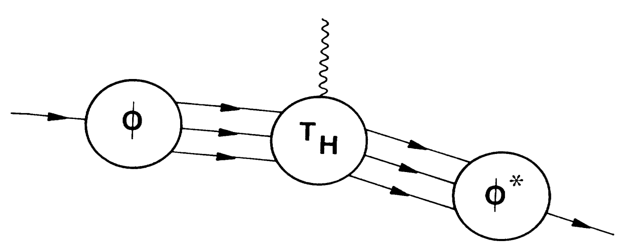

FIG. 2. The process giving G_{Mp}. Three parallel moving quarks enter the circle labeled T_H where one of them absorbs the photon entering mainly from a transverse direction, and then shares the momentum with its fellows so that three parallel quarks emerge.

$$x_i = k_i^+ / p^+ \ , \tag{5}$$

where $k_i^+ = k_i^0 + k_i^3$ and p is the proton momentum. For the three quarks in the initial proton

$$\sum \mathbf{k}_{iT} = 0 \tag{6}$$

and

$$\sum x_i = 1 \ . \tag{7}$$

In the expression for the form factor, the distribution amplitude ϕ is

$$\phi(x,Q) = \int^Q [dk_T] \psi(x,k_T) \ , \tag{8}$$

where ψ is the three-quark wave function. The differentials are

$$[dx] = \prod dx_i \delta \left[1 - \sum x_j \right] \tag{9}$$

and

$$[dk_T] = \prod \left[\frac{d^2 k_{iT}}{16\pi^3} \right] 16\pi^3 \delta^{(2)} \left[\sum k_{jT} \right] \ . \tag{10}$$

The wave function is normalized by

$$\int [dx][dk_T] \, | \psi(x,k_T) |^2 = P_{3q} \ . \tag{11}$$

The "hard-scattering amplitude" T_H is the scattering amplitude for three parallel quarks going into three parallel quarks. There are 42 diagrams that can be drawn form T_H, but only 14 are nonzero and only the four drawn in Fig. 3 need be calculated, the others being obtained by symmetries. If e_j is the operator which gives the charge of quark j, then[1,2]

$$T_H = \left[\frac{8\pi C_B \alpha_s(Q^2)}{Q^2} \right]^2 \sum_{j=1}^{3} [e_j T_j + (x \leftrightarrow y)] \ , \tag{12}$$

with $C_B = \frac{2}{3}$, and

$$T_1 = \frac{1}{x_3(1-x_1)^2} \frac{1}{y_3(1-y_1)^2} + \frac{1}{x_2(1-x_1)^2} \frac{1}{y_2(1-y_1)^2} - \frac{1}{x_2 x_3 (1-x_3)} \frac{1}{y_2 y_3 (1-y_1)} = T_3(1 \leftrightarrow 3) \tag{13a}$$

and

$$T_2 = \frac{1}{x_1 x_3 (1-x_1)} \frac{1}{y_1 y_3 (1-y_3)} \ . \tag{13b}$$

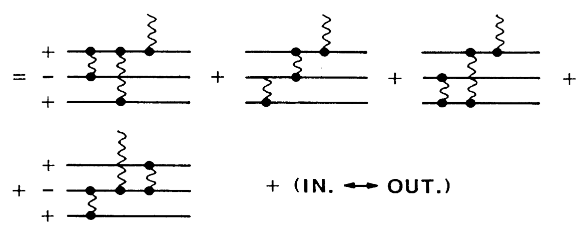

FIG. 3. Lowest-order perturbation diagrams for T_H. The small signs indicate quark helicities.

Note the $1/Q^4$, the $\ln(Q^2/\Lambda^2)$ dependence within the strong-coupling parameters α_s, and the singularities near the kinematic boundaries of x_i and y_i $(0 \le x_i, y_j \le 1)$.

The proton wave function is not at present calculable. However, QCD sum rules allow some moments of the proton wave function to be determined, which in turn sets conditions on model wave functions,[5] and lattice gauge theories may eventually give an *ab initio* calculation. Some progress is being made on the pion wave function using lattice gauge theory.[10]

Accordingly G_M will be calculated for two general classes of wave functions, starting with a simple one-parameter symmetric wave function. This will show what is necessary to get the right normalization and will also display the limitations of symmetric spatial wave functions which seem to persist even in more sophisticated versions of the same.

A. Simple symmetric wave function

A simple and factorable form of the wave function is[4]

$$\psi(x,k_T) = N(x_1 x_2 x_3)^\eta \exp \left[- \sum k_{iT}{}^2 / 2\alpha_0{}^2 \right] \tag{14}$$

from which

$$\phi(x) = N'(x_1 x_2 x_3)^\eta \ . \tag{15}$$

This is a one-parameter family of wave functions, the parameter being the power η. The constants N and N' are fixed by the wave-function normalization condition, and the parameter α_0 by the rms value of k_T, which should be some reasonable value; one value suggested[11] is $\alpha = 0.32$ GeV. A Gaussian in the transverse momentum is incorrect at high k_T and Sec. III will show the effect of additional terms in the transverse-momentum wave function. However, the Gaussian is useful for now to show how the usual normalization condition on the wave function implies constraints between the size of G_{Mp} and the rms quark transverse momenta.

