Time-dependent Hartree-Fock formalism and the excitations of the Dirac sea in the Nambu–Jona-Lasinio model

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(Received 27 March 1987)

The dynamical chiral-symmetry breaking, the light-meson spectrum, and the properties of the pion are investigated in the Nambu–Jona-Lasinio model. The time-dependent Hartree-Fock formalism is used. The pion dispersion relation is derived in a semiclassical approximation. The stationarity of the quark droplet matter is investigated in the Thomas-Fermi approximation.

I. INTRODUCTION

In recent years there has been a revival of interest in chiral effective models in connection with chiral-symmetry breaking and low-energy properties of hadrons. The motivation for this interest relies mainly upon the following considerations. (1) In the theory of quantum chromodynamics (QCD), widely recognized as the fundamental theory of strong interactions, chiral symmetry is spontaneously broken leading to a condensate of $q\bar{q}$ pairs in the QCD vacuum. In this context the problem of understanding the mechanism of symmetry breaking naturally arises as well as its consequences for the properties of hadrons. (2) Chiral effective models, namely, those with interaction between pairs of fermions (i.e., with four-fermion field operators) of the Nambu–Jona-Lasinio (NJL) type,[1] have been shown to provide reasonable approximations to QCD for intermediate length scales,[2] with the advantage of being easier to handle then QCD itself.

Although the original model of NJL contains only the minimal prescription for a chiral effective interaction between quarks it has been largely studied in connection with QCD. Schematic models which illustrate this connection have been developed.[3–7] More recently, extended versions of the NJL model were implemented,[8] providing also a connection between this model and the Skyrme model. The techniques and the formulation used are generally those of quantum field theory.

The NJL model remains a manageable approximation to dynamical symmetry breaking and may very usefully describe low-energy properties of hadrons: namely, the light-meson spectrum and pion properties. In fact, above a critical value of the coupling constant the system undergoes a phase transition: the normal vacuum is replaced by a chirally deformed vacuum with a condensate of $q\bar{q}$ pairs of massless quarks; the quarks acquire a mass and a mode with a zero frequency appears. This mode is interpreted as a massless boson in the spirit of the Goldstone theorem.[9] Chiral symmetry is therefore restored in the average through the Nambu-Goldstone mode.

The light-meson spectrum can be obtained as collective excitations of the Dirac sea of massive particles.

The strong analogy with the theoretical description of low-lying collective particle-hole states in many-body systems suggests the use of techniques developed in nuclear physics.

The aim of this paper is, in this spirit, to use the time-dependent Hartree-Fock (TDHF) formalism to investigate the mechanism of breaking and restoration of chiral symmetry, the spectrum of light mesons, the pion dispersion relation, and the pion decay constant in the framework of the NJL model. These and other techniques, traditionally applied in many-body problems are, in fact, used nowadays in chiral models.[10]

So, after presenting the equilibrium configuration in the context of a mean-field approximation we consider small deviations from equilibrium, which lead to collective excitations.

A convenient way to deal with extended many-body systems is to use the Wigner transform[11] for the density matrix. Such a method leads in a natural way to a semiclassical approximation similar to the static Thomas-Fermi approximation.

The outline of the rest of this paper is as follows. In Sec. II we summarize the TDHF method in the density-matrix formalism. In Sec. III we give a brief account of the NJL model and we present a study of the properties of the physical vacuum. In Sec. IV we study homogeneous collective excitations which give rise to the masses of the pseudoscalar, scalar, and vector mesons. In Sec. V we investigate some static and dynamical properties of the pion and finally we present our conclusions in Sec. VI.

II. TIME-DEPENDENT HARTREE-FOCK FORMALISM

In the framework of the TDHF approximation the wave function of a system of fermions is a Slater determinant, the solution of the variational problem[12]

$$\delta S=\delta\int_{t_1}^{t_2}\left\langle\Psi\left|i\hbar\frac{\partial}{\partial t}-H\right|\Psi\right\rangle dt=0\ , \tag{2.1}$$

where $|\Psi\rangle$ is a Slater determinant of single-particle states $\psi_i(\mathbf{r},t)$, $i\in(1,2,\ldots,N)$ and H is the many-body Hamiltonian

$$H=\sum_{i=1}^{N}t(i)+\tfrac{1}{2}\sum_{i\neq j}v(ij)\ . \tag{2.2}$$

In terms of the density matrix

$$\rho(\mathbf{r},\mathbf{r}',t)=\sum_{i=1}^{N}\psi_i^*(\mathbf{r},t)\psi_i(\mathbf{r}',t)\;, \tag{2.3}$$

the TDHF equation has the form

$$i\hbar\frac{\partial\rho}{\partial t}=[h,\rho]\;. \tag{2.4}$$

The density matrix has the property $\rho^2=\rho$ and h is the single-particle Hamiltonian

$$h(1)=t(1)+\mathrm{tr}_2\rho(2)v^A(12)\;, \tag{2.5}$$

where $v^A(12)$ is the antisymmetrized two-body interaction. It is clear that, in general, h is a functional of the density matrix.

The state of equilibrium is described by a single-particle density matrix satisfying the stability condition

$$\mathcal{E}(\rho_0)\le\mathcal{E}(U\rho_0U^\dagger)\;, \tag{2.6}$$

where

$$\mathcal{E}(\rho)=\mathrm{tr}_1\rho(1)t(1)+\tfrac{1}{2}\mathrm{tr}_1\mathrm{tr}_2\rho(1)\rho(2)v^A(12)\;, \tag{2.7}$$

and U is an arbitrary unitary operator. The most general single-particle density matrix may be written $\rho=U\rho_0U^\dagger$.

From Eq. (2.1) it follows directly that the time evolution of the density matrix may be derived from the Lagrangian

$$L=i\hbar\,\mathrm{tr}(\dot U\rho_0U^\dagger)-\mathcal{E}(U\rho_0U^\dagger)\;. \tag{2.8}$$

We may choose

$$U=\exp\left[\frac{i}{\hbar}s\right]\;, \tag{2.9}$$

where s is a time-dependent single-particle operator, which is the generator of deviations from equilibrium.

Expanding (2.8) up to second order, we obtain the Lagrangian

$$L^{(2)}=\mathrm{tr}\left[\rho_0\left[\frac{i}{2\hbar}[s,\dot s]-\frac{1}{2\hbar^2}[s,[h_0,s]]\right]\right]-\frac{1}{2\hbar^2}\mathrm{tr}_1\mathrm{tr}_2\{\rho_0(1)\rho_0(2)[s(1),[v^A(12),s(2)]]\}\;, \tag{2.10}$$

which describes the dynamics of the state in the neighborhood of a state of equilibrium characterized by the density matrix ρ_0 and the self-consistent HF Hamiltonian h_0.

The following expansion of the density matrix holds in the neighborhood of ρ_0:

$$\rho=\rho_0-\frac{i}{\hbar}[\rho_0,s]-\frac{1}{2\hbar^2}[[\rho_0,s],s]+\cdots\;. \tag{2.11}$$

In order to reduce the computational labor involved in the solution of the linearized equations of motion we replace them by their leading order in a Wigner-Kirkwood expansion in powers of $\hbar$.

In the present case the method is applied to a system of Dirac fermions and so the Wigner transform of one-body operators are 4×4 matrices, which may be expanded in any basis, $\{\Gamma_1,\Gamma_2,\ldots,\Gamma_{16}\}$, of the 4×4 matrices,

$$A=\sum_{\alpha=1}^{16}\Gamma_\alpha f_\alpha\;, \tag{2.12}$$

where f_α are functions of momentum and space coordinates $(\mathbf{x},\mathbf{p})$ and time t.

For completeness we state the well-known expression for the Wigner transform of the product of two single-particle operators A and B in terms of the Wigner transform of these operators:

$$(AB)_W(\mathbf{x},\mathbf{p})=A_W(\mathbf{x},\mathbf{p})\exp(\tfrac{1}{2}i\hbar\overleftrightarrow{\Lambda})B_W(\mathbf{x},\mathbf{p})\;, \tag{2.13}$$

with

$$\overleftrightarrow{\Lambda}=\overleftarrow{\nabla}_{\mathbf{x}}\overrightarrow{\nabla}_{\mathbf{p}}-\overleftarrow{\nabla}_{\mathbf{p}}\overrightarrow{\nabla}_{\mathbf{x}}\;.$$

The following expression for the Wigner transform of a commutator is easily derived, if only the leading order of Eq. (2.13) is retained;

$$[\Gamma_\alpha f_\alpha,\Gamma_\beta f_\beta]_W=[\Gamma_\alpha,\Gamma_\beta]f_\alpha f_\beta+\frac{i\hbar}{2}(\Gamma_\alpha\Gamma_\beta+\Gamma_\beta\Gamma_\alpha)\{f_\alpha,f_\beta\}\;, \tag{2.14}$$

where $\{f,g\}$ denotes the Poisson brackets:

$$\{f,g\}=\frac{\partial f}{\partial\mathbf{x}}\frac{\partial g}{\partial\mathbf{p}}-\frac{\partial f}{\partial\mathbf{p}}\frac{\partial g}{\partial\mathbf{x}}\;.$$

These expressions are largely used in Sec. V B. It is also implicit that we will consider the classical limit of the Lagrangian (2.10) retaining only the leading-order terms in a Wigner-Kirkwood expansion.

From now on we will suppress the index W when it is obvious that we are using a Wigner transform and take $\hbar=1$.

III. SELF-CONSISTENT MEAN-FIELD TREATMENT OF THE PHYSICAL VACUUM

The NJL model describes a system of many fermions, its dynamics being described by the chiral-invariant Hamiltonian

$$H=\sum_{i=1}^{N}\gamma_5(i)\boldsymbol{\sigma}(i)\cdot\mathbf{p}_i-g\sum_{i\neq j}\delta(\mathbf{x}_i-\mathbf{x}_j)[\beta(i)\beta(j)-\beta(i)\gamma_5(i)\beta(j)\gamma_5(j)]\;, \tag{3.1}$$

where $\boldsymbol{\sigma}$, γ_5, and β are 4×4 matrices defined by

$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{bmatrix},\quad \gamma_5=\begin{bmatrix}0 & I_2\\ I_2 & 0\end{bmatrix},$$

$$\beta=\begin{bmatrix}I_2 & 0\\ 0 & -I_2\end{bmatrix},$$

and g is the coupling constant. We denote by I_2 the 2×2 identity matrix and by $\boldsymbol{\sigma}$ the vector having the

Pauli matrices for components. Here we consider the fermions to be quarks and we use the former version of the NJL model,[1] taking into account only one single flavor.

It is a well-known fact that, in this model, at zero positive-energy states occupied, the interaction couples $q\bar{q}$ pairs into the vacuum leading to a spontaneous breaking of chiral symmetry, the quarks acquiring a dynamical mass. We will discuss this mechanism and next we will show that by adding positive-energy states the gap between positive- and negative-energy levels is reduced leading to a restoration of chiral symmetry. This behavior will be explored in order to discuss the stability of quark droplets.

