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String field theory at finite temperature

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We present a real-time formalism enabling us to compute statistical averages of various quantum string field correlation functions and scattering amplitudes in the context of perturbation theory and functional methods. Second quantization of the string fields is carried out in the path-integral representation. Our finite-temperature formalism is a generalization of thermo field dynamics to string field theory for which a doubling of the physical string field degrees of freedom is required. Temperature formally enters our construction by imposing boundary conditions (Kubo-Martin-Schwinger condition) in the generating functional for the string field. Some explicit computations for the case of covariant open bosonic string theory are carried out.

I. INTRODUCTION

Recently, some interest has been devoted to the thermodynamical behavior of superstring theories[1] at high temperature and high energy density.[2] The obvious applications of such studies may lie in cosmology[3] and in the very early Universe. Standard scenarios for the evolution of the Universe (such as inflationary[4] or new inflationary[5] models) may require modifications if superstrings truly describe the dynamics of the known fundamental interactions.

Although such recent works have made use of the canonical as well as the more fundamental microcanonical ensemble for the computation of the partition function of an ideal gas of string and superstring excitations at high energy density, we wish rather, in this paper, to lay the basis for a real-time interacting (bosonic) string field theory at finite temperature in the grand canonical ensemble. Our finite-temperature formalism is a generalization of the so-called thermo field dynamics[6–20] (TFD) to string field theory. Such a formalism has already been developed very satisfactorily in the usual quantum field theory (QFT) for thermodynamical equilibrium[6–14] and is being extended to nonequilibrium[15–20] cases. For a recent general review on real- and imaginary-time field theory at finite temperature and density, see Ref. 21.

String field theory[22–30] differs fundamentally from ordinary field theory by the fact that it is multilocal. However, it can be consistently formulated without violating relativity, causality, or unitarity, and has usually softer ultraviolet behavior. Historically, canonical quantization of string field theory has been carried out in the so-called light-cone gauge[22–27] where it is manifestly unitary since unphysical longitudinal modes have been eliminated by the gauge-fixing procedure. On the other hand, a covariant canonical quantization of string field theory may be more difficult and is not immediately at hand. This state of affairs is rather unfortunate at the present time since the light-cone gauge is not a convenient one to introduce thermal effects into the theory. We are therefore prevented, for the time being, from using all the operator machinery of thermo field dynamics to treat the thermal degrees of freedom in a canonical way.

We shall therefore rely on the path-integral quantization of covariant string field theory[27–29] and we will make use of functional methods to carry out perturbation theory at finite temperature. Functional methods in the context of TFD have been considered[11] for the case of point-particle field theory, and there the temperature formally enters the formalism through boundary conditions [the so-called Kubo-Martin-Schwinger (KMS) condition[31,32]] imposed on the propagator for free fields together with the necessary doubling of the physical degrees of freedom.

One should also mention the existence of geometric approaches to the quantization of string theories and of possible application to string field theory.[33,34] This may well lead to a third formalism for string field quantization. Inclusion of temperature from such geometric approaches may also be an interesting problem.

As a practical example of our method, we will derive the Feynman rules at finite temperature of recently proposed covariant bosonic string field theories[29,30] with cubic and perhaps also quartic interactions, and obtain general expressions for the tachyon four-point amplitude in the tree and one-loop approximations. A brief mention of a possible extension to the finite-temperature theory of closed bosonic strings, fermionic strings, and various types of superstrings is also made. Finally, we comment on the renormalization problem[1,9,10] and the zero-slope limit[1,29,30] of the open bosonic string at finite temperature.

In the following we choose the signature $\mathrm{diag}\, g_{\mu\nu} = (-1,1,\ldots,1)$ for the metric tensor.

II. REVIEW OF TFD FORMALISM AND FUNCTIONAL METHODS IN FIELD THEORY

In this section we briefly review thermo field dynamics as a canonical formalism for quantum statistical mechanics as well as its formulation in the path-integral representation and functional methods.[11]

The main idea and basic problem of canonical TFD is to express the statistical average of a dynamical variable A as a vacuum expectation value, that is,

$$\langle A \rangle \equiv Z^{-1}(\beta)\mathrm{Tr}(Ae^{-\beta H}) = \langle 0(\beta) | A | 0(\beta) \rangle , \qquad (2.1)$$

where β is the inverse temperature, $Z(\beta)$ the partition function defined as

$$Z(\beta) \equiv \mathrm{Tr}(e^{-\beta H}) , \qquad (2.2)$$

and H is the Hamiltonian of the system with eigenvalues ω_n,

$$H | n \rangle = \omega_n | n \rangle, \quad \langle n | m \rangle = \delta_{nm} . \qquad (2.3)$$

