Some global charges in classical Yang-Mills theory

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Three classes of boundary conditions allowing the definition of a global field strength ("global color") are presented. A definition of global color of the sources and of the Yang-Mills field is proposed. Some exact solutions of Yang-Mills equations with point sources and with "topologically nontrivial electric color" are presented.

I. INTRODUCTION

In classical Yang-Mills theory the concept of the total strength of a field configuration seems to be generically ill defined: the Lie-algebra-valued integral[1]

$$Q=\left[\int_{S_\infty} {}^*F\right]\Big/4\pi \tag{1.1}$$

does not intrinsically characterize a field configuration, as it is gauge dependent.[2] The proposed solutions to this problem seem either to be specific to some peculiar properties of a field configuration[3] or to require the introduction of some arbitrary elements,[4] both,[5] or, finally, to assume the existence of Lie-algebra-valued Higgs fields[6] in the model under consideration. When trying to make sense of (1.1), one has to define $\int_{S_\infty}$ —if this symbol is understood as the limit $R\to\infty$ of integrals over a family of surfaces $S(R)$, the requirement of convergence imposes restrictions on the asymptotic behavior of F and on the gauge in which the calculation is performed. If we demand

$$F=O(r^{-2}) \tag{1.2}$$

and choose a gauge in which the limit exists, it will in general cease to exist if we perform a gauge transformation $F\to g^{-1}Fg$, with $g=\exp[f(r)\Sigma]$, where Σ is a constant element of the Lie algebra. The purpose of this paper is to present some conditions under which (1.1) acquires unambiguous meaning.

In Sec. II we recall the Hamiltonian meaning of (1.1). In Secs. III and IV we present three classes of boundary conditions which allow us to define (1.1). In each case we derive the "asymptotic gauge group," i.e., the group of gauge transformations which preserve our conditions. It turns out that these conditions do not all single out a family of gauge transformations which are asymptotically constant (point independent)—this makes (1.1) still ill defined, although finite and well convergent in every admissible gauge.[7]

In Sec. V we propose how to define the global strength of sources, and how to extract gauge-invariant information from (1.1). The global magnetic color is shortly discussed in Sec. VI. In Sec. VII the problem of time evolution and of Lorentz invariance of the charges is briefly discussed. Finally some solutions with "topologically nontrivial electric color" are presented in Sec. VIII. In this paper we shall consider spatial infinity only (i.e., the limit r tending to infinity for fixed t), some of the constructions presented here can be carried over to null infinity by means of the asymptotic expansions of Tafel and Trautman.[8] The symmetry group is assumed to be compact and semisimple throughout, although in many places these assumptions can be weakened. Unless stated otherwise, we consider the fields on a fixed Cauchy surface only. The topological analysis presented here is standard; it seems to us, however, that the context in which it is presented is new.

II. THE CHARGE AS THE MOMENTUM MAPPING

The charges we shall consider can be defined in a standard way by "Noether currents"; in this section we shall recall a slightly more sophisticated symplectic derivation of these quantities (the reader may skip this section without damage in the understanding of the remaining ones). Consider the infinitesimal phase space $P_i=\{(A_\mu, A_{\mu,\nu}, P^{\mu\nu}, P^{\mu\nu}{}_{,\nu})\}$ for the Yang-Mills theory with the canonical symplectic form[9]

$$\omega=(k_{ab}dP^{a\mu\nu}\wedge dA^b_\mu)_{,\nu}\ , \tag{2.1}$$

where k is the Killing metric. G acts on P_i via a family of diffeomorphisms ϕ_g:

$$\phi_g(A_\mu,P^{\mu\nu})=(g^{-1}A_\mu g+g^{-1}g_{,\mu},g^{-1}P^{\mu\nu}g)\ .$$

The action is symplectic so the one-forms $\phi_g' \lrcorner\, \omega$ are closed which allows us to define (at least locally) the momentum mapping j:

$$0=\mathcal{L}_{\phi_e'}\omega=d(\phi_e'\lrcorner\,\omega)\Longrightarrow\phi_e'\lrcorner\,\omega=-d[j(X)]\ , \tag{2.2}$$

where $X=g'(e)$. From the definition of the action we have

$$\phi_e'=D_\mu X^a\partial/\partial A^a_\mu-([X,P^{\mu\nu}])_a\partial/\partial P^{\mu\nu}_a+\cdots\ ,$$

so that

$$\phi_e'\lrcorner\,\omega=-d[k(D_\mu X,P^{\mu\nu})]_{,\nu}\ .$$

$j(X)$ is therefore defined globally and we have

$$j(X)=k(D_\mu X,P^{\mu\nu})_{,\nu}$$

which, after going to integrated structures, leads to the charges[10]

$$Q(X)=\int_{\Sigma} k(D_\mu X,P^{\mu\nu})dV_\nu \ . \tag{2.3}$$

In standard Yang-Mills theory $P^{\mu\nu}=F^{\mu\nu}/4\pi$ on dynamics and so, if vacuum Yang-Mills constraint equations are satisfied and X is asymptotically constant,

$$Q(X)=k\left[X,\int_{S_\infty} {}^*F\right]\Big/4\pi \ . \tag{2.4}$$

If sources are present, a supplementary current term appears in (2.4).