A. Dynamical chiral-symmetry breaking

We will present a brief description of the mechanism of chiral-symmetry breaking in terms of the transition from a Dirac sea (Slater determinant) of massless particles to a Dirac sea of massive particles. In the first case the single-particle states are eigenfunctions of a chirally symmetric single-particle Hamiltonian, while in the second case the single-particle states are eigenfunctions of an asymmetric self-consistent single-particle Hamiltonian. Within the mean-field approach we define the simplest single-particle effective Hamiltonian

$$h=\gamma_5\sigma\cdot\mathbf{p}+\beta M \ , \tag{3.2}$$

where βM is a mean-field potential.

In order to select a model of the physical vacuum we consider, as possible candidates, trial Slater determinants $|\Phi\rangle$ with the upper N negative-energy states occupied. We denote by Λ the highest momentum of the occupied states. This vacuum state will be referred to as (I) and is illustrated in Fig. 1.

We may write

$$|\Phi\rangle=\prod_{i=1}^{N} d_i^\dagger|0\rangle \ , \tag{3.3}$$

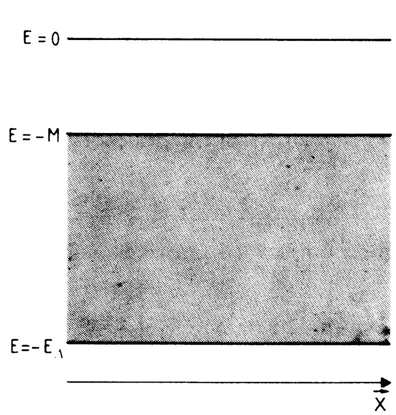

FIG. 1. Illustration of the Dirac sea with the upper N negative-energy states occupied.

where $|0\rangle$ is the absolute vacuum and i stands for the momentum and helicity $|\mathbf{p}_i|\le\Lambda$.

The density matrix is defined by

$$\rho_{\mu\mu'}(\mathbf{x},\mathbf{x}')=\langle\Phi|\psi^\dagger_{\mu'}(\mathbf{x}')\psi_\mu(\mathbf{x})|\Phi\rangle \ .$$

Its Wigner transform is $\mathbf{x}$ independent and may be written

$$\rho=\rho_W(\mathbf{p})=\frac{1}{2}\left[I-\beta\frac{M}{E}-\gamma_5\frac{\sigma\cdot\mathbf{p}}{E}\right]\theta(\Lambda^2-p^2) \ , \tag{3.4}$$

where $E=(p^2+M^2)^{1/2}$ and $\theta(x)$ is the usual step function

$$\begin{aligned}\theta(x)&=1, \quad x\ge 0 \ ,\\ \theta(x)&=0, \quad x<0 \ .\end{aligned} \tag{3.5}$$

The stability condition (see Appendix A) leads to a self-consistent condition for the self-consistent mass M that may be written in the form

$$M=4gM\sum_{\mathbf{p}}\frac{1}{E}\theta(\Lambda^2-p^2) \ . \tag{3.6}$$

known in the literature as the "gap equation."

Aside from the trivial solution $M=0$, Eq. (3.6) has solutions for $M>0$ above a critical value of the coupling strength $g_{\rm cr}=\pi^2/\Lambda^2$. This can be easily seen by performing the summation over $\mathbf{p}$ in Eq. (3.6) which allows us to write

$$\frac{\pi^2}{g\Lambda^2}=\left[1+\frac{M^2}{\Lambda^2}\right]^{1/2}-\frac{M^2}{\Lambda^2}\ln\left[\frac{\Lambda}{M}+\left[1+\frac{\Lambda^2}{M^2}\right]^{1/2}\right] \ . \tag{3.7}$$

In Fig. 2 the dimensionless quantity $g\Lambda^2$ is plotted against Λ/M. We observe that chiral symmetry is broken

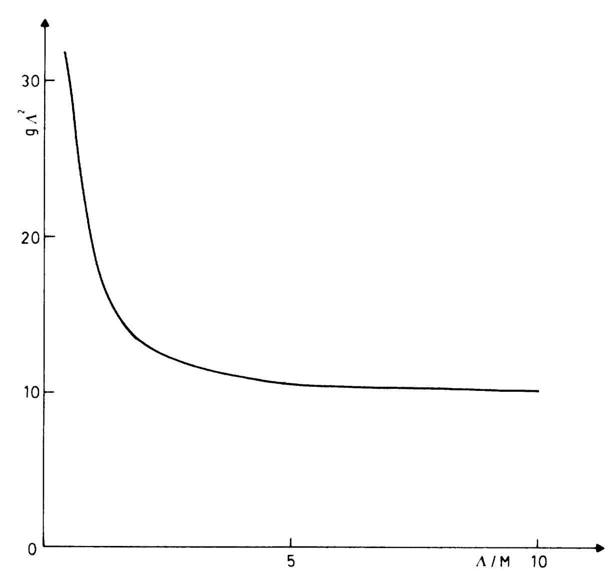

FIG. 2. Dimensionless quantity $g\Lambda^2$ as a function of Λ/M. The chiral symmetry is broken ($M\neq 0$) for $g\Lambda^2>\pi^2$.

ken if $g\Lambda^2$ is larger than π^2. The system undergoes, then, a phase transition into a state of broken symmetry. The most important features are as follows.

(1) When $M=0$ the effective single-particle Hamiltonian describes a massless Dirac free particle, the state vector

$$|\Phi_0\rangle = \prod_{i=1}^{N} d_i^{\dagger(0)} |0\rangle$$

(where $d_i^{\dagger(0)}$ is the creation operator of a massless negative-energy state), has zero chirality and the expectation value

$$\langle \Phi_0 | \bar{\psi}\psi | \Phi_0 \rangle = \left\langle \Phi_0 \left| \sum_{\mu\nu} \psi_\mu^\dagger(\mathbf{x})\beta_{\mu\nu}\psi_\nu(\mathbf{x}) \right| \Phi_0 \right\rangle = \sum_{\mu\nu} \beta_{\mu\nu}\rho_{\nu\mu}^{(0)}(\mathbf{x},\mathbf{x}) ,$$

which plays the role of an order parameter, is zero.

(2) When the mass M is different from zero the HF state,

$$|\Phi\rangle = \prod_{i=1}^{N} d_i^\dagger |0\rangle ,$$

is no longer an eigenstate of chirality and the order parameter, which measures the chiral deformation, is given by

$$\langle \bar{\psi}\psi \rangle = \langle \Phi | \bar{\psi}\psi | \Phi \rangle = \mathrm{Tr}\beta\rho_0 = -\frac{M}{2g} , \tag{3.8}$$

where ρ_0 is the equilibrium density given by Eq. (3.4).

Equations (3.8) and (3.7) enable us to evaluate the dimensionless ratio $-\langle \bar{\psi}\psi \rangle^{1/3}/M$ which is plotted in Fig. 3 as a function of Λ/M.

As is referred to in Appendix A the creation operators of massive particles are related to those of massless particles by a canonical transformation. In fact the new vacuum (which is a Slater determinant of massive particles in negative-energy states), may be written as a coherent superposition of $q\bar{q}$ pairs of massless particles (as usual, we interpret the creation of antiparticles with positive energy as the annihilation of particles in the negative Dirac sea) created in the old vacuum (which is a Slater determinant of massless particles in negative-energy states).

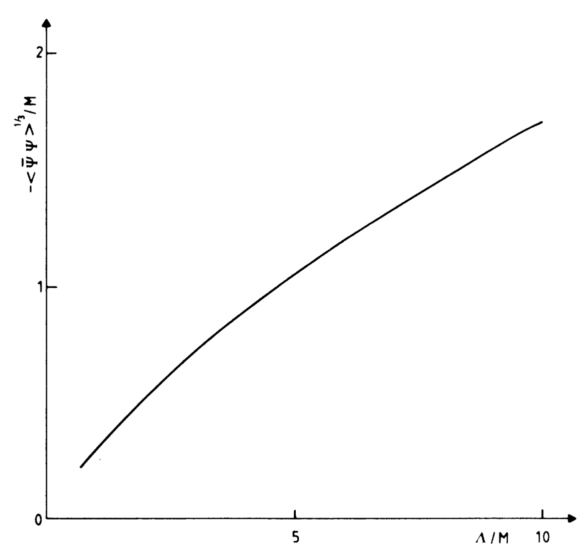

FIG. 3. Dimensionless ratio $-\langle \bar{\psi}\psi \rangle^{1/3}/M$ as a function of Λ/M.

Those facts and the similarity between the self-consistency condition and the gap equation in the BCS theory, are a manifestation of the analogy with this theory which motivated the model.

(3) According to the Goldstone theorem a massless boson, the Goldstone boson, should appear as a result of the breaking of a continuous symmetry.

As a matter of fact, if we perform a chiral rotation of the mean-field vacuum as is shown in Appendix A, a state with the same mean energy as the nonrotated vacuum is obtained.

Chiral symmetry is therefore restored in the average. Since the chiral rotation relating the two Slater determinants is generated by a pseudoscalar operator, a pseudoscalar zero-frequency mode appears in agreement with the Thouless-Valatin theory.[13] This mode is interpreted in the spirit of the Goldstone theorem, as the massless pseudoscalar mode which represents the pion in the chiral limit. A detailed and formal explanation of this subject is given in Sec. V.

B. Restoration of chiral symmetry at finite density

Nowadays it is accepted that when the baryon density is increased hadronic matter undergoes a phase transition to a quark-gluon plasma.[14] This form of matter has been investigated by Monte Carlo simulations. Chiral phase transitions in the NJL model at finite temperature and density[15] can provide an alternative framework to Monte Carlo simulations.

In this concern we discuss a mechanism of chiral phase transition with density. This will be done by introducing a finite number of occupied positive-energy states, where the highest momentum of the occupied state is denoted by λ.

So, we consider as a physical vacuum Slater determinants, $|\Psi\rangle$, with n positive-energy states on the top of the previous state $|\Phi\rangle$ [Eq. (3.3)].

We may write

$$|\Psi\rangle = \prod_{i=1}^{n} b_i^\dagger |\Phi\rangle , \tag{3.9}$$

and refer this vacuum state by (II) (see Fig. 4).

The corresponding Wigner transform of the density matrix is

$$\rho(\mathbf{p}) = I\theta(\lambda^2-p^2) + \frac{1}{2}\left[I - \beta\frac{M}{E} - \gamma_5\frac{\sigma\cdot\mathbf{p}}{E} \right] \times [\theta(\Lambda^2-p^2) - \theta(\lambda^2-p^2)] . \tag{3.10}$$

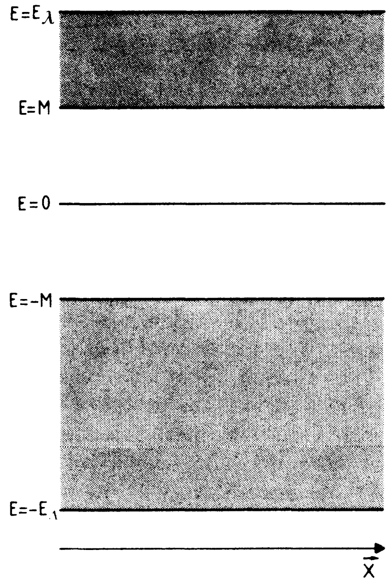

FIG. 4. Illustration of the vacuum state with positive-energy states on the top of the Dirac sea of Fig. 1.