The integrals for G_M can be done analytically. The power must satisfy $\eta > \frac{1}{2}$ to make those integrals converge. One way to begin looking at the results is to examine the ratio G_{Mp}/G_{Mn}, plotted in Fig. 4. The proton form factor has a zero at $\eta = 1$ and the neutron form factor has a zero at

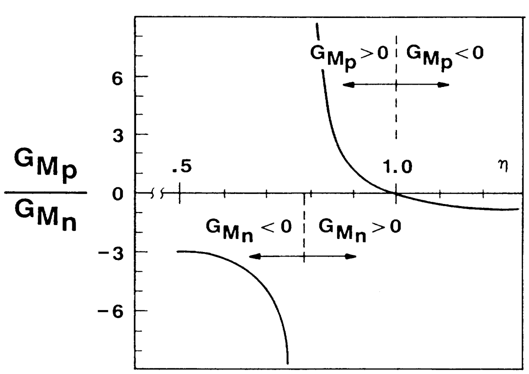

FIG. 4. G_{Mp}/G_{Mn} for the simple symmetric wave function, plotted vs the power parameter for η.

$$\eta=\tfrac{1}{2}+\frac{1}{\sqrt{12}}\simeq 0.79\ . \tag{16}$$

Of course, the neutron and proton form factors have opposite signs and this constrains the values of η which may be chosen. Further, since it is the proton form factor that is positive, η is constrained to $\frac{1}{2}<\eta\leq 0.79$. (Incidentally, there was for a while, an error in the literature in the overall sign of T_H and hence, of the calculated G_{Mp}. This had the effect of requiring $\eta>1$ with consequent large effect on the normalization.)

The result for G_{Mp} using the wave function (14) is

$$Q^4G_{Mp}(Q^2)=\frac{1}{\pi^2}\alpha_s{}^2\alpha_0{}^4\frac{64}{27}\frac{1-\eta}{\eta}N_\eta{}^2X^2\ , \tag{17}$$

with

$$N_\eta{}^2=\frac{(6\eta+2)!}{[(2\eta)!]^3} \tag{18}$$

and

$$X=\frac{(\eta!)^2(\eta-1)!(2\eta-2)!}{(2\eta)!(3\eta-1)!}\ . \tag{19}$$

For $\eta=0.6$, $\alpha_s=0.3$, and $\alpha_0=0.32$ GeV this gives

$$Q^4G_{Mp}\simeq 1.1\ \text{GeV}^4\ , \tag{20}$$

which is near the data for $Q^2\geq 5$ GeV^2.

This wave function demonstrates that the observed size of the proton form factor can be obtained, at least at high Q^2, from PQCD. It shows that the experimental normalization requires a broad ($\eta<1$) rather than a narrow ($\eta>1$) distribution amplitude. It also shows the possibility of a catastrophe that did not occur: what could have happened was the distribution amplitudes which gave a normalization compatible with experiment also gave the wrong sign for G_{Mp}, but this was not the case.

On the other hand, this wave function may be too simple, and whether it can give the value of G_{Mn}/G_{Mp} seen experimentally depends delicately on just what that ratio is. Some comments on the G_{Mn} data are in order.

Strictly speaking, there is no G_{Mn} data away from $Q^2=0$. There is data on σ_n, the differential cross section for e-n elastic scattering, at $Q^2=2.5$, 4, 6, 8, and 10 GeV^2 but only at one angle.[12] Hence a separation of G_{En} and G_{Mn} is impossible. However, several things can be learned or are suggestive about G_{Mn} by analyzing the data which is available.

The salient observation is that σ_n/σ_p falls roughly like $1/Q^2$ from (say) 5 GeV^2 to 10 GeV^2 and is roughly $\frac{1}{4}$ at the upper Q^2. Neglecting G_E gives $\sqrt{\sigma_n|\sigma_p}=|G_{Mn}/G_{Mp}|$ and since the contributions of G_{Ep} to σ_p really appear negligible at high Q^2, we can safely say

$$\left|\frac{G_{Mn}}{G_{Mp}}\right|\leq\frac{1}{2}\ ,$$

at $Q^2=10$ GeV^2. (The sign is known to be negative at $Q^2=0$.)

The falling cross-section ratio suggests more. Possibly, σ_n/σ_p falls because G_{Mn} is dominating σ_n but Q^4G_{Mn} is not yet constant. However, the observed approximate constancy of Q^4G_{Mp} makes an alternative explanation plausible. Possibly, G_{Mn} is small, i.e., its leading term at high Q^2 is small, and then the cross section σ_n is dominated by G_{En}. This requires G_{En} to be about the same size as F_{1p} or G_{Mp} and leads naturally to $\sigma_n/\sigma_p\sim 1/Q^2$. Incidentally, since the leading terms of G_{En} and F_{1n} are the same, one can tie the high- and low-Q^2 data together with the suggestion that $F_{1n}\simeq 0$ at all Q^2.

Returning to the simple symmetric wave function, we can consider three possibilities.