III. QUASI-ABELIAN BOUNDARY CONDITIONS

In this section we shall present our first set of boundary conditions for which (1.1) make sense. This is a subtle problem because the imposition of too-strong boundary conditions may make the whole discussion trivial; moreover, certain boundary conditions may simply be incompatible with the field equations. Suppose that for some reasons we only wish to consider fields satisfying

$$F_{ij}=O_k(r^{-2-\beta}),\quad \beta\geq 0,\ k\geq 0 \ . \tag{3.1}$$

If $\beta>0$, proposition 4.1 of the next section shows that (3.1) implies the existence of a gauge in which

$$A_i=O_{k+1}(r^{-1-\beta}) \ . \tag{3.2}$$

If $\beta=0$, (3.1) implies the existence of gauges, eventually with string singularities, for which (3.2) holds in appropriate domains (cf. the remark following proposition 4.1). If $\pi_1(G)\neq\{e\}$, the standard U(1) magnetic monopole shows that strings cannot be generically removed, while if $\pi_1(G)=\{e\}$, global gauges in $\mathbb{R}^3\setminus B(R)$ always exist but we have not been able to show that the two potentials with strings on, say, the positive z axis and the negative z axis, respectively, can be patched together to a gauge satisfying (3.2). Nevertheless, if $\beta=0$, the above discussion suggests that it is natural to consider potentials satisfying (3.2)[11]. This being the case, a singled-out class of gauge transformations preserving (3.2) appears and it is simple to show that this class consists of transformations satisfying

$$g_{,i}=O_{k+1}(r^{-1-\beta}) \ . \tag{3.3}$$

In particular, if $\beta=0$, (3.3) allows for transformations of the form

$$g(x^i)=g(x^i/|x|) \ ,$$

but there is also the analogue of what is called the "logarithmic ambiguity" in the general-relativistic case:

$$g=\exp(X\ln r) \ ,$$

where X is any fixed element of the Lie algebra. If S_∞ is understood as the limit of a family of coordinate spheres, the simplest way of ensuring convergence of the integral (1.1) is to assume

$$F={}^0F(\theta,\phi)/r^2+o(r^{-2}) \ ; \tag{3.4}$$

the problem with such a hypothesis is that one can obtain another (nevertheless well-convergent) result if the spheres are replaced by, e.g., a family of ellipses. To find boundary conditions ensuring a well-defined charge it is useful to write (1.1) in the form

$$Q=\int_{\Sigma}\{j^\nu-([A_\mu,F^{\mu\nu}])/4\pi\}dV_\nu \tag{3.5}$$

(and we have assumed that the "constraint part" of the field equations is satisfied). Q will not suffer from any of the above-mentioned troubles if we require

$$j^\nu\in L_1(\Sigma) \ , \tag{3.6}$$

$$[A_\mu,F^{\mu\nu}]\in L_1(\Sigma) \ . \tag{3.7}$$

In what follows we will assume that (3.6) is satisfied. We shall analyze three classes of boundary conditions which guarantee the satisfaction of (3.7). A possible hypothesis which exploits the bracket structure in (3.7) is

$$A={}^0A+O_1(r^{-1-\epsilon}),\quad {}^0A=O_1(r^{-1}) \ ,$$
$$F={}^0F+O(r^{-2-\epsilon}),\quad {}^0F=O(r^{-2}),\quad \epsilon>0 \ , \tag{3.8a}$$

$${}^0A\in\mathfrak{H},\quad {}^0F\in\mathfrak{H} \ , \tag{3.8b}$$

where $\mathfrak{H}$ is any chosen Cartan subalgebra of $\mathfrak{G}$ (the field is Abelian in the leading order). We have not been able to prove in general that these boundary conditions guarantee that the charge is well defined; we conjecture that this is indeed the case. To give support to this conjecture we shall establish it under some supplementary conditions on ${}^0F$ (proposition 3.1) and without any supplementary hypotheses for infinitesimal gauge transformations (proposition 3.2). In the noninfinitesimal case we shall require that, in natural coordinates, ${}^0F_{\mu\nu}=\tilde{F}_{\mu\nu}(\theta,\phi)r^{-2}$. We shall call such an $\tilde{F}_{\mu\nu}$ generic if the set of (θ,ϕ) for which $\tilde{F}^{0r}$ belongs to the boundaries of Weyl domains[12] is of measure zero. We have the following proposition.

Proposition 3.1. For generic $\tilde{F}_{\mu\nu}$, the gauge transformations for which $\tilde{g}(\theta,\phi)=\lim_{r\to\infty}g(r,\theta,\phi)$ exists and which preserve the property $\tilde{F}\in\mathfrak{H}$ are of the form

$$\tilde{g}(x)=wh(x),\quad x\in S_\infty \ ,$$

where w is a fixed element of the Weyl group[13] and $h(x)$ has values in T, the maximal torus associated with $\mathfrak{H}$.

Proof. The condition $\tilde{g}(x)^{-1}\tilde{F}(x)\tilde{g}(x)\in\mathfrak{H}$, $x\in S_\infty$, implies that $\tilde{g}(x)=w(x)h(x)$, where $w(x)$ is an element of the Weyl group W and $h(x)$ belongs to the isotropy group of $\tilde{F}(x)$. As $\tilde{F}(x)$ is generic, its isotropy group is equal to T, the maximal torus of which $\mathfrak{H}$ is the Lie algebra; $w(x)=$ const follows from discreteness of W and continuity of $\tilde{g}(x)$ (we always assume gauge transformations of class C_2).

Proposition 3.2. The space of infinitesimal gauge transformations preserving (3.8) is finite dimensional, modulo infinitesimal $\mathfrak{H}$-valued gauge transformations and infinitesimal gauge transformations which vanish at infinity.