Performing the calculation of the energy functional [Eq. (2.7)] and using the stability condition [Eq. (2.6)], as shown in Appendix A, we obtain the following self-consistent condition for the mass M:

$$1=4g\sum_{\mathbf{p}}\frac{1}{E}[\theta(\Lambda^2-p^2)-\theta(\lambda^2-p^2)] \,. \tag{3.11}$$

After performing the summation over $\mathbf{p}$ it takes the form

$$\frac{\pi^2}{g}=\Lambda^2\left[1+\frac{M^2}{\Lambda^2}\right]^{1/2}-M^2\ln\left[\frac{\Lambda}{M}+\left[1+\frac{\Lambda^2}{M^2}\right]^{1/2}\right]$$
$$-\lambda^2\left[1+\frac{M^2}{\lambda^2}\right]^{1/2}+M^2\ln\left[\frac{\lambda}{M}+\left[1+\frac{\lambda^2}{M^2}\right]^{1/2}\right] . \tag{3.12}$$

The nontrivial solution ($M\neq0$) exists only if

$$0<\frac{\pi^2}{g}<\Lambda^2-\lambda^2 \,. \tag{3.13}$$

When λ increases, for a given value of g and Λ, the value of M becomes again equal to zero. This occurs for

$$(\lambda_{\rm cr})^2=\Lambda^2-\frac{\pi^2}{g} \,, \tag{3.14}$$

which corresponds to a critical density.

This shows that it is no longer energetically advantageous to endow the quarks with a finite mass if there are enough positive-energy levels occupied.

This behavior is shown in Fig. 5, which illustrates the restoration of chiral symmetry for two values of the dimensionless factor $g\Lambda^2$.

We also plot for the same values of $g\Lambda^2$, the dimensionless ratio λ/Λ as a function of Λ/M in Fig. 6. This

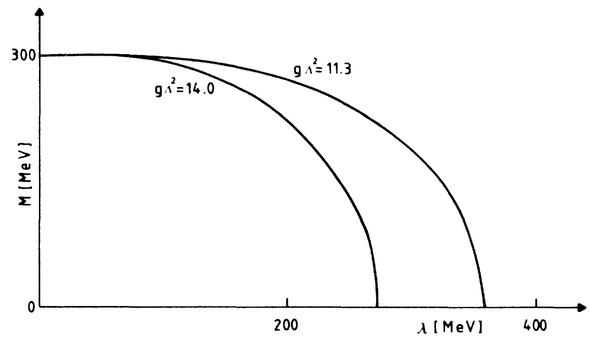

FIG. 5. Quark mass as a function of the parameter λ for different values of $g\Lambda^2$. The transition from the asymmetric chiral state ($M\neq0$) to a symmetric one ($M=0$) occurs for $\lambda_{\rm cr}=\Lambda^2-\pi^2/g$.

figure shows that λ/Λ must always be lower than 1 and attains its maximum value when $M\rightarrow0$.

Under these conditions the order parameter calculated self-consistently is again zero and the stable HF vacuum (the state of minimum energy) is an eigenstate of chirality. Chiral symmetry is then realized again in the Wigner-Weyl mode.

C. Stability of quark droplets

The basic assumption of the bag model[16] is that the quarks are confined to a finite-space region. This model is manifestly phenomenological and it is expected that the net effects of the QCD confinement mechanism are taken into account by the principles of minimum energy.[17]

It is believed that, in principle, there is no confinement mechanism in the NJL model. It is, however, our intention to explore this possibility. In Sec. III B we studied infinite quark matter defined by the vacuum state (II), which was stable against variations of the quark mass. The same is true for vacuum state (I) defined in Sec. III A. We are now interested in discussing the stability of a system with a given number of quarks confined to a variable space region. So, we consider a definite number

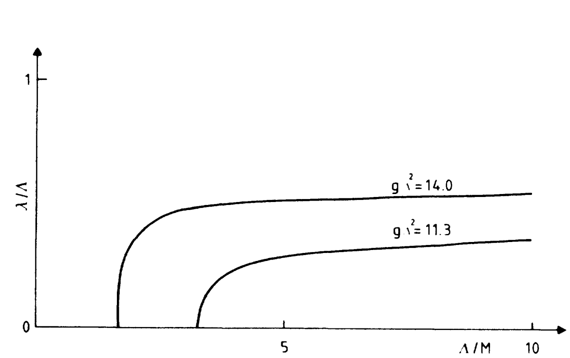

FIG. 6. Dimensionless ratio λ/Λ as a function of Λ/M for two different values of $g\Lambda^2$. For the range of values considered the mass decreases from 300 to 0 MeV. The curves are always below one since $\lambda^2\leq\Lambda^2-\pi^2/g$.

of particles with positive energy in a region Ω of volume $V_b \ll V$, where V is the total normalization volume. From this assumption, it follows that the vacuum (I) is polarized in such a way that the self-consistent mass $M(\mathbf{x})$ depends on $\mathbf{x}$ and is given by

$$M(\mathbf{x})=M \quad \text{if } \mathbf{x}\notin\Omega \;,$$
$$M(\mathbf{x})=M' \quad \text{if } \mathbf{x}\in\Omega \;. \tag{3.15}$$

This means that in a given region Ω of the space we have particles of mass M' and the rest of the space is filled with particles of mass M (in negative-energy states). This defines a vacuum state referred to as (III) which is illustrated in Fig. 7.

We will show that this version of the NJL model does not lead to the formation of a bag by minimization of the energy with respect to V_b.

The volume V_b is related to λ and the number of particles in positive-energy states through the equation

$$N_+ = \frac{\lambda^3}{3\pi^2} V_b \;. \tag{3.16}$$

The number of particles in negative-energy states is given by

$$N_- = \frac{\Lambda^3}{3\pi^2}(V-V_b) + \frac{\Lambda'^3}{3\pi^2} V_b \;, \tag{3.17}$$

where Λ and Λ' are related by

$$\Lambda^2 + M^2 = \Lambda'^2 + M'^2 \;. \tag{3.18}$$

The density matrix defined by Eq. (3.10) describes the vacuum state (II), characterized by the mass M of the Dirac particles, the cutoff momentum λ for positive-energy states and the cutoff momentum Λ for negative-energy states ($\lambda < \Lambda$). The energy of this state in given by (see Appendix A)

$$\mathcal{E}(\Lambda,\lambda,M) = -2\mathcal{T}(\Lambda,\lambda,M) - 4gM^2[\mathcal{V}(\Lambda,\lambda,M)]^2 + 2g[\Omega(\Lambda,\lambda)]^2 \;, \tag{3.19}$$

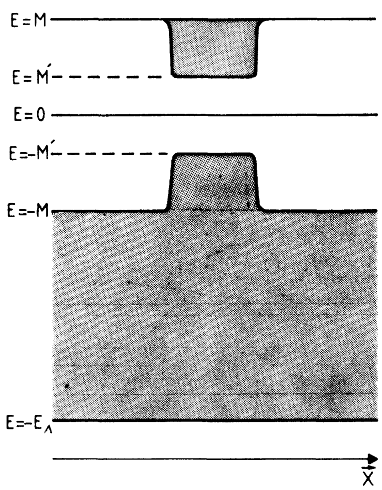

FIG. 7. Illustration of the Dirac sea structure with a finite number of quarks.

where $\mathcal{T}(\Lambda,\lambda,M)$, $\mathcal{V}(\Lambda,\lambda,M)$, and $\Omega(\Lambda,\lambda)$ are defined in Appendix A.

We observe that

$$\mathcal{E}^{(I)}(\Lambda,M) = \mathcal{E}(\Lambda,0,M) \;. \tag{3.20}$$

In this way the energy of the state defined by (III) (positive-energy states inside a region Ω of volume V_b) is given by

$$E = V\mathcal{E}(\Lambda,0,M) + V_b[\mathcal{E}(\Lambda',\lambda,M') - \mathcal{E}(\Lambda,0,M)] \;. \tag{3.21}$$

In the spirit of the Thomas-Fermi method we chose Λ' and M' such that

$$\Lambda'^2 + M'^2 = \Lambda^2 + M^2, \quad \lambda^2 + M'^2 = 0 + M^2 \;, \tag{3.22}$$

which implies

$$M' = (M^2 - \lambda^2)^{1/2}, \quad \Lambda' = (\Lambda^2 + \lambda^2)^{1/2} \;. \tag{3.23}$$

The energy of the quark droplet is negative, but its minimum occurs for $V_b = 3\pi^2 N_+ \lambda^{-3} = V$.

This shows that the simplified Hamiltonian (3.1) with two-body δ forces is not sufficient to produce stability. The possibility remains, however, that stability of the quark droplets will occur for an appropriate quantal treatment.

IV. MESON MASS SPECTRUM

Having characterized in Sec. III the state of stable equilibrium, which we identify with the HF ground state (Dirac sea of massive particles), we wish to investigate now the time evolution of a slightly disturbed state. The stability of the vacuum ensured by the HF prescription is intimately connected with the possible occurrence of stable (undamped) excitations of the chirally deformed vacuum. It is clear that the TDHF approach is equivalent to solving the Bethe-Salpeter equation for particle-hole excitations. For the moment we restrict ourselves to homogeneous fluctuations and will obtain the masses of the collective bosons.

We define the Hermitian and time-dependent generating function $s(\mathbf{p},t)$ (Wigner transform of the generator s):

$$s(\mathbf{p},t) = \gamma_5 L_1 - i\beta\gamma_5 L_2 + \gamma_5 \sigma\cdot\mathbf{V}_1 - i\beta\gamma_5\sigma\cdot\mathbf{V}_2 \;, \tag{4.1}$$

where $\mathbf{L}_i$ and $\mathbf{V}_i$ ($i=1,2$) are variational functions of $\mathbf{p}$ and t.

Expanding the Wigner transform of the density matrix up to first order in s we obtain

$$\rho(\mathbf{p},t)=\rho_0(\mathbf{p})+\left[\gamma_5\frac{M}{E}(L_2+\boldsymbol{\sigma}\cdot\mathbf{V}_2)-i\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E}\times\mathbf{V}_1-\beta\frac{1}{E}(\boldsymbol{\sigma}\cdot\mathbf{p}L_2+\mathbf{p}\cdot\mathbf{V}_2)+i\beta\gamma_5\frac{M}{E}(L_1+\boldsymbol{\sigma}\cdot\mathbf{V}_1)\right]\theta(\Lambda^2-p^2)\ , \tag{4.2}$$

which clearly satisfies the condition $\rho^2=\rho$ to first order in s. $\rho_0(\mathbf{p})$ is the equilibrium density given by Eq. (3.4).