(i) $G_{Mn}/G_{Mp}\simeq -\frac{1}{2}$. More precisely, consider that G_{Mn}/G_{Mp} is falling with Q^2 until $Q^2=10$ GeV^2 but is approximately constant at the value $-\frac{1}{2}$ thereafter. The simple symmetric wave function cannot give this value; values of $|G_{Mn}/G_{Mp}|$ between $-\frac{1}{3}$ and -1 are inaccessible (see Fig. 4).

(ii) G_{Mn}/G_{Mp} small. This possibility means the distribution amplitude gives small G_{Mn}/G_{Mp} at high but experimentally accessible Q^2. (If we let Q^2 be superhigh, then the evolution with changing $\ln Q^2$ of the distribution amplitude must be taken into account and the ultimate consequence of this is known to give $Q^4G_{Mp}\rightarrow O$. On the other hand, the same superhigh-Q^2 limit gives G_{Mn} positive, so that G_{Mn} must have a zero at some finite though possibly superhigh Q^2 and the ratio G_{Mn}/G_{Mp} should fall to zero before it ultimately becomes infinite.) This possibility would require $\eta\simeq 0.79$ in the simple symmetric distribution amplitude, but this has the high price of requiring $\alpha_0=0.67$ GeV (for $\alpha_s=0.3$) to give the observed normalization of Q^4G_{Mp}.

(iii) G_{Mn}/G_{Mp} intermediate. This means G_{Mn}/G_{Mp} about $-\frac{1}{3}$ to $-\frac{1}{4}$. G_{Mn} is small enough that σ_n is still dominated by G_{En}, so the $1/Q^2$ falloff of σ_n/σ_p is still naturally explained. This value of G_{Mn} gives no problem for the simple symmetric distribution amplitude. The example $\eta=0.6$ fits here.

B. More general wave function

For a more general treatment, expand $\phi(x)$ in polynomials (times a weight factor $x_1x_2x_3$ to decrease sensitivity to the end-point singularities[13]). For the present purposes the particular choice of polynomials is not crucial, but it is convenient to use polynomials which are standard[1,2] to the subject and which would be useful for study of logarithmic dependences. The wave equation for $\psi(x,k_T)$ has a kernel which is dominated by one-gluon exchange for high k_T. This observation can be turned into an "evolution equation" which governs how the distribution amplitude $\phi(x,Q^2)$ changes with Q^2 for high Q^2. The evolution equation can be solved by the separation of variables to yield

$$\phi(x,Q^2)=x_1x_2x_3\sum N_i\tilde{\phi}_i(x)\ , \tag{21}$$

where

$$N_i=N_i(Q^2)=n_i\ln^{-\gamma_i}\left[\frac{Q^2}{\Lambda^2}\right]\ . \tag{22}$$

The n_i are noncalculable constants, but the γ_i are calculable[1,2] and are positive and monotonically increasing with i. The first six "Appell polynomials" are

$$\begin{aligned}
&\tilde{\phi}_0=1,\quad \tilde{\phi}_1=x_1-x_3,\quad \tilde{\phi}_2=2-3(x_1+x_3)\\
&\tilde{\phi}_3=2-7(x_1+x_3)+8(x_1{}^2+x_3{}^2)+4x_1x_3\ ,\\
&\tilde{\phi}_4=x_1-x_3-\tfrac{4}{3}(x_1{}^2-x_3{}^2)\ ,\\
&\tilde{\phi}_5=2-7(x_1+x_3)+\tfrac{14}{3}(x_1{}^2+x_3{}^2)+14x_1x_3\ .
\end{aligned} \tag{23}$$

Quarks 1 and 3 are the ones with parallel spin and some of the above are symmetric and some antisymmetric under interchange of quarks 1 and 3. If ϕ is split into parts ϕ_S and ϕ_A which are symmetric and antisymmetric under $1\leftrightarrow3$, then ϕ_S and ϕ_A can be associated with the corresponding symmetry spin-isospin wave functions for the proton and neutron

$$\begin{aligned}
\phi_p(x)=&\phi_S(x)(2u_\uparrow d_\downarrow u_\uparrow-u_\uparrow u_\downarrow d_\uparrow-d_\uparrow u_\downarrow u_\uparrow)/\sqrt{6}\\
&+\phi_A(x)(u_\uparrow u_\downarrow d_\uparrow-d_\uparrow u_\downarrow u_\uparrow)/\sqrt{2}+\text{perm}
\end{aligned} \tag{24a}$$

and

$$\begin{aligned}
\phi_n(x)=&\phi_S(x)(d_\uparrow d_\downarrow u_\uparrow+u_\uparrow d_\downarrow d_\uparrow-2d_\uparrow u_\downarrow d_\uparrow)/\sqrt{6}\\
&+\phi_A(x)(u_\uparrow d_\downarrow d_\uparrow-d_\uparrow d_\downarrow u_\uparrow)/\sqrt{2}+\text{perm}\ .
\end{aligned} \tag{24b}$$