In order to realize such a representation for the thermal vacuum $|0(\beta)\rangle$, it has been recognized[6] that one must first implement an effective doubling of the dynamical variables or degrees of freedom of the theory. This is effected by introducing unphysical states $|\tilde{m}\rangle$ (so-called "tilde" space) orthogonal to the original Fock space. For the latter tilde subsystem one defines the Hamiltonian $\tilde{H}$ which acts on the tilde states as

$$\tilde{H} | \tilde{n} \rangle = \omega_n | \tilde{n} \rangle, \quad \langle \tilde{n} | \tilde{m} \rangle = \delta_{nm} . \qquad (2.4)$$

The total Fock space is now spanned by the direct product of the two orthogonal vector spaces. Defining the thermal vacuum $|0(\beta)\rangle$ as

$$|0(\beta)\rangle \equiv Z^{-1/2}(\beta) \sum_n e^{-\beta\omega_n/2} | n, \tilde{n} \rangle \qquad (2.5)$$

and making use of the orthogonality of the subspaces

$$\langle \tilde{m}, n | A | n', \tilde{m}' \rangle = \langle n | A | n' \rangle = \delta_{mm'} , \qquad (2.6)$$

and

$$\langle \tilde{m}, n | \tilde{A} | n', \tilde{m}' \rangle = \langle \tilde{m} | \tilde{A} | \tilde{m}' \rangle = \delta_{nn'} , \qquad (2.7)$$

one then shows easily that Eq. (2.1) is satisfied.

In ordinary quantum field theory, the TFD formalism is best constructed from the following axioms.[14]

Axiom 1. Dynamical variables belonging to different subspaces are independent:

$$[A(t), \tilde{B}(t)]_{\mp} = 0 . \qquad (2.8)$$

Axiom 2. There is a mapping between the two orthogonal subspaces called tilde conjugation, defined by the following rules:

$$\text{(a)} \quad (AB)^{\sim} = \tilde{A}\tilde{B} , \qquad (2.9)$$

$$\text{(b)} \quad (c_1 A + c_2 B)^{\sim} = c_1^* \tilde{A} + c_2^* \tilde{B} , \qquad (2.10)$$

$$\text{(c)} \quad (\tilde{A})^{\dagger} = (A^{\dagger})^{\sim} , \qquad (2.11)$$

with c numbers c_1 and c_2.

Axiom 3. The thermal vacuum is invariant under the tilde conjugation:

$$|\widetilde{0(\beta)}\rangle = |0(\beta)\rangle . \qquad (2.12)$$

Axiom 4. The thermal vacuum satisfies the following thermal state conditions:

$$A(t,\mathbf{x}) | 0(\beta) \rangle = \sigma \tilde{A}^{\dagger}(t - i\beta/2, \mathbf{x}) | 0(\beta) \rangle , \qquad (2.13)$$

$$\langle 0(\beta) | A(t,\mathbf{x}) = \langle 0(\beta) | \tilde{A}^{\dagger}(t + i\beta/2, \mathbf{x}) \sigma^* , \qquad (3.14)$$

where $|\sigma| = 1$.

Axiom 5. The double tilde conjugation is defined as follows:

$$\tilde{\tilde{A}} = \sigma A . \qquad (2.15)$$

The (anti)commutator in (2.8) is for (fermionic) bosonic $A, \tilde{B}$. Also, there is some freedom with respect to the choice of the phase factor σ in the thermal state conditions (2.13) and (2.14) and the double tilde conjugation (2.15). Note that the thermal state condition has been shown to be equivalent to the KMS condition[31,32] of the axiomatic C^*-algebra approach to statistical mechanics.