After straightforward calculation one arrives at the quadratic Lagrangian

$$\begin{aligned}\mathcal{L}^{(2)}=\int d^3x\Bigg\{&\sum_{\mathbf{p}}\frac{\theta(\Lambda^2-p^2)}{E}\{2M(L_1\dot{L}_2-\dot{L}_1L_2+\mathbf{V}_1\cdot\dot{\mathbf{V}}_2-\dot{\mathbf{V}}_1\cdot\mathbf{V}_2)-4M^2(L_1{}^2+L_2{}^2+\mathbf{V}_1\cdot\mathbf{V}_1+\mathbf{V}_2\cdot\mathbf{V}_2)\\&+4[(\mathbf{p}\cdot\mathbf{V}_1)^2-(\mathbf{p}\cdot\mathbf{V}_2)^2]-4p^2(L_2{}^2+\mathbf{V}_1\cdot\mathbf{V}_1)\}\\&+16gM^2\left[\left[\sum_{\mathbf{p}}\frac{\theta(\Lambda^2-p^2)}{E}L_1\right]^2+\frac{1}{2}\left[\sum_{\mathbf{p}}\frac{\theta(\Lambda^2-p^2)}{E}L_2\right]^2\right]+16g\left[\sum_{\mathbf{p}}\frac{\theta(\Lambda^2-p^2)}{E}\mathbf{p}\cdot\mathbf{V}_2\right]^2\\&+8g\sum_{\mathbf{p}}\frac{\theta(\Lambda^2-p^2)}{E}\left[M^2\mathbf{V}_2\cdot\sum_{\mathbf{p}}\frac{\mathbf{V}_2}{E}\theta(\Lambda^2-p^2)-(\mathbf{p}\times\mathbf{V}_1)\cdot\sum_{\mathbf{p}}\frac{\mathbf{p}\times\mathbf{V}_1}{E}\theta(\Lambda^2-p^2)\right]\Bigg\}\ .\end{aligned} \tag{4.3}$$

Stationarity with respect to arbitrary variations of the functions L_i and $\mathbf{V}_i$ lead to the following equations of motion:

$$\begin{aligned}&\dot{\alpha}_2-2M\alpha_1+8g\frac{M}{E}\sum_{\mathbf{p}}\alpha_1\theta(\Lambda^2-p^2)=0\ ,\\&-M\dot{\alpha}_1-2E^2\alpha_2+4g\frac{M^2}{E}\sum_{\mathbf{p}}\alpha_2\theta(\Lambda^2-p^2)=0\ ,\end{aligned} \tag{4.4a}$$

where $\alpha_i=L_i/E$, and

$$\begin{aligned}&M\dot{\boldsymbol{\beta}}_2-2E^2\boldsymbol{\beta}_1+2\mathbf{p}(\mathbf{p}\cdot\boldsymbol{\beta}_1)+4g\frac{\mathbf{p}}{E}\times\sum_{\mathbf{p}}(\mathbf{p}\times\boldsymbol{\beta}_1)\theta(\Lambda^2-p^2)=0\ ,\\&-M\dot{\boldsymbol{\beta}}_1-2M^2\boldsymbol{\beta}_2-2\mathbf{p}(\mathbf{p}\cdot\boldsymbol{\beta}_2)+8g\frac{\mathbf{p}}{E}\sum_{\mathbf{p}}(\mathbf{p}\cdot\boldsymbol{\beta}_2)\theta(\Lambda^2-p^2)+4g\frac{M^2}{E}\sum_{\mathbf{p}}\boldsymbol{\beta}_2\theta(\Lambda^2-p^2)=0\ ,\end{aligned} \tag{4.4b}$$

where $\boldsymbol{\beta}_i=\mathbf{V}_i/E$.

Eigenmodes are obtained directly by solving the equations of motion (4.4), in addition to the equilibrium condition (3.6) and postulating a time dependence such that

$$\ddot{\alpha}_i=-\omega^2\alpha_i\ , \tag{4.5}$$

$$\ddot{\boldsymbol{\beta}}_i=-\omega^2\boldsymbol{\beta}_i\ . \tag{4.6}$$

(1) The equations of motion (4.4a) may be cast in the form

$$\ddot{\alpha}_1+4E^2\alpha_1-16gE\sum_{\mathbf{p}}\alpha_1\theta(\Lambda^2-p^2)=0\ , \tag{4.7}$$

which leads to the eigenvalue equation for ω:

$$\text{(A)}\quad\left[1-16g\sum_{\mathbf{p}}\frac{E}{4E^2-\omega^2}\theta(\Lambda^2-p^2)\right]\times\sum_{\mathbf{p}}\alpha_1\theta(\Lambda^2-p^2)=0\ .$$

Now, let us consider the system of Eqs. (4.4b). These equations may be decoupled into equations for a vector function, a scalar function, and an axial-vector function.

(2) In order to obtain the scalar solution we assume that the functions $\boldsymbol{\beta}_i$ have the structure $\boldsymbol{\beta}_i=\mathbf{p}f_i(p^2)$ so that $\sum_{\mathbf{p}}\boldsymbol{\beta}_i=0$.

Equations (4.4b) for the functions $\mathbf{p}\cdot\boldsymbol{\beta}_i$ reduce then to

$$\begin{aligned}&M\mathbf{p}\cdot\dot{\boldsymbol{\beta}}_2-2M^2\mathbf{p}\cdot\boldsymbol{\beta}_1=0\ ,\\&-M\mathbf{p}\cdot\dot{\boldsymbol{\beta}}_1-2E^2\mathbf{p}\cdot\boldsymbol{\beta}_2+8g\frac{p^2}{E}\sum_{\mathbf{p}}(\mathbf{p}\cdot\boldsymbol{\beta}_2)\theta(\Lambda^2-p^2)=0\ ,\end{aligned} \tag{4.8}$$

which may be cast in the form

$$\text{(B)}\quad\left[1-16g\sum_{\mathbf{p}}\frac{p^2}{E(4E^2-\omega^2)}\theta(\Lambda^2-p^2)\right]\times\sum_{\mathbf{p}}(\mathbf{p}\cdot\boldsymbol{\beta}_2)\theta(\Lambda^2-p^2)=0\ .$$

(3) Assuming that the functions $\boldsymbol{\beta}_i$ are of the general form $\boldsymbol{\beta}_i=(\mathbf{p}\times\mathbf{a})f_i(p^2)$, where $\mathbf{a}$ is a constant vector, we obtain, for the transverse components,

$$\begin{aligned}&M\mathbf{p}\times\dot{\boldsymbol{\beta}}_1+2M^2\mathbf{p}\times\boldsymbol{\beta}_2=0\ ,\\&-M\mathbf{p}\times\dot{\boldsymbol{\beta}}_2+2E^2\mathbf{p}\times\boldsymbol{\beta}_1\\&\qquad-4g\frac{\mathbf{p}}{E}\times\left[\mathbf{p}\times\sum_{\mathbf{p}}(\mathbf{p}\times\boldsymbol{\beta}_1)\theta(\Lambda^2-p^2)\right]=0\ ,\end{aligned} \tag{4.9}$$

which may be reduced to the form

$$\text{(C)}\quad\left[1+\tfrac{16}{3}g\sum_{\mathbf{p}}\frac{p^2}{E(4E^2-\omega^2)}\theta(\Lambda^2-p^2)\right]\times\sum_{\mathbf{p}}(\mathbf{p}\times\boldsymbol{\beta}_1)\theta(\Lambda^2-p^2)=0\ .$$

(4) Finally we may obtain the equation of motion for the vector function $\boldsymbol{\beta}_2$. By noticing that

$$\boldsymbol{\beta}_2=\hat{\mathbf{p}}(\hat{\mathbf{p}}\cdot\boldsymbol{\beta}_2)-\hat{\mathbf{p}}\times(\hat{\mathbf{p}}\times\boldsymbol{\beta}_2) , \tag{4.10}$$

we can write

$$\text{(D)}\quad \left[1-\tfrac{8}{3}g\sum_{\mathbf{p}}\frac{3M^2+2p^2}{E(4E^2-\omega^2)}\theta(\Lambda^2-p^2)\right]\times\sum_{\mathbf{p}}\boldsymbol{\beta}_2\theta(\Lambda^2-p^2)=0 .$$

In this case the general structure of the functions is given by $\beta_i=\mathbf{p}(\mathbf{p}\cdot\mathbf{a})f_i(p^2)+\mathbf{a}g_i(p^2)$, where $\mathbf{a}$ is a constant vector.

Summarizing, we have a system of four equations [(A), (B), (C), and (D)] from which one can obtain the eigenfrequencies of the stationary solutions.

Equation (A) has only one collective low-energy solution $\omega=0$. This solution is the pseudoscalar Goldstone boson.

Equation (B) has the collective solution $\omega=2M$ and zero binding energy. This frequency corresponds to a scalar boson.

Equation (C), which would correspond to a pseudovector has no low-energy collective solution. However this equation has solutions $\omega>2M$ which depend on the cutoff. A more complete discussion of these modes is left to Sec. V B.

Finally, equation (D) has a stable low-energy collective solution with $0<\omega<2M$ depending on the cutoff. This frequency corresponds to a vector meson.

The numerical values of the masses of the mesons with $0\leq\omega\leq 2M$ are displayed in Fig. 8. One can see that these results are in agreement with those predicted, at a qualitative level, by Nambu and Jona-Lasinio.[1]

All the equations [(A), (B), (C), and (D)] have strongly damped high-energy solutions. These solutions account for the set of infinitely many stationary high-energy solutions. The damping is not a sign of instability but is due to interference effects between the stationary solutions.

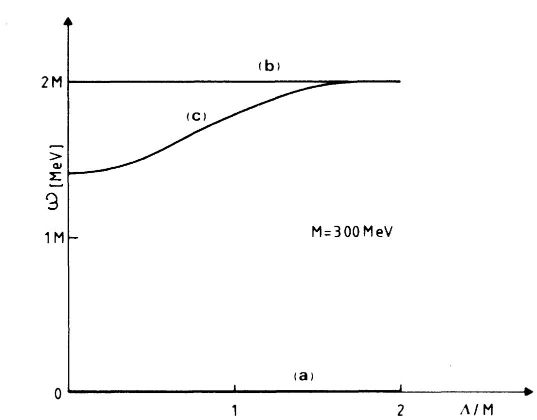

FIG. 8. Mass spectrum. (a) Pseudoscalar mode with $\omega=0$ which is independent of the parameters of the model. (b) Scalar mode with $\omega=2M$, also independent of the parameters of the model. (c) Vector mode which depends on the parameters of the model.

V. STATIC AND DYNAMICAL PROPERTIES OF THE PION

The zero-frequency mode found in the last section is a consequence of the violation of an underlying symmetry which in this case is the chiral symmetry. This mode is predicted by the Thouless-Valatin theory. It will be shown that this mode has also zero energy, in agreement with the Goldstone theorem. The pion dispersion relation and the pion decay constant are calculated in a semiclassical approximation. It is interesting to analyze the energy-weighted sum rules.

