Holonomy transformation, deficit angle, and Aharonov-Bohm effect in a cylindrically symmetric universe

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We obtain the exact expression for the holonomy transformation for a circle that corresponds to a nonstatic and to the most general static, cylindrically symmetric metric. A general formula for the angular deficit is obtained in both cases. A discussion about the Aharanov-Bohm effect in the static case is given.

In this paper, we use the loop variable in the theory of gravity in order to obtain the holonomy transformation and a general formula for the angular deficits for the space-time under consideration. The holonomy transformation enables us to find directly the deficit angles (conical angle) without the necessity of using the Riemann tensor of four-dimensional space-time.

In this context we reproduce the results of Garfinkle,[1] Vilenkin,[2] Gott,[3] and Hiscock.[4] We also obtain a result of Ford and Vilenkin[5] who used the Gauss-Bonnet theorem. In all cases and particularly in this case our method is easier than that of Ref. 5, because to use the Gauss-Bonnet theorem it is necessary to express the Gaussian curvature in terms of the four-dimensional Ricci tensor and hence the momentum-energy tensor of the source, and because in general, it does not seem possible to obtain such relations. We consider a particular case of a cylindrically symmetric metric due to Marder[6] to examine the effect of the parallel transport of vectors around closed paths, showing that this gravitational field provides a gravitational analogue of the Aharonov-Bohm effect.

The loop variables in the theory of gravity are matrices representing parallel transport along contours in a space-time with a given affine connection. This is connected with the holonomy transformation which is a mathematical object that contains information about how vectors change when parallel transported around a closed curve.

Now let us suppose that we have a vector v^α at a point p of a closed curve C in a space-time. Then, one can produce a vector $\bar{v}^\alpha$ at p, which, in general, will be different from v^α by parallel transporting v^α around C. In this case, we associate with the point p and the curve C a linear map $U^\alpha{}_\beta$ such that for any vector v^α at p the vector $\bar{v}^\alpha$ at p results from parallel transporting v^α around C and is given by $\bar{v}^\alpha = U^\alpha{}_\beta v^\beta$. The linear map $U^\alpha{}_\beta$ is called the holonomy transformation associated with the point p and the curve C. If we choose a tetrad frame and a parameter λ for the curve C such that $C(0)=C(1)=p$, then, in parallel transporting a vector v^α from $C(\lambda)$ to $C(\lambda+d\lambda)$, the vector components change by $\delta v^\alpha = M^\alpha{}_\beta v^\beta d\lambda$, where $M^\alpha{}_\beta$ is a linear map which depends on the tetrad, the affine connection of the spacetime and the value of λ. It then follows that the holonomy transformation $U^\alpha{}_\beta$ is given by the ordered matrix product of the N linear maps as

$$U^\alpha{}_\beta = \lim_{N\to\infty} \prod_{i=1}^{N} \left[\delta^\alpha{}_\beta + \frac{1}{N} M^\alpha{}_\beta \big|_{\lambda=i/N} \right] . \tag{1}$$

One often writes the expression in Eq. (1) as

$$U(C) = P \exp\left[\int_C M \right] \tag{2}$$

which should be understood as simply an abbreviation for the expression in Eq. (1). Note that if $M^\alpha{}_\beta$ is independent of λ, then it follows from Eq. (1) that $U^\alpha{}_\beta$ is given by $U^\alpha{}_\beta = \exp(M^\alpha{}_\beta)$.

In this paper we shall use the notation

$$U(C) = P \exp\left[\int_C \Gamma_\mu dx^\mu \right] , \tag{3}$$

where P means ordered product along a curve C and Γ_μ is the tetradic connection.

Now, consider the most general static cylindrically symmetric metric[7]

$$ds^2 = -e^{2\nu} dt^2 + e^{2\psi} d\phi^2 + e^{2\lambda}(d\rho^2 + dz^2) , \tag{4}$$

where ν, ψ, and λ are functions of ρ.