Proof. For infinitesimal gauge transformations $A\to A+\delta_\phi A$ (3.8) yields

$$\delta_\phi A=d\phi+[A,\phi]=w(x)+O_1(r^{-1-\epsilon}),\quad w\in\mathfrak{H} \ ,$$
$$w=d\phi+[{}^0A,\phi]+O_1(r^{-1-\epsilon}\ln r)$$

[(3.3) implies $\phi=O(\ln r)$, whence the logarithmic term above]. Let Δ denote the set of roots[12] and let $\phi=\sum_{\alpha\in\Delta}\phi_\alpha+\Psi$, $\Psi\in\mathfrak{H}$, be the Cartan decomposition of ϕ ($\Longrightarrow d\phi=\sum d\phi_\alpha+d\Psi$ is the Cartan decomposition of $d\phi$). We have

$$d\phi_\alpha+[{}^0A,\phi_\alpha]=d\phi_\alpha+\alpha({}^0A)\phi_\alpha\ , \tag{3.9}$$

which together with (3.3) implies

$$d\phi_\alpha+\alpha({}^0A)\phi_\alpha=O_1(r^{-1-\epsilon}\ln r)\ , \tag{3.10}$$

$$d\Psi=O_1(r^{-1})\ . \tag{3.11}$$

The integrability conditions of (3.10) are

$$\alpha({}^0F)=O(r^{-2-\epsilon}\ln r) \tag{3.12}$$

and, therefore, for ${}^0F$ for which (3.12) does not hold we recover the infinitesimal version of proposition 3.1 For every α for which (3.12) holds (3.10) determines ϕ_α uniquely up to the value of ϕ_α at, say, the north pole of S_∞ and lower-order terms. As a consequence the $\mathfrak{H}$ part of ϕ is arbitrary [up to (3.11)] and there may be as many nonvanishing ϕ_α's as α's for which (3.12) holds, every such α allowing for at most a one-parameter family of ϕ_α's.

It should be noted that the "ϕ_α gauge transformations," whenever allowed, do not change the value of the integral (1.1).

In specifying (3.8) we have arbitrarily singled out some Cartan subalgebra and the freedom of performing gauge transformations which lead from one Cartan subalgebra to any other restores a rigid action of the gauge group at infinity. This is due to the fact that all Cartan subalgebras are conjugated[14] which leads, at least in the generic case, to the asymptotic gauge group which consists of transformations of the form

$$\begin{aligned} &g(x)=g_0h(x)g_1(x),\quad h(x)\in T,\quad dg_0=0\ ,\\ &dg_1=O_1(r^{-1-\epsilon}),\quad dh=O_1(r^{-1})\ ,\\ &g_1\to e\ \text{for}\ r\to\infty\ . \end{aligned} \tag{3.13}$$

Equation (3.8) can be generalized to admit a particular case of what we shall call a "topologically nontrivial electric charge." Let $\mathcal{H}$ be a subbundle of the trivial bundle $\mathfrak{H}\times S_\infty$, each fiber $\mathcal{H}_x$, $x\in S_\infty$, being a Cartan subalgebra of $\mathfrak{H}$. We can weaken (3.8b) to read

$${}^0A(r,\theta,\phi)\in\mathcal{H}_{\theta,\phi},\quad {}^0F(r,\theta,\phi)\in\mathcal{H}_{\theta,\phi}\ . \tag{3.14}$$

We do not know whether the generalization (3.14) is an interesting one because (3.14) and ${}^0F=d\,{}^0A$ may be incompatible with a nontrivial ${}^0F$ and a nontrivial $\mathcal{H}$ bundle. For further purposes it is useful to describe the structure of such bundles. These are in one-to-one correspondence with subbundles $\mathcal{T}$ of the trivial principal bundle $G\times S_\infty$ with typical fiber T, the maximal torus corresponding to, say, $\mathcal{H}_{x_0}$. Each $\mathcal{T}_x$, being a maximal torus, is of the form $g^{-1}(x)\mathcal{T}_{x_0}g(x)$ (with some fixed x_0), $g(x)$ being determined up to multiplication from the left by an element of $N(\mathcal{T}_{x_0})=$ the normalizer of $\mathcal{T}_{x_0}$. Making use of this observation it is not too difficult to show that $\mathcal{T}$ is trivial if and only if

$$[g]_{\pi_2(G/N(\mathcal{T}_{x_0}))}=0\ . \tag{3.15}$$

From the exact homotopy sequence of a fibration we have

$$\pi_2[G/N(\mathcal{T}_{x_0})]=\pi_2(G/T)=\pi_1(T)=Z^n\ ,$$

$n=\mathrm{rank}G=\dim\mathfrak{H}$. If (3.15) holds $\mathcal{H}_{\theta,\phi}$ may be transformed to be equal to $\mathcal{H}_{x_0}$ with the help of the gauge transformation $g(x)$, leading in consequence to the reduction of the gauge group described in proposition 3.1. As we shall see in the following sections, the nontriviality of these bundles leads to some topological invariants which characterize the leading-order behavior of the electric and of the magnetic fields.

To close this section let us note that the conditions ${}^0A=O(r^{-1})$ and ${}^0F=O(r^{-2})$ can be weakened; a finite and well-defined charge will be obtained if ${}^0A=O(r^{-\alpha})$, ${}^0F=O(r^{-\alpha-1})$, $\alpha+\epsilon>1$. Finiteness of energy, if imposed, would yield the supplementary condition ${}^0F\in L_2(\Sigma)$ (which will be satisfied for $\alpha>\frac{1}{2}$).

IV. OTHER BOUNDARY CONDITIONS

The boundary conditions of the previous section had "a group-theoretical character"; the integrability of $[A_\mu,F^{\mu\nu}]$ was ensured by the commutativity of the leading-order terms. In this section we shall present two further sets of boundary conditions guaranteeing

$$[A_i,E^i]\in L_1(\Sigma)\quad (E^i=F^{0i})\ . \tag{4.1}$$

These boundary conditions are clearly sufficient for finiteness and convergence of (1.1). The conditions we shall present in this section suffer from lack of Lorentz covariance; if they are satisfied on some Cauchy surface, they will in general not be satisfied on a boosted one, raising doubts about the possibility of defining the charge on boosted surfaces. Although an unpleasant feature, we do not consider it as eliminating because the problem of definition of charges is a kinematical one, while the behavior of the fields on boosted slices belongs already to the dynamical domain: one obtains the fields on boosted hypersurfaces by evolving the Cauchy data in time. We shall first consider the "electric boundary conditions":

$$B^i=\epsilon_{ijk}F_{jk}/2=O(r^{-2-\epsilon}),\quad \epsilon>0\ , \tag{4.2}$$

$$E^i=O(r^{-2})\ . \tag{4.3}$$

Proposition 4.1. If (4.2) holds, there exists a gauge in which

$$A_i=O_1(r^{-1-\epsilon})\ . \tag{4.4}$$

Remark. If $\epsilon=0$ the construction that follows shows existence of a gauge transformation which leads almost to (4.4): we will have $|A_i|\le C(\theta)/r$, and $C(\theta)$ may go to infinity as θ tends to π, so that a string singularity will be present. A repetition of our construction starting from the negative z axis will lead to a complementary gauge in which the string lies on the positive z axis.