The above axioms can be used to obtain the total generator of space-time translations $\hat{P}_\mu$ in TFD,

$$\hat{P}_\mu \equiv P_\mu - \tilde{P}_\mu , \qquad (2.16)$$

where P_μ is the four-momentum operator. This in turn implies that the total Lagrangian for the system is given as

$$\hat{\mathcal{L}} = \mathcal{L} - \tilde{\mathcal{L}} . \qquad (2.17)$$

The thermal state condition also leads to the existence of operators $\alpha_\beta(t)$ and $\tilde{\alpha}_\beta(t)$ which annihilate the thermal vacuum

$$\begin{aligned} \alpha_\beta(t) | 0(\beta) \rangle &= \tilde{\alpha}_\beta(t) | 0(\beta) \rangle \\ &= \langle 0(\beta) | \alpha_\beta^{\dagger}(t) \\ &= \langle 0(\beta) | \tilde{\alpha}_\beta^{\dagger}(t) = 0 . \end{aligned} \qquad (2.18)$$

Introducing the thermal doublet notation A^α,

$$A^\alpha = \begin{cases} A, & \alpha = 1, \\ \tilde{A}^{\dagger}, & \alpha = 2, \end{cases} \qquad (2.19)$$

the latter operators are related to the zero-temperature operator $\alpha^\alpha(t)$ by the Bogoliubov transformation

$$\alpha_\beta^\alpha(t) = U^{-1}(-i\partial_t)^{\alpha\gamma}\alpha^\gamma(t) , \qquad (2.20)$$

where

$$U^{-1}(\omega) \equiv n^{1/2}(\omega) \begin{bmatrix} e^{\beta\omega/2} & -\sigma \\ -\sigma & e^{\beta\omega/2} \end{bmatrix} , \qquad (2.21)$$

and which is normalized for bosons (B) and fermions (F) as

$$U_B(\omega)\tau U_B(\omega) = \tau, \quad U_F(\omega) U_F^{-1}(\omega) = 1 . \qquad (2.22)$$

In the above equations, we defined

$$\tau^{\alpha\beta} \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad n(\omega) = \frac{1}{e^{\beta\omega} - \rho} , \qquad (2.23)$$

where ρ is $+1$ (-1) for bosons (fermions). One also has the inverse transformation

$$\alpha^{\alpha}(t)=U(-i\partial_t)^{\alpha\beta}\alpha^{\gamma}_{\beta}(t) , \quad (2.24)$$

which can also be expressed in the canonical form

$$\alpha^{(\sim)}=e^{iG(\beta)}\alpha^{(\sim)}_{\beta}e^{-iG(\beta)} , \quad (2.25)$$

with generator $G(\beta)$ satisfying

$$G(\beta)=G^{\dagger}(\beta)=-\tilde{G}(\beta) , \quad (2.26)$$

The thermal vacuum is then given as

$$|0(\beta)\rangle=e^{-iG(\beta)}|0,\tilde{0}\rangle , \quad (2.27)$$

which can be shown to reproduce the expression (2.5) by explicit computation of the generator $G(\beta)$.

The canonical formalism presented so far accommodates a perturbation theory analogous to the zero-temperature case. In particular, N-point correlation functions in the Heisenberg representation are given by the following Gell-Mann–Low formula:[7]

$$\langle 0(\beta)|T\phi_1^{\alpha_1}(x_1)\cdots\phi_n^{\alpha_n}(x_n)|0(\beta)\rangle=\frac{\left\langle\Phi(\beta)\left|T\phi_1^{\alpha_1}(x_1)\cdots\phi_n^{\alpha_n}(x_n)\exp\left[i\int_{-\infty}^{\infty}d^4x\,\hat{\mathcal{L}}_I(x)\right]\right|\Phi(\beta)\right\rangle}{\left\langle\Phi(\beta)\left|T\exp\left[i\int_{-\infty}^{\infty}d^4x\,\hat{\mathcal{L}}_I(x)\right]\right|\Phi(\beta)\right\rangle} , \quad (2.28)$$

in which $|\Phi(\beta)\rangle$ denotes the (free) thermal vacuum since the right-hand side of (2.28) is in the interaction representation. The above relation together with Wick's theorem yield the Feynman rules of the theory at finite temperature. The rules are similar to the zero-temperature case except for the fact that vertices now carry thermal indices which should be summed over and that propagators are now finite-temperature causal propagators. That this is so is due to the normal-ordering procedure in the Wick expansion. Normal ordering should be carried out with respect to the thermal operators α_β and $\tilde{\alpha}_\beta$ instead of the corresponding zero-temperature operators since the vacuum is the thermal vacuum. This is the origin of temperature-dependent factors in the causal propagator of our Feynman rules. It has been shown that these Feynman rules are equivalent to those obtained in the complex-time or path-ordering method under a suitable choice of the complex-time integration contour.[12,13,35–37]