A. Pion mass and rotation energy

The collective mode identified with the pion has a frequency $\omega=0$. This mode is associated with the generating function

$$s(\mathbf{p},t)=\gamma_5 L_1-i\beta\gamma_5 L_2 , \tag{5.1}$$

where the variational functions satisfy the equations of motion (4.4a).

These fields have a linear time dependence as a consequence of the fact that, solutions of the type $\alpha=u\exp(i\omega t)+v\exp(-i\omega t)$ of the equation $\ddot{\alpha}+\omega^2\alpha=0$ of harmonic oscillations, are replaced for $\omega=0$, by solutions of the form $\alpha=m+nt$.

This fact and the equations of motion (4.4a) lead to the following form for the variational functions L_1 and L_2:

$$\begin{aligned} L_1(\mathbf{p},t)&=L_1(t)=A+Bt ,\\ L_2(\mathbf{p},t)&=L_2(\mathbf{p})=\frac{C}{E^2} , \end{aligned} \tag{5.2}$$

where A, B, and C are constants and $E^2=p^2+M^2$.

The classical energy (identified with a chiral rotation of the vacuum) is not given by zero but by $\mathcal{P}^2/2\mathcal{L}$ where $\mathcal{L}$ is the inertial parameter corresponding to the rotation in the chiral space, and $\mathcal{P}$ is the expectation value of the chiral operator in the rotational excited state of the vacuum.

The pion, identified with the rotational excited state $|\Phi_\pi\rangle$, is characterized by the density matrix

$$\rho(\mathbf{p},t)=\rho_0(\mathbf{p})+\left[\gamma_5\frac{M}{E}L_2-\beta\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E}L_2+i\beta\gamma_5\frac{M}{E}L_1\right]\theta(\Lambda^2-p^2) . \tag{5.3}$$

Using the Hamiltonian of the model, Eq. (3.1), we easily obtain

$$\mathcal{E}_\pi = \langle \Phi_\pi | H | \Phi_\pi \rangle = -4M^2 \int \frac{d^3p}{(2\pi)^3} \frac{L_1^2}{E} \theta(\Lambda^2 - p^2) - 4 \int \frac{d^3p}{(2\pi)^3} E L_2^2 \theta(\Lambda^2 - p^2)$$
$$+ 8gM^2 \left[2 \left[\int \frac{d^3p}{(2\pi)^3} \frac{L_1}{E} \theta(\Lambda^2 - p^2) \right]^2 + \left[\int \frac{d^3p}{(2\pi)^3} \frac{L_2}{E} \theta(\Lambda^2 - p^2) \right]^2 \right] . \quad (5.4)$$

Bearing in mind Eqs. (5.2) we obtain

$$\mathcal{E}_\pi = 16M^2C^2 \left[\int \frac{d^3p}{(2\pi)^3} \frac{\theta(\Lambda^2 - p^2)}{E^3} \right]^2 \times \left[\frac{g}{2} - \left[4M^2 \int \frac{d^3p}{(2\pi)^3} \frac{\theta(\Lambda^2 - p^2)}{E^3} \right]^{-1} \right] . \quad (5.5)$$

As we have

$$\langle \Phi_\pi | \gamma_5 | \Phi_\pi \rangle = 4M \int \frac{d^3p}{(2\pi)^3} \frac{L_2}{E} \theta(\Lambda^2 - p^2) = 4MC \int \frac{d^3p}{(2\pi)^3} \frac{\theta(\Lambda^2 - p^2)}{E^3} = \mathcal{P} , \quad (5.6)$$

we conclude that the inertial parameter depends on the parameters of the model and is given by

$$\mathcal{L} = \left[g - \frac{1}{2M^2 e} \right]^{-1} , \quad (5.7)$$

where

$$e = \int \frac{d^3p}{(2\pi)^3} \frac{\theta(\Lambda^2 - p^2)}{E^3} .$$

The computation of $\mathcal{L}$ shows that this parameter is always very large: $> 10^4$ MeV^{-1}. This fact is important since it is consistent with the Goldstone theorem which predicts a zero energy for this mode.

B. Dispersion law of the pion

We shall now be concerned with the derivation of the dispersion law for the pseudoscalar mode. This mode, interpreted as the chiral limit for the pion in the spirit of the Goldstone theorem, obeys a phononlike dispersion law $\omega(k) \sim_{k \to 0} Ck$.

The approach used is based on the classical limit of the quantal action principle which has been outlined in Sec. II.

When we have considered the frequency of the pionic mode in the limit of infinite wavelength the only variational functions that contributed were L_1 and L_2. However, in order to obtain dispersion relations, we must allow for a dependence on $\mathbf{x}$. In this way other functions may couple with L_1 and L_2 and contribute to the frequency of the pionic wave. As a matter of fact when we consider the classical limit of the Lagrangian (2.10) using a generator $s(\mathbf{x},\mathbf{p},t)$ with the structure of the generator (4.1) it can be seen that all the functions L_i and $\mathbf{V}_i$ are coupled to each other [see Eq. (B2) in Appendix B]. From the Lagrangian defined in Appendix B one may obtain dispersion relations for all modes. Since we are mainly interested in the pion dispersion relation and wish to avoid a lengthy and tedious calculation, we choose the following ansatz for the generator s:

$$s(\mathbf{x},\mathbf{p},t) = \chi(\mathbf{x},t) + \mathbf{p} \cdot \boldsymbol{\phi}(\mathbf{x},t) . \quad (5.8)$$

The analysis of the coupling between the different functions in Eq. (B2) of Appendix B shows that the simplest way of having linear terms in the expansion of s is through the ansatz

$$\mathbf{V}_i(\mathbf{x},\mathbf{p},t) = \mathbf{p} \times \mathbf{z}_i(\mathbf{x},t) \quad \text{with } \mathbf{v} \wedge \mathbf{z}_i = 0 . \quad (5.9)$$

Consequently, in the present work, we restrict the generator s to the form

$$s(\mathbf{x},\mathbf{p},t) = \gamma_5 L_1(\mathbf{x},t) - i\beta\gamma_5 L_2(\mathbf{x},t) + \gamma_5 \boldsymbol{\sigma} \cdot (\mathbf{p} \times \mathbf{z}_1(\mathbf{x},t)) - i\beta\gamma_5 \boldsymbol{\sigma} \cdot (\mathbf{p} \times \mathbf{z}_2(\mathbf{x},t)) . \quad (5.10)$$

Using this generator we find, for the classical limit of the Lagrangian (2.10),

$$L^{(2)} = \int d^3x \{ 2Ma(L_1\dot{L}_2 - \dot{L}_1 L_2) - 2(2b - M^2a)L_2^2 - 2g(c+d)^2 \nabla L_1 \cdot \nabla L_1 + \tfrac{4}{3} M a_1 (\mathbf{z}_1 \cdot \dot{\mathbf{z}}_2 - \dot{\mathbf{z}}_1 \cdot \mathbf{z}_2)$$
$$- \tfrac{8}{3}(b_1 + \tfrac{4}{3} g a_1^2) \mathbf{z}_1 \cdot \mathbf{z}_1 - \tfrac{8}{3} M^2 a_1 \mathbf{z}_2 \cdot \mathbf{z}_2 - \tfrac{8}{3}[1 + 2g(e+d)] a_1 \mathbf{z}_1 \cdot \nabla L_1 \} , \quad (5.11)$$

where we have already performed the summation over $\mathbf{p}$ since the variational functions do not depend on $\mathbf{p}$.

The coefficients in Eq. (5.11) depend on the parameters of the model and are defined in Appendix B.

The Lagrangian (5.11) leads to homogeneous equations of motion for the functions L_1 and $\mathbf{V}_1$ which now includes gradient terms:

$$Ma\dot{L}_2 + g(c+d)^2 \nabla^2 L_1 + \tfrac{2}{3}[1 + 2g(e+d)] a_1 \nabla \cdot \mathbf{z}_1 = 0 ,$$
$$Ma\dot{L}_1 + (2b - M^2 a) L_2 = 0 ,$$
$$Ma_1 \dot{\mathbf{z}}_2 - 2(b_1 + \tfrac{4}{3} g a_1^2) \mathbf{z}_1 - [1 + 2g(c+d)] a_1 \nabla L_1 = 0 , \quad (5.12)$$
$$\dot{\mathbf{z}}_1 + 2M \mathbf{z}_2 = 0 .$$

By postulating plane-wave solutions as well as a harmonic time dependence we easily obtain, for low values of $|\mathbf{k}|$, the frequencies

$$\omega_1 = f(\Lambda/M)k \ , \tag{5.13}$$

$$\omega_2 = [\Lambda^2 f_1(\Lambda/M) + f_2(\Lambda/M)k^2]^{1/2} \ , \tag{5.14}$$

where the functions f, f_1, and f_2 are referred to in Appendix B.

The first equation is the typical dispersion law for a Goldstone boson

$$\omega_1(k) \underset{k\to 0}{\sim} Ck \ . \tag{5.15}$$

Equation (5.14) represents the dispersion law of a pseudovector mode. This mode corresponds, for $\mathbf{k}=0$, to the lowest solution of equation (C) of Sec. IV and has a mass which depends on the cutoff and is larger than $2M$.

It is shown in Appendix B that the dispersion law of the pion, Eq. (5.13), is obtained even if we set $\mathbf{V}_2=0$. In this case the fluctuation of the density matrix can be written as

$$\delta\rho = \left[\gamma_5 \frac{M}{E} L_2 + i\beta\gamma_5 \frac{M}{E} L_1 - \beta\frac{\sigma\cdot\mathbf{p}}{E} L_2 + i\beta\gamma_5\frac{M}{E}\sigma\cdot\mathbf{V}_1 - \sigma\cdot\frac{\mathbf{p}\times\mathbf{V}_1}{E}\right]\theta(\Lambda^2-p^2) + \sigma\cdot\left[\left[\frac{\nabla L_1}{2E} - \frac{\mathbf{p}(\mathbf{p}\cdot\nabla L_1)}{2E^3}\right]\theta(\Lambda^2-p^2) + \frac{\mathbf{p}(\mathbf{p}\cdot\nabla L_1)}{2E}\delta(\Lambda^2-p^2)\right] \ , \tag{5.16}$$

which has a similar structure to Eq. (8.25) of Le Yaouanc *et al.*[6]

From Eq. (5.13) we obtain, for values of $\Lambda/M \leq 2$, a pion velocity $C \simeq 1$. However, for high values of Λ/M, the solution exhibits an unstable behavior, which is due to the coupling between L_1 and $\mathbf{Z}_1$. As a matter of fact, a real pion velocity is obtained for all values of Λ/M, as is shown in Fig. 9, provided we set $\mathbf{Z}_1=0$. For relevant values of M and $-\langle\psi\bar\psi\rangle^{1/3}$, the values of C lie around 2. This is shown in Fig. 10. From now on we will use the expression of $w(\mathbf{k})$ so obtained.