Proof. We shall construct the gauge transformation which will lead to a potential satisfying (4.4). Let x_0 be an arbitrary point on the positive z axis: $x_0=(0,0,r_0)$.

For $r>r_0$ let $g(0,0,r)$ be uniquely defined as follows.

$\forall X\in \mathfrak{G}$, $g(0,0,r)^{-1}Xg(0,0,r)$ is obtained by parallel transport of X from r_0 to r along the z axis.

For all (θ,ϕ), $\theta\neq\pi$, we define $g(x)$ as follows. $\forall X\in\mathfrak{H}$, $g(r,\theta,\phi)^{-1}Xg(r,\theta,\phi)$ is obtained by parallel transport of X from the north pole to x along the meridian.

This defines $g(x)$ for all x with $|x|\geq r_0$ except on the negative z axis. Let $X_{X_0}(x)=g(x)^{-1}X_0g(x)$, $|X_0|=1$. The "parallel-transport deviation equation,"

$$D_\theta X=0 \Longrightarrow D_\theta D_a X=[F_{a\theta},X],\quad a=r,\phi\ ,$$

the preservation of the length of the vectors under parallel transport, and $D_r X_{X_0}(\theta=0)=D_\phi X_{X_0}(\theta=0)=0$ imply, for $\theta<\pi$, the existence of a constant C such that

$$|D_r X_{X_0}|\leq Cr^{-1-\epsilon}\ ,$$

$$|D_\phi X_{X_0}|\leq Cr^{-\epsilon}\quad (D_\theta X_{X_0}=0 \text{ by construction})\ .$$

Consider now the parallel-transport operator $T^{\theta}_{\phi_1,\phi_2}$ along parallels with given θ on $S(R)$ from ϕ_1 to ϕ_2. We have

$$T^{\theta}_{\phi_1,\phi_2}X_{X_0}(\phi_1,\theta)-X_{X_0}(\phi_2,\theta)=\int_{\phi_2}^{\phi_1}T^{\theta}_{\phi,\phi_2}D_\phi X_{X_0}(\phi,\theta)d\phi\ ,$$

which for $\theta=\pi$ leads to

$$|X_{X_0}(\phi_1,\theta)-X_{X_0}(\phi,\theta)|\leq C'r^{-\epsilon}\ ,$$

in virtue of the previous estimates and because $T^{\pi}_{\phi,\phi_1}=id$. It is easy to show now that this last inequality holds also for all $\theta\in(\pi-\tilde{\epsilon},\pi)$, with some $\tilde{\epsilon}>0$ and with a possibly larger constant C'. For a semisimple Lie group there exists a neighborhood of unity which is smoothly diffeomorphic with its image under the adjoint representation; thus, this last estimate shows that for sufficiently large r, say, $r\geq R_0$, and for $\theta\in(\pi-\tilde{\epsilon},\pi)$ all the $g(r,\theta,\phi)$ lie within the radius of injectivity of the exponential mapping at $g(r,\pi,0)$:

$$g(r,\theta,\phi)=\exp[\tilde{\Sigma}(r,\theta,\phi)]g(r,\pi,0)\ .$$

$\tilde{\Sigma}$ is defined uniquely and satisfies

$$\begin{aligned}&|\tilde{\Sigma}(r,\theta,\phi)|\leq\tilde{C}r^{-\epsilon},\quad |D_\theta\tilde{\Sigma}|\leq\tilde{C}r^{-\epsilon}\ ,\\ &|D_\phi\tilde{\Sigma}|\leq\tilde{C}r^{-\epsilon},\quad |D_r\tilde{\Sigma}|\leq\tilde{C}r^{-\epsilon-1}\ .\end{aligned}\tag{4.5}$$

Let Φ be any smooth function satisfying $\Phi(x)=1$ for $x\leq\pi-2\tilde{\epsilon}/3$, $\Phi(x)=0$ for $x\geq\pi-\tilde{\epsilon}/3$. For $r\geq R_0$ define a C_2 G-valued function $\tilde{g}$:

$$\begin{aligned}&\tilde{g}=g,\quad \theta\notin(\pi-\tilde{\epsilon},\pi)\ ,\\ &\tilde{g}(r,\theta,\phi)=\exp[\tilde{\Sigma}(r,\theta,\phi)\Phi(\theta)]g(r,\pi,0)\\ &\qquad\qquad\text{for } \pi-\tilde{\epsilon}<\theta<\pi\ .\end{aligned}\tag{4.6}$$

Define $\tilde{X}_{X_0}(x)=\tilde{g}(x)^{-1}X_0\tilde{g}(x)$. From (4.5) and (4.6) it follows that

$$|D_i\tilde{X}_{X_0}|\leq\tilde{\tilde{C}}r^{-1-\epsilon}\ .\tag{4.7}$$

If we perform a gauge transformation $X\to\tilde{g}(x)X\tilde{g}(x)^{-1}$, the fields $\tilde{X}_{X_0}$ will be represented by point-independent fields X_0, (4.7) still holding (being gauge independent). We therefore have

$$D_i\tilde{X}_{X_0}=\partial_i\tilde{X}_{X_0}+[A_i,\tilde{X}_{X_0}]=[A_i,X_0]=O(r^{-1-\epsilon})\ .\tag{4.8}$$

As (4.8) holds for all X_0, the compactness and semisimplicity of $\mathfrak{G}$ imply

$$A_i=O(r^{-1-\epsilon})\ ,\tag{4.9}$$

which had to be established [the higher differentiability of A follows from standard theorems on differentiability of solutions of differential equations and similar considerations, if F is assumed $O_k(r^{-2-\epsilon})$].