Finite-temperature free propagators have the following matrix from and spectral representation in TFD:

$$\Delta_0^{\alpha\beta}(x,y)\equiv i\langle\Phi(\beta)|T\phi^{\alpha}(x)\phi^{\beta}(y)|\Phi(\beta)\rangle=\frac{1}{(2\pi)^4}\int d^4p\,e^{ipx}\Delta_0^{\alpha\beta}(p) , \quad (2.29)$$

where

$$\Delta_0^{\alpha\beta}(p)\left[U_B(|p_0|)\frac{\tau}{p^2+M_B{}^2-i\tau\delta}U_B(|p_0|)\right]^{\alpha\beta}=-\int_{-\infty}^{\infty}d\omega\,\sigma(\omega,\mathbf{p};M_b)\left[U_B(\omega)\frac{\tau}{p_0-\omega+i\tau\delta}U_B(\omega)\right]^{\alpha\beta} , \quad (2.30)$$

for real bosons, and

$$S_0^{\alpha\beta}(x,y)\equiv i\langle\Phi(\beta)|\psi^{\alpha}(x)\bar{\psi}^{\beta}(y)|\Phi(\beta)\rangle=\frac{1}{(2\pi)^4}\int d^4p\,e^{ipx}S^{\alpha\beta}(p) , \quad (2.31)$$

where

$$\begin{aligned}S_0^{\alpha\beta}(p)&=(-i\not{p}+M_F)\left[U_F(|p_0|)\frac{1}{p^2+M_F{}^2-i\tau\delta}U_F{}^{-1}(|p_0)\right]^{\alpha\beta}\\&=-\int_{-\infty}^{\infty}d\omega\,\sigma(\omega,\mathbf{p};M_F)(-\not{\omega}-\not{\mathbf{p}}+M_F)\left[U_F(\omega)\frac{1}{p_0-\omega+i\tau\delta}U_F{}^{-1}(\omega)\right]^{\alpha\beta} ,\end{aligned} \quad (2.32)$$

for fermions. The Bogoliubov matrices $U_B(\omega)$ and $U_F(\omega)$ are given by Eq. (2.21) as

$$U_B(\omega)=n_B{}^{-1/2}(\omega)\begin{bmatrix}e^{\beta\omega/2} & 1\\ 1 & e^{\beta\omega/2}\end{bmatrix} \quad (2.33)$$

and

$$U_F(\omega)=n_F{}^{-1/2}(\omega)\begin{bmatrix}e^{\beta\omega/2} & -i\\ -i & e^{\beta\omega/2}\end{bmatrix} , \quad (2.34)$$

with the choice $\sigma=-1$ $(-i)$ for bosons (fermions).[38]

The spectral function $\sigma(\omega,\mathbf{p};M)$ is also obtained as

$$\begin{aligned}\sigma(\omega,\mathbf{p};M)&=\frac{1}{2\omega_p}[\delta(\omega-\omega_p)-\delta(\omega+\omega_p)] ,\\ \omega_p{}^2&\equiv\mathbf{p}^2+M^2 .\end{aligned} \quad (2.35)$$

In general, one can show that if a full thermal Green's function possesses a spectral representation

$$\Delta^{11}(k_0,\mathbf{k})=\int_0^{\infty}d\omega\,\rho(\omega,\mathbf{k})\Delta_0{}^{11}(k_0,\omega) , \quad (2.36)$$

with a real spectral function $\rho(\omega,\mathbf{k})$, and $\Delta_0(k)$ given by

(2.30) with ω_k replaced by ω, then the propagator

$$\Delta^{11}(t,\mathbf{x})=\frac{1}{(2\pi)^4}\int d^4k\, e^{ikx}\Delta^{11}(k_0,\mathbf{k}) , \qquad (2.37)$$

satisfies the KMS condition.[11] A similar theorem naturally also holds for fermion propagators.

Before discussing the path-integral quantization problem, one would like to point out that, in general, the free propagators (2.30) and (2.32) can be expressed as a sum of zero-temperature and temperature-dependent parts. Since the latter part of the propagator is on the mass shell with a factor $\delta(k^2+M^2)$, it seems equivalent at first glance to use either one of the matrices $U_{B,F}(|k_0|)$ or $U_{B,F}(\omega_k)$ in (2.30) and (2.32). However, when computing chain diagrams similar to those occurring in the expression for the self-energy, divergences caused by the pinching of the integration contour can be created because of powers of free propagators at the same energy-momenta k. These singularities are multiple poles around the mass shell, $\delta^N(k^2+M^2)$ and occur only at finite temperature. It has been shown,[9,39] however, that the cancellation of these δ^N-singularities occurs if the prescription $U_{B,F}(|k_0|)$ is chosen instead of $U_{B,F}(\omega_k)$. This enables the Bogoliubov transformation matrices to factorize and to appear only at both ends of a chain diagram. This also implies that the full propagators have spectral representations similar to the free case.