It is now straightforward to calculate the form factors and normalization condition. The results are[14,15]

$$\begin{aligned}
Q^4G_{Mp}(Q^2)=\left[\frac{4\pi\alpha_s}{27}\right]^2\Bigg[&20N_1{}^2-42\sqrt{3}N_0N_1+36N_2{}^2+28\sqrt{3}N_1N_2-54N_0N_2+188N_3{}^2-144N_2N_3-\frac{220}{\sqrt{3}}N_1N_3\\
&+198N_0N_3+\frac{26}{27}N_4{}^2+\frac{46}{\sqrt{3}}N_3N_4-\frac{22}{\sqrt{3}}N_2N_4-\frac{26}{3}N_1N_4+10\sqrt{3}N_0N_4+\frac{77}{9}N_5{}^2\\
&-\frac{41}{9\sqrt{3}}N_4N_5-59N_3N_5+11N_2N_5+\frac{35}{\sqrt{3}}N_1N_5-42N_0N_5\Bigg]\ ,
\end{aligned} \tag{25a}$$

$$\begin{aligned}
Q^4G_{Mn}(Q^2)=\left[\frac{4\pi\alpha_s}{27}\right]^2\Bigg[&54N_0{}^2+22N_1{}^2+42\sqrt{3}N_0N_1-6N_2{}^3-28\sqrt{3}N_1N_2+54N_0N_2-\frac{170}{3}N_3{}^2+60N_2N_3\\
&+\frac{220}{\sqrt{3}}N_1N_3-30N_0N_3+\frac{26}{27}N_4{}^2-\frac{46}{\sqrt{3}}N_3N_4+\frac{22}{\sqrt{3}}N_2N_4-\frac{26}{3}N_1N_4-10\sqrt{3}N_0N_4\\
&-\frac{145}{54}N_5{}^2+\frac{41}{9\sqrt{3}}N_4N_5+\frac{65}{3}N_3N_5-\tfrac{5}{3}N_2N_5-\frac{35}{\sqrt{3}}N_1N_5+20N_0N_5\Bigg]\ .
\end{aligned} \tag{25b}$$

The G_M have now been given for a six-parameter family of wave functions. Note that there is no $N_0{}^2$ term for the proton. This is the same as the zero at $\eta=1$ in the previous wave function.

The N_i cannot be made arbitrarily large because there is a wave-function normalization condition to satisfy. Some tradeoff between normalization of the distribution amplitude and normalization of the transverse momentum part of the wave function is possible, but large N_i will generally lead to large and possibly unacceptable rms quark transverse momenta. It can be seen how this happens for a factorizable wave function with a Gaussian k_T dependence. Let

$$\psi(x,k_T)=\phi(x)g(k_T)\ , \tag{26}$$

where

$$\int[dk_T]g(k_T)\equiv\langle g\rangle=1 \tag{27}$$

and

$$g(k_T)=\frac{192\pi^4}{\alpha_0{}^4}\exp\left[-\sum k_{iT}{}^2/2\alpha_0{}^2\right]\ . \tag{28}$$

The normalization condition becomes

$$\int[dx]\phi^2(x)\int[dk_T]g^2(k_T)\equiv\langle\phi^2\rangle\langle g^2\rangle=P_{3q} \tag{29}$$

or

$$\begin{aligned}
\tfrac{1}{165}(&165N_0{}^2+11N_1{}^2+33N_2{}^2+17N_3{}^2+\tfrac{1}{3}N_4{}^2\\
&+\tfrac{53}{9}N_5{}^2-44N_0N_5+2N_2N_3+\tfrac{2}{3}N_1N_4\\
&-\tfrac{22}{3}N_0N_5-\tfrac{14}{3}N_2N_5+\tfrac{2}{3}N_3N_5)=\frac{7!}{48\pi^4}\alpha_0{}^4P_{3q}\ ,
\end{aligned} \tag{30}$$

where P_{3q} is the three-quark probability. If only one N_i is nonzero, the most form factor for a fixed normalization occurs if $i=3$. For this case with $P_{3q}=1$ and $\alpha_s=0.3$, the proton form factor data at $Q^2\geq5$ GeV^2 is fit with $\alpha_0=0.39$ GeV, a reasonable value. Other choices of i would tend to give larger, less acceptable values of α_0.

Let us discuss the requirements on the distribution amplitude if it is to give the observed G_{Mp} and G_{Mn} and not imply via the normalization condition a quark transverse momentum, measured by α_0, which is unacceptably big. The possibilities for the neutron data will match our earlier discussion, and all remarks will apply to Appell polynomial expansions through the quadratic Appell polynomials.

(i) $G_{Mn}/G_{Mp} \simeq -\frac{1}{2}$. Can we obtain this with only the symmetric Appell polynomials contributing to the distribution amplitude and with an acceptable α_0, say α_0 below 700 MeV? The answer is no. (Our search routine is simple: we scan on a tight enough grid all the $\{N_i\}$ which will give α_0 below 700 MeV, searching for sets of $\{N_i\}$ which give Q^4G_{Mp} in the range 0.9–1.1 GeV^4 and Q^4G_{Mn} in the range -0.4–-0.6 GeV^4, both with $\alpha_s=0.3$. There are no such sets.) Using only antisymmetric Appell polynomials is clearly futile, since they give $G_{Mn} \geq G_{Mp}$. A distribution amplitude which works quite well at giving both G_{Mp} and G_{Mn} is (for $P_{3q}=1$, $\alpha_s=0.3$, and $\alpha_0=0.39$ GeV)

$$\phi_M(x)=(0.38\ \mathrm{GeV}^2)x_1x_2x_3(\tilde{\phi}_3-\tilde{\phi}_1)\ . \tag{31}$$

This amplitude is perhaps surprising because it is asymmetric, and Sec. IV below contains some speculation on how this might come about. A more complicated, but better because it fits certain QCD-sum-rule results, distribution amplitude which also gives $G_{Mn}/G_{Mp} \simeq -\frac{1}{2}$ is the one due to Chernyak and Zhitnitsky, and it is given explicitly in Sec. IV.