Proposition (4.1) shows that (4.2) and (4.3) naturally single out a class of gauges for which (4.9) holds. It is trivial to show that C_2 gauge transformations preserving (4.9) must be of the form

$$\begin{aligned}&g(x)=g_0g_1(x),\quad dg_0=0\ ,\\ &dg_1=O_1(r^{-1-\epsilon}),\quad g_1\to e\quad\text{for } r\to\infty\ .\end{aligned}\tag{4.10}$$

An alternative set of conditions ensuring (4.1) are the "asymptotically Coulomb boundary conditions":

$$\begin{aligned}&A_\mu=O_1(r^{-1}),\quad A_r=O_1(r^{-1-\epsilon})\ ,\\ &\epsilon_{ijk}x^jE^k=O(r^{-1-\epsilon})\ .\end{aligned}\tag{4.11}$$

The constraint equations $D_iE^i=4\pi j^0$, (3.6) and (4.11), lead one to expect that

$$E^i=Q(\theta,\phi)x^i/r^3+O(r^{-2})\ .\tag{4.12}$$

If $\partial A_r/\partial t$ vanishes faster than r^{-2}, (4.12) and the field equations imply $A^0=Q(\theta,\phi)/r+O(r^{-1})$, and if, moreover, the magnetic field vanishes faster than r^{-2}, proposition 4.1 implies that we have a solution of the type (3.8) or (3.14). It seems, however, that the classes (3.14), (4.2), (4.3), and (4.11) need not coincide. The solution (4.12) may exhibit the already mentioned "topologically nontrivial electric charge" behavior, which we shall discuss now in some detail. For every θ and ϕ we can define

$$\begin{aligned}&T_{\theta,\phi}=\{g\in G{:}g^{-1}Q(\theta,\phi)g=Q(\theta,\phi)\}\ ,\\ &\mathcal{T}=\{(g,x),x\in S_\infty,g\in T_x\}\ ,\end{aligned}\tag{4.13}$$

each $T_{\theta,\phi}$ being the isotropy group of $Q(\theta,\phi)$. In this way we have defined a subset $\mathcal{T}$ of the trivial bundle $G\times S_\infty$. In what follows we shall assume that all the $T_{\theta,\phi}$ are isomorphic, i.e., $\mathcal{T}$ is a bundle (this is a restrictive assumption, analogous to the genericity assumption in proposition 3.1). We shall also focus our attention on the case where the fibers of $\mathcal{T}$ are maximal tori, a situation described at the end of Sec. III (our further discussion would not be essentially altered without this last hypothesis). As in Sec. III, $Q(\theta,\phi)$ may be written in the form

$$Q(\theta,\phi)=g^{-1}(\theta,\phi)Q_0(\theta,\phi)g(\theta,\phi)\ ,$$

with Q_0 having values in a fixed Cartan subalgebra. If the π_2 class of g in G/T is trivial, one can, via an angle-dependent gauge transformation [which, therefore, preserves (4.11) and (4.12)], obtain a Q which lies in a fixed Cartan subalgebra. The whole gauge group gets asymptotically reduced to a rigid group by requiring that the gauge transformations preserve this structure, as in proposition 3.1. If, however, the bundle $\mathcal{T}$ is not trivial, there are no preferred gauges and conditions (4.11) and (4.12) leave the freedom of performing gauge transformations (3.3) with $\beta=0$, excluding the logarithmic transformations discussed in Sec. III.

In the case of $G=\mathrm{SU}(2)$ the condition of triviality of $\mathcal{T}$ may be expressed in a more direct way in terms of conditions on $Q(\theta,\phi)$. This can be described in a setting considerably more general than (4.11) and (4.12) as follows: consider any family of spheres $S(R)$ such that $\forall x$, $i_R {*F}(x)\neq 0$, for $R_1\leq R\leq R_2$, where i_R is the inclusion $i_R : S(R)\hookrightarrow\Sigma$. Consider the algebra-valued field $Q_R={}^*i_R{*F}$, where $*$ is the Hodge dual with respect to the induced metric on $S(R)$. $e_R=Q_R/|Q_R|$ is a mapping from $S(R)=S_2$ to $\{X\in\mathrm{SU}(2), |X|=1\}=S_2$, so we can define its degree $\deg(e_R)$. It follows from homotopy invariance and from $\pi_2(G)=\{e\}$ for every Lie group that $\deg(e_R)$ is gauge invariant and constant for $R\in[R_1,R_2]$, in particular, if $R_2=\infty$, $\deg(e_\infty)$ defines a discrete topological invariant. It is easily seen that in the case of (4.12) $\deg(e_R)=\deg[Q(\theta,\phi)/|Q(\theta,\phi)|]$. It can be shown that the $\mathcal{T}$ bundle (4.13) is trivial if and only if $\deg(e_R)=0$ [in the SU(2) case the genericity requirement is $Q(\theta,\phi)\neq 0$ for all θ and ϕ]. In Sec. VIII some exact solutions exhibiting the above-mentioned "topologically nontrivial behavior" are presented.

Let us remark that (4.2) and (4.3) can be weakened to $E_i=O(r^{-\gamma})$, $\gamma+\epsilon>2$ (in particular if $B=0$ no restrictions on the asymptotic behavior of E are imposed). (4.11) can be weakened to $A_i=O_1(r^{-\gamma})$, $\epsilon_{ijk}x^jE^k/r=O(r^{-\Psi})$, $\gamma+\Psi>3$, $A_r=O_1(r^{-\chi})$, $E^i=O(r^{-\kappa})$, $\kappa+\chi>3$.