To compute the tetradic connection, we start by defining the one-forms $\theta^D (D=0,1,2,3)$:

$$\begin{aligned} \theta^0 &= e^\nu dt , \\ \theta^1 &= e^\lambda d\rho , \\ \theta^2 &= e^\psi d\phi , \\ \theta^3 &= e^\lambda dz . \end{aligned} \tag{5}$$

The geometry (4) is obtained from (5) by the expression

$$ds^2 = \eta_{DE}\theta^D\theta^E = -(\theta^0)^2 + (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 , \tag{6}$$

where η_{DE} is the Minkowski tensor diag$(-,+,+,+)$.

Then the tetrad frame defined by $\theta^D = e^{(D)}_\alpha dx^\alpha$ is given by

$$e^{(0)}_0 = e^\nu, \quad e^{(1)}_1 = e^\lambda, \quad e^{(2)}_2 = e^\psi, \quad \text{and } e^{(3)}_3 = e^\lambda . \tag{7}$$

A straightforward calculation gives the value of $\Gamma_\mu dx^\mu$ (or $\Gamma^D_{\mu E} dx^\mu$ where D and E are tetradic indices). The unique non-null $\Gamma^D_{\mu E} dx^\mu$ are

$$\Gamma^0_{\mu 1} dx^\mu = e^{-\lambda} \frac{d}{d\rho}(e^\nu) dt = \Gamma^1_{\mu 0} dx^\mu ,$$

$$\Gamma^2_{\mu 1} dx^\mu = e^{-\lambda}\frac{d}{d\rho}(e^\psi)d\phi = -\Gamma^1_{\mu 2}dx^\mu \ , \tag{8}$$

$$\Gamma^3_{\mu 1} dx^\mu = e^{-\lambda}\frac{d}{d\rho}(e^\lambda)dz = -\Gamma^1_{\mu 3}dx^\mu \ .$$

We shall consider circles with center at the origin and a fixed value of ρ. So, in this case

$$\Gamma^D_{\mu E}dx^\mu = \Gamma_\phi d\phi \ , \tag{9}$$

where

$$\Gamma_\phi = \begin{bmatrix} 0 & -e^{-\lambda}\frac{d}{d\rho}(e^\psi) & 0 & 0 \\ e^{-\lambda}\frac{d}{d\rho}(e^\psi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{10}$$

As Γ_ϕ is independent of ϕ, then

$$U(C) = P\exp\left[\int_{\phi_1}^{\phi_2}\Gamma_\phi d\phi\right] = e^{\Gamma_\phi(\phi_2-\phi_1)} \ . \tag{11}$$

From Eq. (10), it is easy to see that

$$\Gamma_\phi{}^3 = -e^{-2\lambda}\left[\frac{d}{d\rho}(e^\psi)\right]^2\Gamma_\phi \equiv -\beta^2\Gamma_\phi$$

$$\text{(definition of } \beta) \ . \tag{12}$$

Equation (12) implies that for a complete circle we have

$$U(C) = 1 + \frac{\Gamma_\phi}{\beta}\sin(2\pi\beta) + \frac{\Gamma_\phi{}^2}{\beta^2}[1-\cos(2\pi\beta)] \ . \tag{13}$$

Equation (13) is the exact expression for the holonomy transformation that corresponds to the class of solution given by Eq. (4).

Now, let us define the deficit angle and establish its connection with the holonomy transformation. The deficit angle is one number and the holonomy transformation is a set of linear maps (one for each point and closed curve). One must then obtain from the linear map a single number, the deficit angle (in our case) which is a property of axially symmetric, asymptotically conical space-times (at infinity, these space-times are asymptotically a cone rather than a plane). To obtain the single linear map we consider a point p on the curve C (since the space-time is axially symmetric, it does not matter which point we choose). Then $U^\alpha{}_\beta$, as defined previously, is the holonomy transformation associated with the point p and curve C, where C is an integral curve of the axial Killing field in the asymptotic region. With $U^\alpha{}_\beta$, the deficit angle χ can be defined by

$$\cos\chi = U^\alpha{}_\beta \hat{\mathbf{A}}_\alpha \hat{\mathbf{A}}^\beta \ , \tag{14}$$

where $\hat{\mathbf{A}}^\alpha$ is the unit vector in the direction of the axial Killing field. Using tetradic indices we can write

$$\cos\chi = \bar{A}^a \eta A_a \ , \tag{15}$$

where $\bar{A}^a = U^a{}_b A^b$.