By choosing adequate generators, the formalism developed allows us, in principle, to obtain dispersion relations for all bosons. The results of this preliminary work indicate that special attention should be paid to the choice of the generators, which should not produce excitations in violation of the allowed single-particle states ($|\mathbf{p}| \leq \Lambda$). This requirement should ensure the existence of a positive-definite energy. Work along this direction is in progress.

C. Pion decay constant

For a quantal pionic state $|\pi\rangle$ of momentum $\mathbf{k}$ the pion decay constant f_π can be defined computing the time component of the axial-vector matrix element

$$\langle 0 | j_5 | \pi\rangle = f_\pi[\omega(k)]^{1/2} \ , \tag{5.17}$$

where

$$j_5 = \sum_{i=1}^{N} \gamma_5(i) e^{-i\mathbf{p}\cdot\mathbf{x}_i} \ . \tag{5.18}$$

For convenience we write the generator s as

$$s = \mathcal{N}(Ae^{i\omega t} + A^\dagger e^{-i\omega t}) \ , \tag{5.19}$$

where A has the structure of Eq. (5.10).

Defining an excited state $|\nu\rangle$ as $|\nu\rangle = Q_\nu^\dagger |\mathrm{RPA}\rangle$, where $|\mathrm{RPA}\rangle$ is the correlated ground state, such that

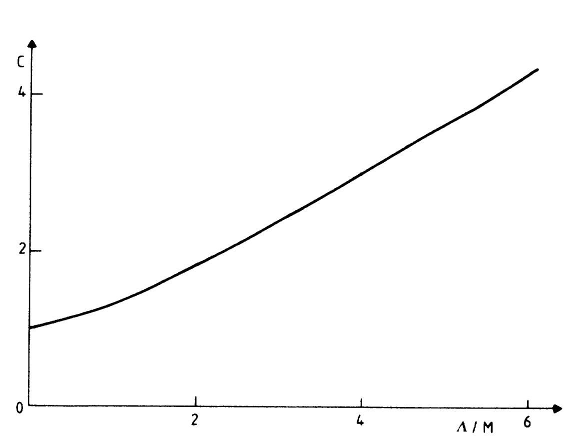

FIG. 9. Goldstone pion velocity as a function of Λ/M.

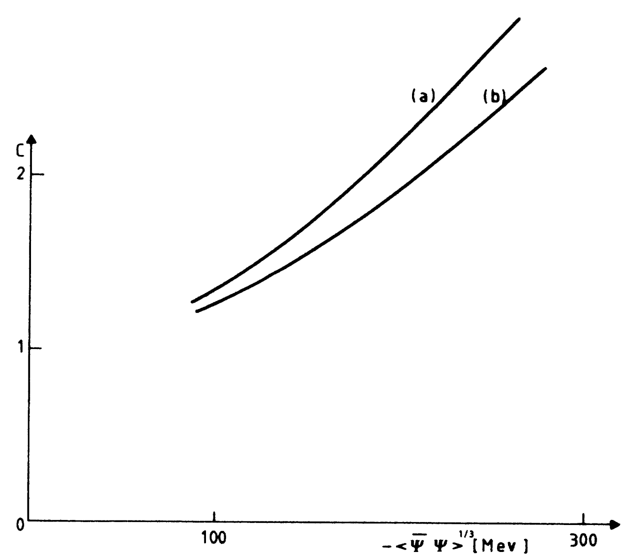

FIG. 10. Goldstone pion velocity as a function of $-\langle\bar\psi\psi\rangle^{1/3}$ for the quark masses (a) $M=300$ MeV and (b) $M=350$ MeV.

$Q_\nu \mid \text{RPA}\rangle = 0$, the normalization condition

$$\langle \nu \mid \nu' \rangle = \delta_{\nu\nu'} = \langle \text{RPA} \mid [Q_\nu, Q_{\nu'}^\dagger] \mid \text{RPA}\rangle \simeq \langle \Phi \mid [Q_\nu, Q_{\nu'}^\dagger] \mid \Phi \rangle \tag{5.20}$$

is easily obtained.

Here we have used the quasiboson approximation[18] assuming that the correlated ground state does not differ significantly from the HF ground state. This allows the calculation of all the relevant transition amplitudes in the HF approximation.

For the particular case of the pion state we have $\mid \pi \rangle = \mathcal{N} A^\dagger \mid \text{RPA}\rangle$. The normalization follows from the requirement $\mathcal{N}^2 \langle \Phi \mid [A, A^\dagger] \mid \Phi \rangle = 1$ and is easily calculated.

We find

$$\mathcal{N}^{-2} = 4iMa\,(l_1^* l_2 - l_1 l_2^*) \,, \tag{5.21}$$

where a as well as b, which is introduced below, depend on the model and are defined in Appendix B. l_i are the amplitudes of the waves, that is,

$$L_i(\mathbf{x}, t) = l_i(t) e^{i\mathbf{k}\cdot\mathbf{x}} \,.$$

Using the equations of motion (5.12) we obtain

$$\mathcal{N} = \left[\frac{\omega}{8(2b - M^2 a) l_2 l_2^*} \right]^{1/2} . \tag{5.22}$$

As

$$\langle 0 \mid j_5 \mid \pi \rangle \simeq \mathcal{N} \langle \Phi \mid [j_5, A^\dagger] \mid \Phi \rangle = -4iMa\mathcal{N} l_2^*$$

and using the expression of $\mathcal{N}$, Eq. (5.22), we obtain

$$\langle 0 \mid j_5 \mid \pi \rangle = \frac{\sqrt{2}Ma}{(2b - M^2 a)^{1/2}} [\omega(k)]^{1/2} \,. \tag{5.23}$$

We get therefore, from Eq. (5.17),

$$f_\pi = Mh(\Lambda/M) \tag{5.24}$$

where

$$h(\Lambda/M) = \frac{\sqrt{2}a}{(2b - M^2 a)^{1/2}} \,.$$

Equation (5.24) shows that a nonzero value of f_π is related to a nonzero value of the mass of the quarks and consequently with a spontaneous breaking of symmetry. The pion decay constant depends also on the cutoff parameter Λ. For the self-consistent values $M = 300$ MeV, $\Lambda/M = 2$, and $g\Lambda^2 = 13$ we obtain $f_\pi = 51$ MeV and $-\langle \bar{\psi}\psi \rangle^{1/3} = 161$ MeV. The set of values $M = 300$ MeV, $\Lambda/M = 4$, and $g\Lambda^2 = 11$ yields to $f_\pi = 61$ MeV and $-\langle \bar{\psi}\psi \rangle^{1/3} = 270$ MeV. In Fig. 11 we plot the dimensionless ratio f_π/M as a function of Λ/M. Figure 12 shows the behavior of the ratio $-\langle \bar{\psi}\psi \rangle^{1/3}/f_\pi$ as a function of Λ/M. These two figures agree with those of Ref. 1.

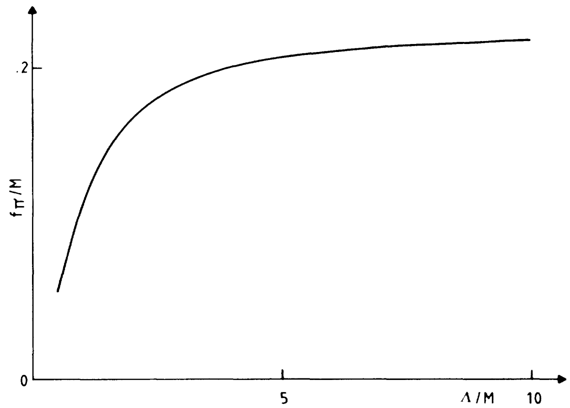

FIG. 11. Dimensionless ratio f_π/M as a function of Λ/M.

D. Sum rules

The sum rules are a very important tool in the theory of collective states and are useful in testing the reliability of a particular model.

The most important sum rule is the energy-weighted sum rule S_1. It can be written as

$$\sum_\nu (\omega_\nu - \omega_0) \mid \langle \nu \mid Q \mid 0 \rangle \mid^2 = \tfrac{1}{2} \langle \Phi \mid [Q, [H, Q]] \mid \Phi \rangle \,, \tag{5.25}$$

where F is a Hermitian single-particle operator.

This relation holds for a set of exact eigenstates $\mid \nu \rangle$ of H. As we usually have only approximate states $\mid \nu \rangle$ and energies ω_ν, it provides a test of the validity of any approximation through the degree of satisfaction of the sum rule.

Equation (5.25) is satisfied if the left-hand side is evaluated with the RPA states included and the right-hand side is calculated using the HF ground state.

Applying Eq. (5.25) when the excited state is a pion of momentum $\mathbf{k}$ and using the matrix element for the transition operator, Eq. (5.17) we can write

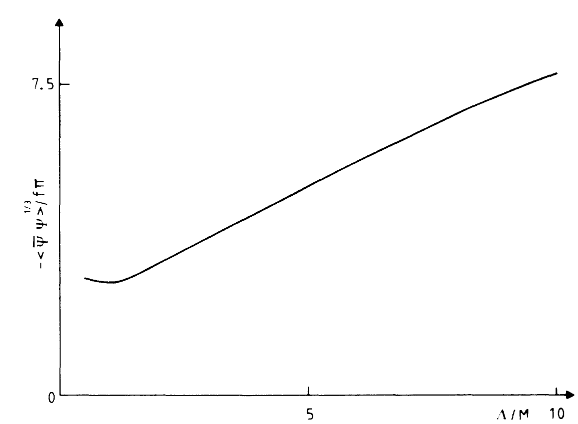

FIG. 12. Dimensionless ratio $-\langle \bar{\psi}\psi \rangle^{1/3}/f_\pi$ as a function of Λ/M.

$$\omega^2(k)f_\pi{}^2 \le \tfrac{1}{2}\langle \Phi \,|\, [j_5,[H,j_5]] \,|\, \Phi \rangle \ , \tag{5.26}$$

where the transition operator is defined by Eq. (5.18).

The second member of Eq. (5.26) is easily computed and we get, using the notation introduced in Appendix B,

$$\omega^2(k)f_\pi{}^2 = 2g(c+d)^2k^2 \ . \tag{5.27}$$

Taking into account the form of f_π given by Eq. (5.24) and comparing with the dispersion law of the pion [Eq. (B7)] we easily conclude that this collective mode exhausts 100% of the energy-weighted sum rule of the transition j_5. This is a good check for the validity of the model and confirms the collective character of this low-lying mode.

VI. CONCLUDING REMARKS

We have investigated, in the framework of the NJL model, the mechanism and implications of chiral phase transitions, as well as collective excitations of the Dirac sea and pion properties, through the TDHF formalism.

We used a self-consistent mean-field approach to treat the physical vacuum and describe the spontaneous breaking of chiral symmetry and its realization in the Nambu-Goldstone mode. By enlarging the phase space to positive-energy states chiral symmetry was shown to be restored and realized again in the Wigner-Weyl mode, the $q\bar{q}$ pairs being decoupled.