Finally, we close this section with some comments on the path-integral quantization and functional methods in TFD.

A generating functional in quantum field theory has the property

$$\langle 0|T[\phi(x_1)\cdots\phi(x_n)\psi(y_1)\cdots\psi(y_m)\bar\psi(z_1)\cdots\bar\psi(z_l)]|0\rangle=\frac{i^{-n}\delta^n}{\delta J(x_n)\cdots\delta J(x_1)}\frac{i^{-m}\vec{\delta}^m}{\delta\bar\eta(y_1)\cdots\delta\bar\eta(y_m)}\times\frac{i^{-l}\overleftarrow{\delta}^l}{\delta\eta(z_l)\cdots\delta\eta(z_1)}Z[J,\eta,\bar\eta]\Bigg|_{J=\eta=\bar\eta=0} , \qquad (2.38)$$

where the external sources J,η commuting and anticommuting c numbers, respectively. Note that the symbols $\vec{\delta}/\delta\bar\eta$ and $\overleftarrow{\delta}/\delta\eta$ mean that the respective functional derivatives operate from the left and the right of the generating functional.

At finite temperature, in the context of the operator formalism of TFD, an explicit expression for the generating functional of interacting bosons and fermions fields has been obtained in Ref. 11. It is given by

$$\hat{Z}[J,\eta,\bar\eta]=\hat{N}^{-1}\exp\left[i\int d^4x\left\{\mathcal{L}_I\left[\frac{-i\delta}{\delta J},\frac{-i\delta}{\delta\eta},\frac{-i\delta}{\delta\bar\eta}\right]-\mathcal{L}_I^*\left[\frac{i\delta}{\delta\tilde J},\frac{i\delta}{\delta\tilde\eta},\frac{i\delta}{\tilde{\bar\eta}}\right]\right\}\right]$$
$$\times\exp\left[i\int d^4x\, d^4y\left\{\tfrac{1}{2}[J(x)\tau\Delta_0(x-y)\tau J(y)]^{\alpha\alpha}+[\bar\eta(x)\tau S_0(x-y)\tau\eta(y)]^{\alpha\alpha}\right\}\right] , \qquad (2.39)$$

in which summation should be carried out over repeated thermal indices and where

$$J^\alpha\equiv\begin{bmatrix}J\\ \tilde J\end{bmatrix},\quad \eta^\alpha\equiv\begin{bmatrix}\eta\\ \tilde\eta^*\end{bmatrix},\quad \bar\eta^\alpha\equiv\begin{bmatrix}\bar\eta\\ \tilde{\bar\eta}^*\end{bmatrix} . \qquad (2.40)$$

Also the propagators and the matrix τ have been defined already in Eqs. (2.23), (2.29), and (2.31). The Feynman rules obtained from (2.39) are very similar to those obtained at zero temperature. This is the main computational advantage of TFD. Considering only the real boson case for simplicity, one also shows that the finite-temperature generating functional of Eq. (2.39) can be rewritten in the following path-integral representation:

$$\hat{Z}[J]\propto\int\prod_{\alpha,x}[d\phi^\alpha(x)]\exp\left[i\int d^4x[\tfrac{1}{2}\phi^\alpha(x)\Delta_0^{-1\alpha\beta}(-i\partial)\phi^\beta(x)+\mathcal{L}_I(\phi^1)-\mathcal{L}_I^*(\phi^2)+J^\alpha(x)\tau^{\alpha\beta}\phi^\beta(x)]\right] . \qquad (2.41)$$

Note that the inverse propagator $\Delta_0^{-1}(-i\partial)$ is defined as

$$\Delta_0^{-1}(-i\partial)e^{ikz}\equiv\Delta_0^{-1}(k)e^{ikx} . \qquad (2.42)$$

At zero temperature, the inverse propagator is diagonal,

$$\Delta_0^{-1}(k)=(k^2+M^2)\tau-i\delta , \qquad (2.43)$$

and consequently the two sectors ϕ_1 and ϕ_2 decouple and lead to two independent zero-temperature theories. What makes the theory described by (2.41) nontrivial with respect to the thermal information is the fact that the Feynman convergence factor ($-i\delta$ term) emerges in a nondiagonal form when inverting the thermal propagator (2.30). It is through this boundary condition that temperature comes in, since this nondiagonal Feynman term mixes both sectors ϕ^1 and ϕ^2,

$$\Delta_0^{-1}(k)=(k^2+M^2)\tau -i\delta[\tau U_B(|k_0|)U_B(|k_0|)\tau] . \qquad (2.44)$$

Similar reasoning also holds for the fermion case.