As A^b is a unit vector the elements of U are the components of the parallel translated vector. From this and Eq. (15), it follows that, the corresponding diagonal element of U is the cosine of the angle between the vectors. Then, we can write, in this case,

$$\cos\chi_a = U^a{}_a \ , \tag{16}$$

where a is a tetradic index.

The non-null angular deviations occur in our case for $a=1$ and 2. Considering $a=1$, we have

$$\cos\chi_1 = \cos 2\pi\beta \tag{17}$$

or

$$|\chi_1| = |2\pi\beta + 2\pi n| \ .$$

As $\beta \to 1$ we must have $\chi_1 \to 0$ we choose $n=-1$ so that

$$|\chi_1| = |2\pi(\beta-1)| \ , \tag{18}$$

where

$$\beta^2 \equiv e^{-2\lambda}\left[\frac{d}{d\rho}(e^\psi)\right]^2 \ .$$

Equation (18) with a definition of β given by Eq. (12) corresponds to the general formula for the angular deficit for a class of static cylindrically symmetric space-time given by metric (4).

If we made a different choice for the coordinate ρ one can write the metric given by Eq. (4) as

$$ds^2 = -e^A dt^2 + e^B dz^2 + e^C d\phi^2 + d\rho^2 \ . \tag{19}$$

If we identify all the coefficients of the metrics (4) and (19) and use our formula (18), we get exactly

$$\chi = 2\pi\int_0^\infty \frac{d^2}{d\rho^2}(e^{C/2})d\rho \ . \tag{20}$$

Equation (20) is the Garfinkle expression for the deficit angle in which we assume certain boundary conditions such as (1) at the axis the metric is smooth and (2) at infinity, space-time is asymptotically conical. Proceeding as Garfinkle did we obtain Eq. (82) of Ref. 2. Then, if we consider Vilenkin's solution (cosmic string) we get from Eq. (18) that $\chi = 8\pi\mu$.

Now, assume that the coordinates can be chosen so that $e^\lambda \to 1$ and $e^\psi \to \rho$ as $\rho \to 0$ and $e^\lambda \to 1$ and $e^\psi \to b\rho$ as $\rho \to \infty$. Therefore, in the neighborhood of the origin, the space is flat and for large ρ it becomes a cone with conical angle πb. As the space is asymptotically conical, if we transport a vector parallel to a given closed curve around it, in general, it will not return to itself but will undergo a rotation by an angle which is expressible as the area integral of the Gaussian curvature over the subsurface of the space enclosed by the curve. Here, we get this angle with the use of Eq. (18).

Now, setting the values of e^λ and e^ψ when $\rho \to 0$ and $\rho \to \infty$ in Eq. (18) we get

$$\chi = 2\pi(1-b) \tag{21}$$

which is the same result obtained by Ford and Vilenkin[5] with the use of the Gauss-Bonnet theorem.

Then, we conclude that with the expression for the transport operator (holonomy transformation) for the space-time given by metric (4) we can get a general expression for the deficit angle that undergoes a vector parallel transported around a circle.

Consider a particular case of metric (4) which is given by[6]

$$ds^2 = dt^2 - \rho^2 d\phi^2 - A^2(d\rho^2 + dz^2) . \tag{22}$$

Then we obtain, for the deficit angle,

$$|\chi_{1,2}| = 2\pi \left| \frac{1}{A} - 1 \right| . \tag{23}$$

From Eq. (23) we conclude that there will be no Aharonov-Bohm effect if and only if $1/A$ is an odd integer. This condition is not always satisfied because $1/A$ is not necessarily an odd integer. Then, this flat space-time, describing a nontrivial gravitational field, provides a gravitational analog of the Aharonov-Bohm effect, as pointed out by Dowker.[8]

Now, consider a metric describing a nonstatic cylindrically symmetric space-time, which is given by

$$ds^2 = A^2(\rho)[dt^2 - B^2(t)dz^2] - D^2(t)[d\rho^2 + G^2(\rho)d\phi^2] , \tag{24}$$

where ρ, ϕ, and z are cylindrical coordinates defined by $0 < \rho < \infty$, $0 < \phi < 2\pi$, $0 < z < \infty$, and $t > 0$.