This behavior was explored in order to discuss the stability of droplets of quarks. Although the result was a negative one, we think that it might be a starting point to build a tentative model for a stable solution. Work along this direction is in progress.

Collective excitations of the chirally deformed vacuum were described. Besides the pseudoscalar Goldstone boson we obtain a scalar boson, with mass $\omega=2M$ and a vector boson with a mass slightly displaced from the unperturbed masses. It is also possible to obtain other solutions from the RPA-type eigenvalue equations (A), (B), (C), and (D), which have masses $\omega>2M$ and describe Landau damped high-energy modes.

The Goldstone boson, interpreted as the chiral limit for the pion, was the object of detailed study; namely, its energy in zero momentum frame, dispersion relation, and decay constant were investigated. We shall emphasize that the TDHF formalism in a semiclassical approximation provides an adequate tool to evaluate dispersion relations for the collective modes, allowing a unified treatment for the dispersion relations of all bosons. Here we restrict ourselves to the dispersion relation for the pion in the long-wavelength limit.

We notice that our treatment is semiclassical. The interpretation of the excitation modes is also semiclassical. In a full quantum theory the wave modes become particles.

ACKNOWLEDGMENTS

We wish to thank M. Rosina, D. Schutte, J. N. Urbano, V. Bernard, and U.-G. Meissner for helpful discussions. This work was supported by Instituto Nacional de Investigação Científica and by Junta Nacional de Investigação Cientifica e Tecnológica.

APPENDIX A: VACUUM STATE AND VACUUM ENERGY IN THE THOMAS-FERMI APPROXIMATION

We treat here, for completeness, the structure of the HF stationary state of the chirally symmetric and asymmetric phases. We calculate also the expectation value of the Hamiltonian in the asymmetric phase and perform the restoration of chiral symmetry in the average through the Nambu-Goldstone mode.

The density matrices in the HF states for massless and massive particles are, respectively,

$$\begin{aligned}\rho^{(0)}_{\mu\mu'}(\mathbf{x},\mathbf{x}') &= \langle \Phi_0 \,|\, \psi^{\dagger(0)}_{\mu'}(\mathbf{x}')\psi^{(0)}_{\mu}(\mathbf{x}) \,|\, \Phi_0 \rangle \ ,\\ \rho_{\mu\mu'}(\mathbf{x},\mathbf{x}') &= \langle \Phi \,|\, \psi^{\dagger}_{u'}(\mathbf{x}')\psi_{\mu}(\mathbf{x}) \,|\, \Phi \rangle \ ,\end{aligned} \tag{A1}$$

where

$$|\Phi_0\rangle = \prod_i d_i^{(0)\dagger}\,|\,0\rangle, \quad |\Phi\rangle = \prod_i d_i^{\dagger}\,|\,0\rangle \ , \tag{A2}$$

and $\psi^{(0)}(\mathbf{x})$ and $\psi(\mathbf{x})$ are field solutions of the Dirac equation for massless and massive particles, respectively.

By imposing the condition that $M=0$ for $x_0=0$ (see Ref. 1) one can show that the operators b_i ($b_i^\dagger$) and d_i ($d_i^\dagger$) are related to $b_i^{(0)}$ ($b_i^{(0)\dagger}$) and $d_i^{(0)}$ ($d_i^{(0)\dagger}$) by means of the canonical transformation

$$\begin{aligned} b_{\mathbf{p},s} &= u(|\mathbf{p}|)b^{(0)}_{\mathbf{p},s} + v(|\mathbf{p}|)d^{(0)}_{\mathbf{p},s} \ ,\\ d_{\mathbf{p},s} &= u(|\mathbf{p}|)d^{(0)}_{\mathbf{p},s} - v(|\mathbf{p}|)b^{(0)}_{\mathbf{p},s} \ ,\\ b^{\dagger}_{\mathbf{p},s} &= u(|\mathbf{p}|)b^{(0)}_{\mathbf{p},s} + v(|\mathbf{p}|)d^{(0)\dagger}_{\mathbf{p},s} \ ,\\ d^{\dagger}_{\mathbf{p},s} &= u(|\mathbf{p}|)d^{(0)\dagger}_{\mathbf{p},s} - v(|\mathbf{p}|)b^{(0)\dagger}_{\mathbf{p},s} \ ,\end{aligned} \tag{A3}$$

where

$$u(|\mathbf{p}|) = \left[\frac{1}{2}\left[1+\frac{|\mathbf{p}|}{E+M}\right]\right]^{1/2}$$

and

$$v(|\mathbf{p}|) = \left[\frac{1}{2}\left[1-\frac{|\mathbf{p}|}{E+M}\right]\right]^{1/2} .$$

The new HF state written as

$$|\Phi\rangle = \prod_{\mathbf{p},s}[u(|\mathbf{p}|)d^{(0)\dagger}_{\mathbf{p},s} - v(|\mathbf{p}|)b^{(0)\dagger}_{\mathbf{p},s}]\,|\,0\rangle \ ,$$

may be cast in the form

$$|\Phi\rangle = \prod_{\mathbf{p},s} u(|\mathbf{p}|)\left[1-\frac{v(|\mathbf{p}|)}{u(|\mathbf{p}|)}b^{(0)\dagger}_{\mathbf{p},s}d^{(0)}_{\mathbf{p},s}\right]|\,\Phi_0\rangle$$

or

$$|\Phi\rangle \propto \exp\left[\sum_{\mathbf{p},s}\gamma(|\mathbf{p}|)b^{(0)\dagger}_{\mathbf{p},s}d^{(0)}_{\mathbf{p},s}\right]|\,\Phi_0\rangle \ , \tag{A4}$$

which is a coherent state of $q\bar{q}$ pairs of massless quarks.

In order to calculate the expectation value of the Hamiltonian in the HF vacuum $|\Phi\rangle$, referred to as (I), we use Eq. (2.7) that can be written in the form

$$\mathcal{E}(\rho)=\int d^6\zeta\,\mathrm{tr}(\gamma_5\sigma\cdot\mathbf{p}\rho)+\tfrac{1}{2}\int d^6\zeta_1 d^6\zeta_2\,\mathrm{tr}_1\mathrm{tr}_2 v^A(12)\rho(1)\rho(2)\ , \qquad (A5)$$

where

$$d^6\zeta=\frac{d^3p}{(2\pi)^3}d^3x\ ,$$

$$v(12)=-2g\delta(\mathbf{x}_1-\mathbf{x}_2)[\beta(1)\beta(2)-\beta(1)\gamma_5(1)\beta(2)\gamma_5(2)]\ , \qquad (A6)$$

and the Wigner transform of the density matrix is given by Eq. (3.4).

As we have

$$\mathrm{tr}(\gamma_5\sigma\cdot\mathbf{p}\rho)=-2\frac{p^2}{E}\theta(\Lambda^2-p^2)\ ,$$

the kinetic energy is

$$T=-2\mathcal{T}(\Lambda,M)V\ . \qquad (A7)$$

Performing the summation over $\mathbf{p}$ we have

$$\mathcal{T}(\Lambda,M)=\frac{1}{8\pi^2}\Lambda^4[(1+u^2)^{1/2}(1-\tfrac{3}{2}u^2)+\tfrac{3}{2}u^4\ln f(u)]\ , \qquad (A8)$$

with

$$f(u)=\frac{1}{u}+\left[1+\frac{1}{u^2}\right]^{1/2},\quad u=M/\Lambda\ .$$

The potential energy is also easily calculated.

(i) For the direct term we have

$$\mathrm{tr}(\beta\rho)=-\frac{2M}{E}\theta(\Lambda^2-p^2)\ ,$$

$$\mathrm{tr}(\beta\gamma_5\rho)=0\ ,$$

so

$$V_{\mathrm{dir}}=-4gM^2[\mathcal{V}(\Lambda,M)]^2V\ , \qquad (A9)$$

where

$$\mathcal{V}(\Lambda,M)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{E}\theta(\Lambda^2-p^2)\ . \qquad (A10)$$

That is,

$$\mathcal{V}(\Lambda,M)=\frac{1}{4\pi^2}\Lambda^2[(1+u^2)^{1/2}-u^2\ln f(u)]\ . \qquad (A11)$$

(ii) For the exchange term we have

$$\mathrm{tr}[\beta\rho(1)\beta\rho(2)]=\left[1+\frac{M^2}{E(1)E(2)}+\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{E(1)E(2)}\right]\theta(\Lambda^2-p^2)\ ,$$

$$\int\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\mathrm{tr}[\beta\rho(1)\beta\rho(2)]=[\Omega(\Lambda)]^2+M^2[\mathcal{V}(\Lambda,M)]^2\ ,$$

where

$$\Omega(\Lambda)=\int\frac{d^3p}{(2\pi)^3}\theta(\Lambda^2-p^2)=\frac{\Lambda^3}{6\pi^2}\ .$$

In the same way

$$\int\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\mathrm{tr}[\beta\gamma_5\rho(1)\beta\gamma_5\rho(2)]=-[\Omega(\Lambda)]^2+M^2[\mathcal{V}(\Lambda,M)]^2$$

and

$$V_{\mathrm{ex}}=2g[\Omega(\Lambda)]^2V\ . \qquad (A12)$$

Considering a unitary volume we have

$$\mathcal{E}(\rho)=\mathcal{E}(\Lambda,M)=-2\mathcal{T}(\Lambda,M)-4gM^2[\mathcal{V}(\Lambda,M)]^2+2g[\Omega(\Lambda)]^2\ . \qquad (A13)$$

This energy is referred as $\mathcal{E}^{(\mathrm{I})}(\Lambda,M)$ in Sec. III C.

The minimization of the energy with respect to M provides a stability condition that can be put in the form

$$1=4g\sum_{\mathbf{p}}\frac{1}{E}\theta(\Lambda^2-p^2)\ . \qquad (A14)$$

It is easy to prove that the vacuum energy $\mathcal{E}(\rho)$, Eq. (A13), is invariant under the transformation generated by $\exp(i\gamma_5\theta)$.

We have

$$\langle\Phi|H|\Phi\rangle=\langle\Phi(\theta)|H|\Phi(\theta)\rangle\ ,$$

where

$$|\Phi(\theta)\rangle=\exp\left[i\theta\sum_{j=1}^{N}\gamma_5(j)\right]|\Phi\rangle\ .$$

We easily find

$$\mathcal{E}(\rho(\theta))=\mathcal{E}(\rho)\ ,$$

where

$$\rho(\theta)=e^{i\theta\gamma_5}\rho e^{-i\theta\gamma_5}\ .$$

In fact we have successively

$$\int\frac{d^3p}{(2\pi)^3}\mathrm{tr}(\gamma_5\sigma\cdot\mathbf{p}e^{i\theta\gamma_5}\rho e^{-i\theta\gamma_5})=-2\mathcal{T}(\Lambda,M)\ ,$$

$$\int\frac{d^3p}{(2\pi)^3}\mathrm{tr}(\beta e^{i\theta\gamma_5}\rho e^{-i\theta\gamma_5})=-2M\mathcal{V}(\Lambda,M)(\cos^2\theta-\sin^2\theta)\ , \qquad (A15)$$

$$\int\frac{d^3p}{(2\pi)^3}\mathrm{tr}(\beta\gamma_5 e^{i\theta\gamma_5}\rho e^{-i\theta\gamma_5})=-4iM\sum_{\mathbf{p}}\frac{1}{E}\theta(\Lambda^2-p^2)\sin\theta\cos\theta\ ,$$

and

$$V_{\mathrm{dir}}=-4gM^2[\mathcal{V}(\Lambda,M)]^2\ ,$$

which in addition with the exchange contributions allow us to write

$$\mathcal{E}(\rho(\theta))=-2\mathcal{T}(\Lambda,M)-4gM^2[\mathcal{V}(\Lambda,M)]^2+2g\,[\Omega(\Lambda)]^2\ , \tag{A16}$$

thus showing that the vacuum energy is invariant under a chiral transformation.