V. GLOBAL CHARGES

In this section we shall propose how to extract gauge-invariant information from the Lie-algebra-valued integrals of the kind (1.1). Consider first the "source-strength" integral:

$$Q_j(g,\Sigma)=\int_\Sigma g^{-1}j^\mu g\,dV_\mu,\quad g=g(x)\ , \tag{5.1}$$

and we have written $Q_j(g,\Sigma)$ to emphasize the dependence of (5.1) upon the gauge (cf. Ref. 15 for some interesting remarks). Our procedure can be described as follows: let $\mathfrak{H}$ be any Cartan subalgebra of $\mathfrak{G}$, Δ_0 any base of the root system,[12] and $\mathfrak{H}_0$ the related Weyl domain. There is a unique element j_0^0 of the orbit of j^0 lying in $\mathfrak{H}_0$ (Ref. 12). For further purposes let us note that j_0^0 is uniquely defined by the $n=\mathrm{rank}G=\dim\mathfrak{H}$ numbers $\{\alpha(j_0^0)\}_{\alpha\in\Delta_0}$. For $\alpha\in\Delta_0$ we define

$$\alpha[j,\Sigma]=\int_\Sigma \alpha(j_0^0)d^3x\ . \tag{5.2}$$

The set of numbers $\{\alpha[j,\Sigma]\}_{\alpha\in\Delta_0}$ is defined up to the action of the Weyl group on Δ_0 (which acts on it as a subgroup of the permutation group), irrespective of the choice of $\mathfrak{H}$ and $\mathfrak{H}_0$. As for every x there exists ${}^0g(x)$ such that

$${}^0g(x)^{-1}j^0\,{}^0g(x)=j_0^0\ , \tag{5.3}$$

(5.2) may be looked upon as the value of $\alpha(Q)$, where Q is given by (5.1) in the gauge where all the j^0 belong to $\mathfrak{H}_0$. The trouble is that there may not exist a continuous ${}^0g(x)$ for which (5.3) holds. Nevertheless the integrals (5.2) can be related to the family of integrals (5.1) as follows: for $X\in\mathfrak{G}$ define

$$Q_j(X,\Sigma)=\sup_{g(x)}\int_\Sigma k(X,g^{-1}(x)j^\mu g(x))dV_\mu\ , \tag{5.4}$$

where the supremum is taken over all C_2 gauges. Clearly

$$|Q_j(X,\Sigma)|\leq|X|\int_\Sigma|j^0|d^3x\ ,$$

and (3.6) implies that $Q_j(X,\Sigma)$ is indeed well defined. From (5.4) it follows

$$\forall h\in G,\quad Q_j(h^{-1}Xh,\Sigma)=Q_j(X,\Sigma)\ ,$$

which shows that it is sufficient to consider (5.4) for X belonging to a fixed Weyl domain $\mathfrak{H}_0$ of a fixed Cartan subalgebra $\mathfrak{H}$. The integrand $k(X,g^{-1}(x)j^0g(x))$ takes its maximal value when $g^{-1}j^0g\in\mathfrak{H}_0$, because, as has been shown by Kostant,[16] the orthogonal projection of the adjoint orbit of any element of $\mathfrak{G}$ on some Cartan subalgebra is the convex hull of the intersection points of the orbit with $\mathfrak{H}$. Therefore, if a C_2 gauge transformation g_0 exists for which (5.3) holds we have

$$Q_j(X,\Sigma)=k(X,Q_0(\Sigma))\ ,\qquad Q_0(\Sigma)=\int_\Sigma g_0{}^{-1}j^\mu g_0\,dV_\mu\ . \tag{5.5}$$

$Q_0(\Sigma)$ is in one-to-one correspondence with the set of numbers (5.2). By showing that the supremum commutes with the integral it can be proved that (5.4) contains exactly the same information as (5.2), if the symmetry group G is connected.

It is worth noting that formula (5.4) leads to the interpretation of our charges as quantities describing some extremal properties of the set of vectors obtained from the integrals (1.1), and this is in fact the way we came to consider the charges (5.2).

Let us illustrate our construction in the case of $\mathrm{SU}(n)$: $\mathfrak{H}$ can be chosen as the set of diagonal matrices (in some basis), and $\mathfrak{H}_0$ as the set of matrices for which $\lambda_1\leq\cdots\leq\lambda_n$, with λ_i the ith eigenvalue, a base in the system of roots may be chosen as $\alpha_i(h)=\lambda_{i+1}-\lambda_i$. Our prescription therefore requires the ordering, pointwise, of all the eigenvalues of j^0 in increasing order and the global charges are given by the integrals $\Lambda_i=\int_\Sigma\lambda_i(x)d^3x$. In the simplest case $G=\mathrm{SU}(2)$ we have

$$Q_j(X,\Sigma)=k(X,X)^{1/2}\int_\Sigma k(j^0,j^0)^{1/2}d^3x\ .$$

In the previous sections we have analyzed some conditions which led to a finite charge (1.1), with $\int_{S_\infty}$ well convergent and surface independent. In the case of boundary conditions (4.11) and (4.12) one was left with all angle-dependent asymptotic gauge transformations in the reduced gauge group. In such a case, in a manner completely analogous to (5.2) and (5.4), we propose to define

$$\alpha[F,\Sigma]=\left[\int_{S_\infty}\alpha({}^0F^{r0})d^2S\right]\Big/4\pi,\quad \alpha\in\Delta\ , \tag{5.6}$$

$$Q(X,\Sigma)=\left[\sup_{g(x)}k\left[X,\int_{S_\infty}g^{-1}{}^*Fg\right]\right]\Big/4\pi\ , \tag{5.7}$$

where the supremum is taken over angle-dependent C_2 gauge transformations,[17] and ${}^0F^{r0}$ is the unique element of the gauge orbit of F^{r0} lying in $\mathfrak{H}_0$.

From what has been said it is not too difficult to show what follows.

Proposition 5.1. (5.7) contains the same information as (5.6).

From (3.5) one obtains the following.