As in the preceding case, to compute the Γ's, we define the one-forms θ^C $(C = 0,1,2,3)$:

$$\begin{aligned} \theta^0 &= A(\rho)dt , \\ \theta^1 &= D(t)d\rho , \\ \theta^2 &= D(t)G(\rho)d\phi , \\ \theta^3 &= A(\rho)B(t)dz . \end{aligned} \tag{25}$$

Then the tetrad frame defined by $\theta^C = e_\alpha^{(C)} dx^\alpha$ is given by

$$e_0^{(0)} = A(\rho), \quad e_1^{(1)} = D(t), \quad e_2^{(2)} = D(t)G(\rho) ,$$

and (26)

$$e_3^{(3)} = A(\rho)B(t) .$$

A straightforward calculation gives us the unique non-null tetradic connection $\Gamma^C_{\mu E} dx^\mu$ (where C and E are tetradic indices):

$$\begin{aligned} \Gamma^1_{\mu 0} dx^\mu &= \frac{A'}{D} dt + \frac{\dot{D}}{A} d\rho = \Gamma^0_{\mu 1} dx^\mu , \\ \Gamma^2_{\mu 0} dx^\mu &= \frac{\dot{D}G}{A} d\phi = \Gamma^0_{\mu 2} dx^\mu , \\ \Gamma^2_{\mu 1} dx^\mu &= G' d\phi = -\Gamma^1_{\mu 2} dx^\mu , \\ \Gamma^3_{\mu 1} dx^\mu &= \frac{A'B}{D} dz = -\Gamma^1_{\mu 3} dx^\mu , \end{aligned} \tag{27}$$

in which the overdot denotes d/dt and a prime denotes $d/d\rho$.

As in the first case, we shall consider circles with the center at the origin with a fixed value of ρ. So, in this case

$$\Gamma^C_{\mu E} dx^\mu = \Gamma_\phi d\phi , \tag{28}$$

where

$$\Gamma_\phi = \begin{pmatrix} 0 & -G' & 0 & 0 \\ G' & 0 & 0 & \frac{\dot{D}G}{A} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\dot{D}G}{A} & 0 & 0 \end{pmatrix} . \tag{29}$$

From Eq. (29) it is easy to see that

$$\Gamma_\phi{}^3 = -\left[(G')^2 - \left[\frac{\dot{D}G}{A} \right]^2 \right] \Gamma_\phi \equiv -A_\phi{}^2 \Gamma_\phi \quad \text{(definition of } A_\phi) . \tag{30}$$

Using the fact that Γ_ϕ is independent of ϕ together with Eq. (30), we obtain from Eq. (3) that, for a complete circle,

$$U(C) = 1 + \frac{\Gamma_\phi}{A_\phi} \sin(2\pi A_\phi) + \frac{\Gamma_\phi{}^2}{A_\phi{}^2} [1 - \cos(2\pi A_\phi)] . \tag{31}$$

Equation (31) is the exact expression for the holonomy transformation that corresponds to the solution given by Eq. (24).

The non-null angular deviations occur, in this case, for $a = 0, 1$, and 2. Considering $a = 1$, we get

$$|\chi_1| = |2\pi(A_\phi - 1)| . \tag{32}$$

Equation (32) with the definition of A_ϕ corresponds to the angular deficit expression for $a = 1$, for a nonstatic cylindrically symmetric spacetime.

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[1]D. Garfinkle, Phys. Rev. D **32**, 1323 (1985).
[2]A. Vilenkin, Phys. Rev. D **23**, 825 (1981).
[3]J. R. Gott III, Astrophys. J. **288**, 422 (1985).
[4]W. A. Hiscock, Phys. Rev. D **31**, 3288 (1985).
[5]L. H. Ford and A. Vilenkin, J. Phys. A **14**, 2353 (1981).
[6]L. Marder, Proc. R. Soc. London **A252**, 45 (1959).
[7]S. Chandrasekhar and J. L. Friedman, Astrophys. J. **175**, 379 (1972).
[8]J. S. Dowker, Nuovo Cimento **52B**, 129 (1967).