(ii) G_{Mn}/G_{Mp} small. Here we can succeed with symmetric Appell polynomials. The smallest α_0 we can find for $Q^4G_{Mp}=1.0$ GeV^4 and Q^4G_{Mn} between ± 0.1 GeV^4 is $\alpha_0=0.48$ GeV (for the record, $N_\phi=0.2$, $N_2=0.2$, $N_3=0.57$, $N_5=0.5$, all in GeV^2, and $Q^4G_{Mn}=-0.07$ GeV^4). This α_0 however is only borderline acceptable as it gives a somewhat large $\langle k_T \rangle$ for the quarks and the difficulties of escaping some problems associated with the "soft contributions" (see Sec. III) go like ${\alpha_0}^4$ or higher.

(iii) G_{Mn}/G_{Mp} intermediate. As there was no problem with the simple symmetric distribution amplitude, there is none here. The smallest α_0 for the symmetric case and $Q^4G_{Mp}=1.0$ GeV^4 is here and is $\alpha_0=0.37$ GeV. (Again for the record, $N_0=0.1$, $N_2=-0.1$, $N_3=0.43$, $N_5=0.0$, all in GeV^2, and $Q^4G_{Mn}=-0.28$ GeV^4.)

For now it should be emphasized that the apparent asymptotic G_M has a size as well as a Q^2 falloff which can be matched in PQCD with reasonable values of the QCD coupling constant and quark transverse momenta.

III. SOFT CONTRIBUTIONS

The PQCD expression for the form factor can be derived as an approximation to the impulse approximation.

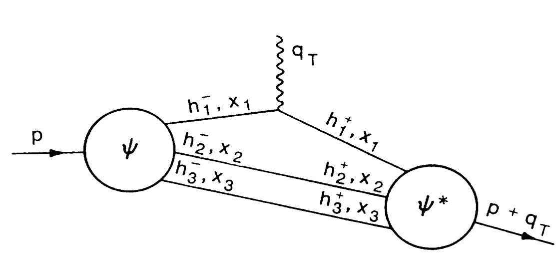

FIG. 5. The impulse approximation which generates both hard and soft contributions to the form factor.

At high Q^2 only the "tail" or high-k_T part of the wave function is important and this is the piece that can be calculated in PQCD and substituted into the usual impulse approximation to obtain the PQCD result for G_M. Note that while the impulse approximation is the dominant contribution[1] to the form factor at high Q^2, the same is not necessarily true at low Q^2.

There remain low-k_T parts of the wave function which can make contributions to G_M. These are the "soft contributions"[4] and are not included in PQCD. It is (or will be) clear enough that they fall faster with Q^2 than the PQCD or "hard" contributions. How big are they at Q^2's where experiments are done? For the wave functions defined by Eqs. (21) and (26) the answer is that they are big—unfortunately or fortunately—but modifications can be made in the k_T dependence to make them small. This will be discussed in this section.

The impulse approximation in the infinite-momentum frame formalism can be written in symmetric form:

$$Q^4G(Q^2)=Q^4\sum_i \int [dx][d^2k_T]\psi(x,\underline{h}_T^{(i)-})e_i\psi^*(x,\underline{h}_T^{(i)+})\ , \tag{32}$$

where ψ includes the spin-isospin part of the wave function, e_i is the charge of the struck quark, and $\mathbf{h}_T^{(i)\pm}$ are the transverse momenta of the quarks in the case where the ith quark is struck (see Fig. 5); for i being one we have

$$\mathbf{h}_{1T}^{(1)\pm}=\mathbf{k}_{1T}\pm\tfrac{1}{2}(1-x_1)\mathbf{q},\quad \mathbf{h}_{2T}^{(1)\pm}=\mathbf{k}_{2T}\mp\tfrac{1}{2}x_2\mathbf{q}\ ,$$
$$\mathbf{h}_{3T}^{(1)\pm}=\mathbf{k}_{3T}\mp\tfrac{1}{2}x_3\mathbf{q}\ . \tag{33}$$

For the proton this becomes

$$G(Q^2)=\int [dx][d^2k_T]\left[\left(\psi_S-\frac{1}{\sqrt{3}}\psi_A[x,h_T^{(1)-}]\right)\left(\psi_S-\frac{1}{\sqrt{3}}\psi_A[x,h_T^{(1)+}]\right)+\tfrac{2}{3}\psi_A(x,h_T^{(2)-})\psi_A(x,h_T^{(2)+})\right]\ . \tag{34}$$