Finally, we can use the same procedure to calculate the expectation value of the Hamiltonian in the HF vacuum referred to as (II).

In this case we have

$$\mathcal{E}(\rho)=\mathcal{E}(\Lambda,\lambda,M)=-2\mathcal{T}(\Lambda,\lambda,M)-4gM^2[\mathcal{V}(\Lambda,\lambda,M)]^2+2g\,[\Omega(\Lambda,\lambda)]^2\ , \tag{A17}$$

where

$$\begin{aligned}\mathcal{T}(\Lambda,\lambda,M)&=\mathcal{T}(\Lambda,M)-\mathcal{T}(\lambda,M)\ ,\\ \mathcal{V}(\Lambda,\lambda,M)&=\mathcal{V}(\Lambda,M)-\mathcal{V}(\lambda,M)\ ,\\ \Omega(\Lambda,\lambda)&=\Omega(\Lambda)+\Omega(\lambda)\ .\end{aligned} \tag{A18}$$

The minimization of Eq. (A17) with respect to M allows us to obtain the stability condition (3.12).

APPENDIX B: DISPERSION LAWS

Using a generating function $s(\mathbf{x},\mathbf{p},t)$ of the type

$$s(\mathbf{x},\mathbf{p},t)=\gamma_5L_1(\mathbf{x},\mathbf{p},t)-i\beta\gamma_5L_2(\mathbf{x},\mathbf{p},t)+\gamma_5\boldsymbol{\sigma}\cdot\mathbf{V}_1(\mathbf{x},\mathbf{p},t)-i\beta\gamma_5\boldsymbol{\sigma}\cdot\mathbf{V}_2(\mathbf{x},\mathbf{p},t)\ , \tag{B1}$$

we find, for the classical limit of the Lagrangian (2.10),

$$\begin{aligned}L'^{(2)}=L^{(2)}+\int d^3x\Bigg[&2g\sum_{\mathbf{p}}\frac{\partial}{\partial p_j}\theta(\Lambda^2-p^2)\frac{\partial}{\partial x_j}V_{1i}\sum_{\mathbf{p}}\frac{\partial}{\partial p_k}\theta(\Lambda^2-p^2)\frac{\partial}{\partial x_k}V_{1i}-8g\left[\sum_{\mathbf{p}}F(|\mathbf{p}|)\nabla L_1\right]^2\\ &-8g\left[\sum_{\mathbf{p}}F(|\mathbf{p}|)\nabla\mathbf{V}_1\right]^2-8g\sum_{\mathbf{p}}\frac{\partial}{\partial p_j}\theta(\Lambda^2-p^2)\frac{\partial}{\partial x_j}V_{2i}\sum_{\mathbf{p}}\frac{\partial}{\partial x_k}\theta(\Lambda^2-p^2)\frac{\partial}{\partial x_k}V_{2i}\\ &-8g\sum_{\mathbf{p}}\frac{M}{E}\theta(\Lambda^2-p^2)V_{2k}\sum_{\mathbf{p}}\frac{\partial}{\partial p_j}\theta(\Lambda^2-p^2)\frac{\partial}{\partial x_j}V_{1k}+16g\sum_{\mathbf{p}}\frac{\mathbf{p}\times\mathbf{V}_1}{E}\theta(\Lambda^2-p^2)\sum_{\mathbf{p}}F(|\mathbf{p}|)\nabla L_1\Bigg]\ ,\end{aligned} \tag{B2}$$

where $L^{(2)}$ is given by Eq. (4.3) and

$$F(|\mathbf{p}|)=\frac{1}{2E}\left[\left[1-\frac{p^2}{3E^2}\right]\theta(\Lambda^2-p^2)+\frac{p}{3}\delta(\Lambda^2-p^2)\right]\ .$$

From this general Lagrangian one can derive homogeneous equations for all the mesons. Equation (B2) allows also the analysis of the coupling between the fields and the structure of the successive terms of the expansion of the generating function s. In this way we easily state that the ansatz (5.9) is the simplest hypothesis leading to linear terms in $\mathbf{p}$ in the expansion of s.

If we now introduce the ansatz (5.9) in the Lagrangian (B2) we obtain Eq. (5.11) where the coefficients are defined by

$$\begin{aligned}a&=\sum_{\mathbf{p}}\frac{1}{E}\theta(\Lambda^2-p^2)\\ &=\frac{\Lambda^2}{4\pi^2}[(1+u^2)^{1/2}-u^2\ln f(u)]\ ,\\ b&=\sum_{\mathbf{p}}E\,\theta(\Lambda^2-p^2)\\ &=\frac{\Lambda^4}{8\pi^2}\left[(1+u^2)^{1/2}\left[1+\frac{u^2}{2}\right]-\frac{u^4}{2}\ln f(u)\right]\ ,\\ c&=\sum_{\mathbf{p}}\frac{1}{E}\left[1-\frac{p^2}{3E^2}\right]\theta(\Lambda^2-p^2)\\ &=\frac{\Lambda^2}{6\pi^2}(1+u^2)^{-1/2}\ ,\\ d&=\tfrac{1}{3}\sum_{\mathbf{p}}\frac{p}{E}\delta(\Lambda^2-p^2)=c\ ,\\ a_1&=\sum_{\mathbf{p}}\frac{p^2}{E}\theta(\Lambda^2-p^2)\\ &=\frac{\Lambda^4}{8\pi^2}[(1+u^2)^{1/2}(1-\tfrac{3}{2}u^2)+\tfrac{3}{2}u^4\ln f(u)]\ ,\\ b_1&=\sum_{\mathbf{p}}p^2E\,\theta(\Lambda^2-p^2)\\ &=\frac{\Lambda^6}{4\pi^2}\left[(1+u^2)^{1/2}\left[\frac{1}{3}+\frac{u^2}{12}-\frac{u^4}{8}\right]+\frac{u^6}{8}\ln f(u)\right]\ ,\end{aligned} \tag{B3}$$

where $f(u)$ and u are defined in Appendix A.

Solving the equation of motion we obtain, for low values of $|\mathbf{k}|$, the frequencies

$$\omega_1 = k\left[\frac{g(c+d)^2}{M^2a^2}(2b-M^2a)\right.$$
$$\left.\times\left[1-a_1{}^2\frac{[1+2g(c+d)]^2}{g(c+d)(3b_1+4ga_1{}^2)}\right]\right], \quad \text{(B4)}$$

$$\omega_2 = \left[\frac{4}{a_1}(b_1+\tfrac{4}{3}ga_1{}^2)\right.$$
$$\left.+\tfrac{1}{3}[1+2g(c+d)]^2\frac{2b-M^2a}{M^2a^2}\frac{a_1{}^2}{b_1+\tfrac{4}{3}ga_1{}^2}k^2\right]^{1/2}, \quad \text{(B5)}$$

that we have written in the text in a condensed form.

If we set $\mathbf{V}_2=0$, Eq. (5.12) becomes replaced by the following equations of motion:

$$Ma\dot{L}_2+g(c+d)^2\nabla^2L_1+\tfrac{2}{3}[1+2g(c+d)]a_1\nabla\cdot\mathbf{z}_1=0\ ,$$
$$Ma\dot{L}_1+(2b-M^2a)L_2=0\ , \quad \text{(B6)}$$
$$(b_1+\tfrac{4}{3}a_1{}^2)\mathbf{z}_1+[1+2g(c+d)]a_1\nabla L_1=0\ .$$

This assumption leads to a dispersion law for the pion of the same form as that given by Eq. (B4).

If we also set $\mathbf{V}_1=0$, Eq. (B4) reduces to

$$\omega_1 = k\left[\frac{g(c+d)^2}{M^2a^2}(2b-M^2a)\right]^{1/2}. \quad \text{(B7)}$$

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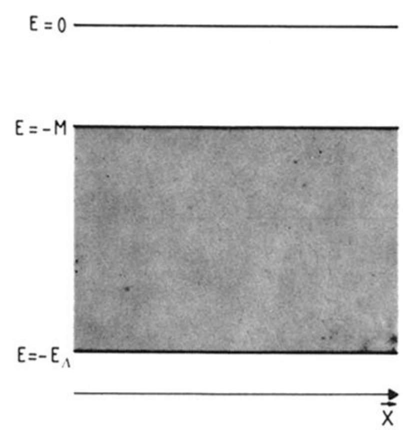

FIG. 1. Illustration of the Dirac sea with the upper N negative-energy states occupied.

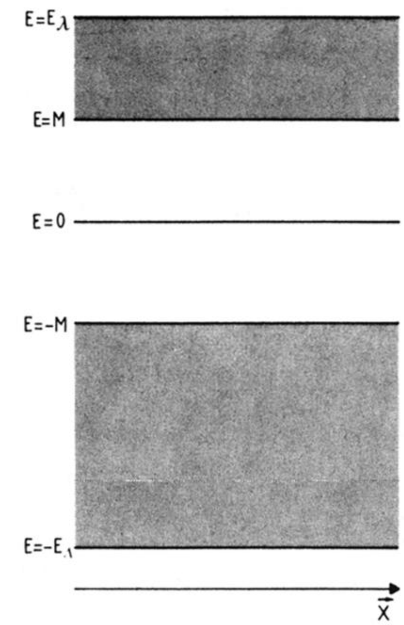

FIG. 4. Illustration of the vacuum state with positive-energy states on the top of the Dirac sea of Fig. 1.

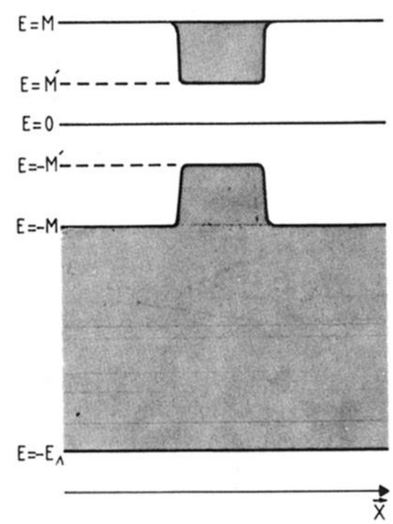

FIG. 7. Illustration of the Dirac sea structure with a finite number of quarks.