Proposition 5.2. Suppose that A has values in a fixed Weyl domain $\mathfrak{H}_0(\Longrightarrow j^\mu$ has values in $\mathfrak{H}_0$). We have $\alpha[F,\Sigma]=\alpha[j,\Sigma]$, for all $\alpha\in\Delta_0$.

It is clear that in this last case $\{\alpha[j,\Sigma]\}_{\alpha\in\Delta_0}$ describes the classical Abelian electric charges of the fields F and j.

As discussed above, prescriptions (1.1) (in an admissible gauge) and (5.5) give the same result in the case of boundary conditions (3.8); they may however give different results in case of (4.2). We shall not analyze this problem any further.

Having defined the charges $\alpha[F,\Sigma]$ and $\alpha[j,\Sigma]$ one can consider the deficit charges $\alpha[F,\Sigma]-\alpha[j,\Sigma]$, and they vanish if the field is quasi-Abelian in the sense of proposition 5.2. This difference may be thought of as the color carried by the Yang-Mills field itself, a phenomenon related to the so-called color-screening behavior (cf., e.g., Ref. 18).

VI. MAGNETIC CHARGES

It is straightforward to include magnetic charges in our discussion. The appropriate boundary integral takes of course the form

$$Q_M=\left[\int_{S_\infty}F\right]\Big/4\pi\ ,$$

(3.8) or (3.14) guarantee convergence, finiteness, and uniqueness of definition of Q_M up to the action of the Weyl group, as in Sec. III. The equivalent of (4.1) in the magnetic case takes the form

$$[A_i,B^i]\in L_1(\Sigma)\ ;$$

(4.11) can be supplemented by

$$\epsilon_{ijk}x^iB^j/r=O(r^{-2-\epsilon})$$

to give again a well-convergent Q_M; the topological discussion of the previous sections can be carried over word for word to the magnetic case, and the construction of Sec. V may also be used to define global magnetic charges. In the special case considered by Goddard, Nuyts, and Olive,[19] our global charges reproduce the "magnetic weights." 't Hooft's definition of magnetic strength leads to the same number as ours for the 't Hooft–Polyakov[6,20] monopole. It should be noted that neither the standard monopole topological number, related to the topological properties of the asymptotic Higgs field, nor the topological invariants discussed by Lubkin[21] and Chan and Tsou[22] coincide with our discrete invariants.

VII. DYNAMICS OF THE COLOR CHARGES

If there exists a gauge in which

$$[A_\mu,F^{\mu\nu}]=o(r^{-3}),\quad j^\mu=o(r^{-3})\ , \tag{7.1}$$

the charges (1.1) (calculated in this gauge) are time independent and Lorentz invariant, which is readily demonstrated with the help of the Stokes theorem and the field equations. (In the general-relativistic case, for space-times asymptotically flat in the sense of Christodoulou and O'Murchadha,[23] they are also invariant under "supertranslations.") This shows that (3.8) leads indeed to conserved, Lorentz-invariant $\mathfrak{G}$-valued charges at spatial infinity. In the case of the boundary conditions of Sec. IV there is no guarantee of obtaining well-defined charges on boosted hypersurfaces, but one can still inquire about time evolution. One obtains time-independent charges if

$$[A_\mu,F^{\mu i}]=o(r^{-2}),\quad j^i=o(r^{-2}) \tag{7.2}$$

holds together with (4.1). If, in addition, (4.2) and (4.3) are satisfied Q is well defined and time independent. The generalized charges (5.7) will be time independent if (7.2) and (4.11) are satisfied in a gauge in which the supremum of (5.7) is attained.

In opposition to the Abelian case the source charges (5.2) or (5.4) need not be conserved in time even if the global charges (5.6) or (5.7) are. There may be "a flow of color" between the sources and the fields.

VIII. SOME EXAMPLES

In this section we shall present some solutions of field equations with point sources exhibiting the "topologically nontrivial electric charge behavior." Consider the Abelian point-particle solution:

$$\begin{aligned}&A=Q_0r^{-1}dt,\quad Q_0\in\mathfrak{G}\ ,\\&dQ_0=0,\quad F=Q_0r^{-2}dt\wedge dr\ .\end{aligned} \tag{8.1}$$

For every mapping $g{:}S_2\to G$ the fields

$$\begin{aligned}&A_g=g^{-1}Ag+g^{-1}dg\ ,\\&F_g=g^{-1}Fg=Q(\theta,\phi)r^{-2}dt\wedge dr\end{aligned} \tag{8.2}$$

are solutions of vacuum Yang-Mills equations in $\mathbb{R}^4\setminus\{r=0\}$,[24] $Q(\theta,\phi)$ being a mapping from S_2 to the G orbit of Q_0: $Q{:}S_2\to\{g^{-1}Q_0g,g\in G\}=G/I_{Q_0}$ (I_{Q_0} is the isotropy group of Q_0).

If Q_0 is generic then $\pi_2(G/I_{Q_0})=Z^{\text{rank}G}$, as discussed at the end of Sec. III, and so the π_2 class of Q may be nontrivial—in such a case g necessarily has a singularity at, say, either the south or the north pole and (8.2) gives a solution of Yang-Mills equations in $\mathbb{R}^4\setminus\{r=0\}$ with smooth F but a potential having a string singularity. It is interesting to ask whether F given by (8.2), with $[Q]_{\pi_2}\neq0$, can admit some smooth potential. The answer is negative, as can be seen from the following argument: if a smooth potential exists, the equations $F_{ab}=0$, $a,b=t,\theta,\phi$, and the simple connectedness of S_2 imply the existence of a smooth gauge transformation g_1 which leads to $A_a=0$, so that $A_{r,t}=g_1{}^{-1}Qg_1r^{-2}=\tilde{Q}r^{-2}$. The integration in time of this last equation and a simple analysis of the equations $F=dA+[A,A]/2$ show then that $\tilde{Q}=\tilde{Q}(r,t)$, which is in contradiction with $[Q]_{\pi_2}\neq 0$ and the smoothness of g_0.