If we use a factored form [Eq. (26)] for the wave function and suppose initially that just a Gaussian [Eq. (28)] can give the transverse-momentum distribution, then

$$Q^4G(Q^2)=\frac{48\pi^4Q^4}{{\alpha_0}^4}\int [dx]\left[\left|\phi_S-\frac{1}{\sqrt{3}}\phi_A\right|^2 e^{-x_{23}Q^2/2{\alpha_0}^2}+\tfrac{2}{3}{\phi_A}^2e^{-x_{13}Q^2/2{\alpha_0}^2}\right]\equiv f(Q^2/2{\alpha_0}^2)\ , \tag{35}$$

where $x_{ij}=x_i^2+x_j^2+x_ix_j$. The function $f(\xi)$ is shown in Fig. 6 calculated using the Chernyak-Zhitnitsky distribution amplitude; qualitatively the results are the same for all distribution amplitudes we have considered. Note that $f(\xi)$ peaks at $\xi\simeq 60$, which corresponds to $Q\simeq 20$ $(\text{GeV}/c)^2$ for $\alpha_0=0.39$ GeV, and that it has a maximum value of about 5 GeV^4, about five times larger than the experimental value of 1 GeV^4. Furthermore, $f(\xi)$ does not fall to the experimental value until Q^2 is greater than 300 $(\text{GeV}/c)^2$. This shows that the low-k_T components of this wave function are dominating over the high-k_T parts; to make (35) smaller than the hard calculation (25) for $Q^2\geq 10$ $(\text{GeV}/c)^2$ would require that α_0^2 be 30 times smaller, which would in turn violate the normalization condition (30). The result may appear surprising, since the Gaussian wave function (28) has an rms transverse momenta of only $\sqrt{2}\,\alpha_0$ and falls off rapidly with k_T.

However, one knows that a Gaussian cannot properly represent the high-transverse-momentum tail of the wave function, and it is not difficult to find a wave function $g(k_T)$ which will give a smaller result for the impulse approximation, and will at the same time be consistent with the hard-scattering calculation and the normalization condition. Let

$$g(k_T)=3(16\pi^2)^2\left[\frac{A}{4\alpha^4}e^{-(p_\rho{}^2+p_\lambda{}^2)/2\alpha^2}+\frac{B\theta}{(p_\rho{}^2+\epsilon\kappa^2)(p_\rho{}^2+\epsilon p_{\lambda T}{}^2)}\right]=g_{\text{soft}}(k_T)+g_{\text{hard}}(k_T)\;, \qquad (36)$$

where $\mathbf{p}_\rho=(\mathbf{k}_{1T}-\mathbf{k}_{3T})/\sqrt{2}$ and $\mathbf{p}_\lambda=(\mathbf{k}_{1T}+\mathbf{k}_{3T}-2\mathbf{k}_{2T})/\sqrt{6}$. Choosing $\epsilon\simeq\frac{1}{16}$ will prove convenient, and $\theta\equiv\theta(p_\rho-\kappa)\theta(p_\lambda-\kappa)$ so that θ is 1 if both p_ρ and p_λ are above κ and otherwise is zero. The tail falls asymptotically like four powers of momentum as it should but otherwise is chosen only for purposes of illustration.

The linear normalization condition for g tells us that

$$A+BI=1\;, \qquad (37)$$

with

$$I=\frac{1}{\epsilon}\int_1^{Q^2/\kappa^2}\frac{dx}{x+\epsilon}\ln\frac{x+\epsilon Q^2/\kappa^2}{x+\epsilon}\;. \qquad (38)$$

We will evaluate I numerically (e.g., for $\epsilon=\frac{1}{16}$, $Q^2=10$ GeV^2, and $\kappa=300$ MeV we get $I=51$), but note that

$$\lim_{Q^2\to\infty} I(\epsilon,Q^2/\kappa^2)\sim\frac{1}{2\epsilon}\ln^2\frac{Q^2}{\kappa^2}\;.$$

The usual normalization condition takes the form

$$\frac{A^2}{\alpha^4}+\frac{8AB}{\alpha^4}R+\frac{8B^2}{(1+\epsilon)^2\kappa^4}=\frac{1}{48\pi^4\langle\phi^2\rangle}=\frac{1}{\alpha_0^4}=44.1\ \text{GeV}^{-4}\;; \qquad (39)$$

the numerical result is for the Chernyak-Zhitnitsky wave function. The overlap term involves

$$R=\int_1^\infty dx\,dy\frac{e^{-(x+y)\kappa^2/2\alpha^2}}{(x+\epsilon)(x+\epsilon y)} \qquad (40)$$

and R is easily bounded:

$$R<\frac{2\alpha^2}{\kappa^2}\left[1+\frac{\kappa^2}{2\alpha^2}\right]^{-1}e^{-\kappa^2/\alpha^2}\;.$$

This suffices to make the overlap term negligible for the parameters we will work with below.

At high Q^2, a simple formula can be obtained for the impulse approximation (32), if we use the approximate relations

$$\int[d^2k_T]g_{\text{hard}}(h_T^{(i)+})g_{\text{hard}}(h_T^{(i)-})\simeq 2g_{\text{hard}}(xq^{(i)})\int[d^2k_T]g_{\text{hard}}(k_T)\;, \qquad (41a)$$

$$\int[d^2k_T]g_{\text{soft}}(h_T^{(i)+})g_{\text{hard}}(h_T^{(i)-})\simeq g_{\text{hard}}(xq^{(i)})\int[d^2k_T]g_{\text{soft}}(k_T)\;, \qquad (41b)$$

in which the notation $xq^{(i)}$ for the arguments of g refer to the substitutions, using the first quark as an example:

$$\mathbf{k}_{1T}\to(1-x_1)\mathbf{q},\quad \mathbf{k}_{2T}\to -x_2\mathbf{q},\quad \mathbf{k}_{3T}\to -x_3\mathbf{q}\;, \qquad (41c)$$

which are the values of the transverse momenta assumed for one of the wave functions in the integrand when the other is at its peak (where its arguments $h_T^{\pm}$ are zero). [Note that the approximations (41) hold only at high Q^2 when the integrands peak sharply at $h^{\pm}=0$; they do not work for the product of two Gaussians which do not peak sharply even at high Q^2.] Using (41) and (37) the impulse approximation becomes

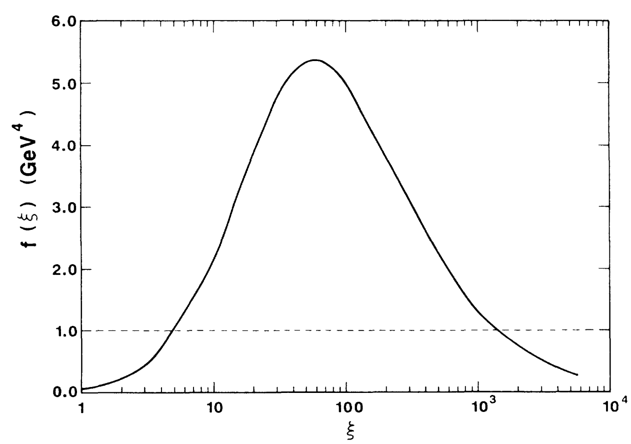

FIG. 6. The function $f(\xi)$ defined in Eq. (35).

$$Q^4 G_{Mp}(Q^2) = A^2 f\left[\frac{Q^2}{2\alpha^2}\right] + B\,24(16\pi^2)^2(J_1+J_2)\ . \tag{42}$$

The A^2 term is usually referred to as the "soft contribution," and falls asymptotically like Q^{-2}. The term linear in B contains the effects of the high-momentum tail of the wave function, and its asymptotic Q^2 dependence is a reflection of the high-momentum behavior of this tail. Consistency with the hard calculation requires that B be chosen to reproduce the results (25). The integrals J_i are given by

$$J_1 = \left\langle \frac{\left[\phi_S - \frac{1}{\sqrt{3}}\phi_A\right]^2 \theta(1-x_1+x_3 \geq \sqrt{2}\kappa/Q)\theta(x_2 \geq (\frac{2}{3})^{1/2}\kappa/Q)}{(1-x_1+x_3)^2[(1-x_1+x_3)^2+3\epsilon x_2{}^2]} \right\rangle = 8.3\times 10^{-5}\ \mathrm{GeV}^4\ ,$$
$$J_2 = \left\langle \frac{\frac{2}{3}\phi_A{}^2\theta(\,|x_1-x_3|\, \geq \sqrt{2}\kappa/Q)\theta(1-x_2 \geq (\frac{2}{3})^{1/2}\kappa/Q)}{(x_1-x_3)^2[(x_1-x_3)^2+3\epsilon(1-x_2)^2]} \right\rangle = 3.2\times 10^{-5}\ \mathrm{GeV}^4\ , \tag{43}$$

where again the numerical values are for the Chernyak-Zhitnitsky distribution amplitude. The consistency condition gives

$$B = 1.3\times 10^{-2} \tag{44}$$

if $\alpha_s = 0.3$. Now the linear condition (37) gives

$$A = 1 - BI = 0.34 \tag{45}$$

for $\epsilon = \frac{1}{16}$, $Q^2 = 10\ \mathrm{GeV}^2$, and $\kappa = 300$ MeV. Finally, the normalization condition (38) determines a new value of α

$$\alpha \simeq 0.23\ \mathrm{GeV}\ . \tag{46}$$

With this small value of A, the soft contribution to (42) is about $\frac{6}{10}$ of the data level, even at its maximum value, which occurs at about $Q^2 \simeq 6\ (\mathrm{GeV}/c)^2$.

The hard contributions plus soft contributions then give something more than the data. We should point out that our main goal in this section was to show that the soft contributions do not dominate the hard ones, and in this we have succeeded, but now something more may be added. The excess of cross section could have several easy explanations. One is that we have used a factorized form of the wave function, which must be unrealistic, and abandoning this would give us still more flexibility to reduce the soft contributions. Another is that perhaps the soft contributions really do make some significant but not dominant contribution to the form factor at present Q^2 and that some other distribution amplitude which proportionately shrinks both the hard and soft contributions should be used. Chernyak and Zhitnitsky's distribution amplitude was based on their QCD-sum-rule moments, but their distribution amplitude is not the only one that fits those moments, and many of the others give a smaller hard-scattering contribution.