In a similar way one can obtain magnetic monopoles:

$$F=Q(\theta,\phi)\sin\theta\, d\theta\wedge d\phi,\quad [Q]_{\pi_2}\neq 0\ , \tag{8.3}$$

or monopoles with a Higgs field ϕ in the adjoint representation (and eventually string singularities in the potential). In this case the field configurations are numbered by two sets of integer invariants, $[Q]_{\pi_2}$ and $[\lim_{r\to\infty}\phi]_{\pi_2}$. (The asymptotic leading-order terms of the Polyakov–'t Hooft monopole can be obtained in this way (cf. Ref. 22) starting from the standard Dirac monopole and a constant Higgs field, so that in this case $[Q]_{\pi_2(G/T)}=[\lim\phi]_{\pi_2(G/T)}$. This last equality holds for monopoles considered by Goddard, Nuyts, and Olive.[19] It also holds for the monopoles constructed by Jaffe and Taubes[25] provided $r^2k(\phi,F_{0r})$ is bounded from below for large r.) Clearly, the three above-mentioned families of fields are gauge equivalent to the Abelian point particle (8.1), the standard magnetic U(1) monopole, or the Polyakov–'t Hooft monopole, respectively, if one admits singular gauge transformations. If not (which is the commonly accepted point of view) the foregoing solutions are distinct classes of solutions. The problem of whether or not these are different solutions is, therefore, equivalent to the problem of what one calls a gauge transformation.

To end this section let us give explicit examples of solutions (8.2) for $G=\mathrm{SU}(2)$. For

$$F=Q_1 r^{-2}dt\wedge dr\ ,$$

$$Q_1=x^i\sigma^i r^{-1}\quad (\Longrightarrow \deg Q=1)$$

we have two potentials, singular on either the negative or the positive z axis:

$$\begin{aligned}
&A_1=Q_1 r^{-1}dt+g_1^{-1}dg_1\ ,\\
&g_1=\cos(\theta/2)+i[\cos(\phi)\sigma_y-\sin(\phi)\sigma_x]\sin(\theta/2)\ ,\\
&A_2=Q_1 r^{-1}dt+g_2^{-1}dg_2\ ,\\
&g_2=\sin(\theta/2)\sigma_x+\cos(\theta/2)\exp(i\phi\sigma_z)\ .
\end{aligned}\tag{8.4}$$

For the fields

$$F_n=Q_n r^{-2}dt\wedge dr,\quad Q_n=\mathbf{f}_n\sigma\ ,$$

$$\mathbf{f}_n=(\sin\theta\cos n\phi,\ \sin\theta\sin n\phi,\ \cos\theta)\quad (\Longrightarrow \deg Q_n=n)$$

which are only Lipschitz continuous on the z axis (and not differentiable) one can obtain potentials replacing ϕ by $n\phi$ in (8.4) (these potentials will be singular on the whole z axis, which can be cured by smoothing out $\mathbf{f}_n$ on the z axis).

IX. CONCLUSIONS

We have presented three classes of boundary conditions allowing the introduction of the concept of "global color." Two of them lead to the asymptotic reduction of the gauge group to a "rigid" group and, consequently, a definition of Lie-algebra-valued charges. In the case of the third class we have shown how to define $n=\mathrm{rank}G$ scalars carrying global gauge-invariant information about the fields. We have pointed out the existence of solutions exhibiting a "topologically nontrivial color behavior." Our results show that one can in some situations associate to a configuration of Yang-Mills-Higgs fields three sets of topological invariants—the invariants associated with the asymptotic behavior of the electric field, of the magnetic field, and of the Higgs field. In the general case these invariants may differ, as is demonstrated by the trivial example $G=\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$, $F=F_1+F_2+F_3$, $\phi=\phi_3$, F_1 purely electric with $\lim_{r\to\infty}[F_1^{0r}r^2]=n_1$, F_2 purely magnetic with $\lim_{r\to\infty}[{}^*F_2^{0r}r^2]=n_2$, and (F_3,ϕ_3) a Polyakov–'t Hooft-type monopole,[25] F_i having values in the ith SU(2) factor of G. It would be interesting to find out whether these invariants may differ in less trivial cases. It seems also of some interest to study the stability of the "topologically nontrivial electric" solutions. Our results are disappointing in some sense, because one has to impose rather strong boundary conditions in order to be able to define the global charges; it is however obvious that the boundary conditions we have presented, in each case, hardly can be weakened without spoiling at least the convergence of the charge integrals.

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[2]$A=A_\mu dx^\mu=A_{a\mu}e^a dx^\mu$, $F=F_{\mu\nu}dx^\mu\wedge dx^\nu/2=F_a e^a=F_{a\mu\nu}e^a dx^\mu\wedge dx^\nu/2$, e^a are the generators of the Lie algebra, S_∞ denotes a two-sphere at infinity (r going to infinity with fixed t), a star denotes the Hodge dual, $dV_\mu=\epsilon_{\mu\alpha\beta\gamma}dx^\alpha\wedge dx^\beta\wedge dx^\gamma/6$, greek indices run from 0 to 3, $a,b,\ldots$ are Lie-algebra indices running from 1 to dim$\mathfrak{G}$, where $\mathfrak{G}$ is the Lie algebra of the group G, $i,j,\ldots$ are space indices running from 1 to 3. k_{ab} is the Killing metric on G; e is the identity of the group. Connections are always assumed to be C_1; we consider C_2 gauge transformations only. $f=O_n(r^{-\gamma})$ is a shorthand for $|f|\leq C(r+1)^{-\gamma}$, $|\nabla f|\leq C(r+1)^{-\gamma-1},\ldots,|\nabla_1\cdots\nabla_n f|\leq C(r+1)^{-\gamma-n}$. $f=o(r^{-\gamma})$ if $\lim_{r\to\infty}r^\gamma f=0$; d (D) denotes exterior (covariant) differentiation.

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