Not only does the new wave function (36) show that the hard-scattering calculation dominates for Q^2 of physical interest, it is a more realistic model wave function for the transverse-momentum dependence of the proton. The term proportional to B produces a tiny high-k_T tail for the wave function which more accurately models the power-law behavior of the proton wave function expected for large k_T. Adding such a behavior decouples the two conditions (37) and (39). It is not surprising that this small "tail" (a) contributes the major part of the strength required for the hard-scattering result, (b) dominates the impulse approximation at large Q^2 [if the wave function were exact, the extra term in (42) should reproduce the result (25) exactly], and (c) that it plays no role in the normalization of the wave function, which is dominates by small momentum components. Furthermore, the tail is so small that it also plays no role in the rms value of k_T.

The first term in the sum (36) is the only term which contains truly small values of k_T, and the contribution of this term to the impulse approximation is the "soft contribution" referred to above. By moving some of the strength in Eq. (37) to the tail, the soft contribution has been reduced by a factor of about 9 and no longer dominates the form factor.

Summarizing, consideration of the tail has resulted in a reduction of the soft contributions to G_{Mp} by a factor of about 9, so that at their peak they are below the data, and decoupling of the normalization condition (a nonperturbative low-momentum effect) from the asymptotic calculation (a high-momentum effect).

Thus, the soft contributions do not necessarily dominate the PQCD contributions to the form factor.

IV. COMMENTS AND CONCLUSIONS

In conclusion, the following comments are offered.

(1) Factored wave function. For simplicity, a factored form for the wave function has been used in this paper. This is probably an oversimplification; the correct wave function is not likely to be factorizable. The hard and soft regions of transverse momenta could easily have different x dependencies, and this can give us significant extra freedom to manipulate the hard and soft contributions.

(2) Asymptotic ratio G_{Mp}/G_{Mn}. It has been noted[1] that at very, very high Q^2 (that is, $\ln\ln Q^2 \gg 1$) only the zeroth Appell polynomial survives and the proton form factor goes to zero relative to the neutron form factor. It should also be noted that in this limit the neutron

form factor is positive,[5] so that the neutron form factor must have a zero[14] at some large but finite Q^2.

(3) Chernyak and Zhitnitsky distribution amplitude.[5] Chernyak and Zhitnitsky have proposed a distribution amplitude for the proton. Their distribution amplitude is gotten by supposing an expansion in terms of the six lowest Appell polynomials and fitting to six moments that are calculated using QCD sum rules, and is

$$\phi(x)=x_1x_2x_3(0.111\tilde{\phi}_0-0.274\tilde{\phi}_1-0.212\tilde{\phi}_2 +0.248\tilde{\phi}_3+0.221\tilde{\phi}_4 +0.002\tilde{\phi}_5)\ \mathrm{GeV}^2 .$$

This distribution amplitude gives a good account of G_{Mp}, gives $G_{Mn}\simeq-\frac{1}{2}G_{Mp}$, and is quite asymmetric. While this distribution amplitude is not uniquely forced by the calculated moments, those moments do not allow the possibility of no asymmetry.

As an amusement, examine the hard-scattering expression for G_{Mp} and note that every single term there is positive if N_0, N_3, and N_4 have one sign and N_1, N_2, and N_5 have the opposite sign. This is just the sign pattern in the Chernyak-Zhitnitsky distribution amplitude, excepting the last term whose coefficient is too small to be significant. The QCD sum rules have thus led to a distribution amplitude which satisfies one clear criterion for maximizing G_{Mp}.

(4) Asymmetric wave function. The sorts of wave function which fit both the neutron and proton form factors are quite asymmetric in the three quarks. This is perhaps a surprise and it may be worth speculating how it may come about. First, note that the distribution amplitude, which is a transverse-momentum integrated wave function, is dominated by the high-k_T part of the wave function (if the wave function falls as a power of k_T as expected from PQCD). At the same time, the normalization (which unlike the distribution amplitude is gotten by squaring the wave function before integrating) is dominated by low k_T. The expectation of near symmetry among the quarks comes from calculations of things such as the charge radius or magnetic moment that are like the normalization in being dominated by the low-k_T part of the wave function, and the x dependence associated with this could be quite symmetric.

Why, then, might one expect an asymmetry at high k_T? Think of quark-quark scattering, or equally well, electron-electron scattering at very high energies. There is a large, angle-dependent spin dependence. At 90° in the c.m., the amplitude for scattering two same helicity electron is twice the magnitude of the amplitude for opposite helicity electrons. High-k_T quarks result from a hard scattering of low-k_T quarks, and this amplitude is spin dependent. The pair of quarks with same helicity are more likely to scatter each other out to high k_T than other pairs of quarks, and this same scattering will likely also scatter the quarks forward and backward so that one of the same helicity quarks will have a large share of the longitudinal momentum.[16] This is just what is seen.

To conclude, the magnitude and sign of either nucleon magnetic form factor can be fit with a broad distribution amplitude, and consideration of both nucleons together suggests an asymmetric spatial part of the distribution amplitude. Finally, the soft contributions may be below the asymptotic QCD results in the range where experiments may support the latter.

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