We have now described a consistent and very elegant real-time formalism of QFT at finite temperature.

In the next section we discuss general features of gauge fixed Becchi-Rouet-Stora-Tyutin- (BRST) invariant bosonic string field theories and carry out covariant quantization in the path-integral method. We shall see that even though a covariant operator (or canonical) formalism is not available at the present time, the statistical mechanics of string field theory can be formulated in the path-integral representation in a way analogous to the one discussed in this section for ordinary QFT.

III. STRING FIELD THEORY

Thus far, canonical quantization of string field theory has been performed only in the light-cone gauge.[22–26] The unfortunate lack of a manifestly covariant operator formalism for string fields is an obstacle for the study of the statistical mechanics of such theories since the light-cone gauge is not a suitable gauge for which thermal effects can be introduced.

All covariant formulations[27–30] of various kinds of string field theories have been expressed in the path-integral representation. We shall therefore seek a way of describing the quantum statistical mechanics of string field in such a representation. In this section we restrict our discussion to bosonic string fields only.

In general, a (bosonic) string field Φ is a functional of the string coordinate $X_\mu(\sigma)$ (Refs. 22–30), of Faddeev-Popov (FP) ghosts and antighosts $c(\sigma)$ and $b(\sigma)$, and also, in some cases,[29] of an unphysical length parameter α $(-\infty<\alpha<\infty)$,

$$\Phi=\Phi[X_\mu(\sigma),c(\sigma),b(\sigma);(\alpha)] \; . \tag{3.1}$$

The coordinates $X_\mu(\sigma)$, $c(\sigma)$, and $b(\sigma)$ are the first-quantized operators of a string theory with reparametrization invariance of the world sheet. In such a theory, constraints must be imposed to ensure unitarity in the physical sector of the Hilbert space. Such a Hilbert space is the Fock space $\mathcal{F}$ of the states spanned by the oscillator modes of the string and ghost coordinates. The oscillator algebra for open bosonic strings,

$$\begin{aligned}&[\alpha_m^\mu,\alpha_m^\nu]=ng^{\mu\nu}\delta_{m+n,0}, \quad [x^\mu,p^\nu]=ig^{\mu\nu} \; ,\\ &\{c_n,c_m\}=\{b_n,b_m\}=0, \quad \{c_n,b_m\}=\delta_{n+m,0} \; ,\end{aligned} \tag{3.2}$$

which follows directly from the first quantization of $X_\mu(\sigma)$, $c(\sigma)$, and $b(\sigma)$ (with appropriate boundary conditions), usually spans states in a space with an indefinite metric since the time components α_m^0 have a minus sign in their commutation relations. A subsidiary condition (first-class constraint) is then imposed on the Hilbert space in order that the physical subspace be free from the negative-norm states. Such a condition takes the form

$$Q\,|\,\chi_{\text{phys}}\rangle=0 \; , \tag{3.3}$$

where Q is the canonical Becchi-Rouet-Stora-Tyutin (BRST) charge associated with the reparametrization invariance. In a free string theory, the nilpotency of the BRST generator is equivalent to the Virasoro algebra for which the central extension (the anomaly) vanishes, a case known to occur only for $D=26$ dimensions in the bosonic case.[40] Note that an algebra similar to (3.2) holds for the closed-string case, the only difference being the presence of left and right movers in the algebra. In a Lagrangian formulation of string field theory, the BRST charge must be conserved. In other words, the action must be invariant under the BRST transformation. For the interacting case, however, the BRST transformation is a nonlinear one.[29,30] Nilpotency of the nonlinear BRST transformation severely restricts the form of the interaction. Gauge-fixed BRST-invariant actions have been successfully constructed[29,30] for both the open and the closed bosonic string cases.

In general, a field functional of the string coordinates can be viewed as a Dirac bracket,

$$\Phi=\langle X_\mu(\sigma),c(\sigma),b(\sigma)\,|\,\Phi\rangle \; , \tag{3.4}$$

where $|\,\Phi\rangle$ is a state in the Fock space of the BRST first-quantized string in the oscillator basis (3.2). This in turn implies that one can expand the ket $|\,\Phi\rangle$ as

$$|\,\Phi\rangle=\sum_s |\,s\,\rangle\Phi_s \; , \tag{3.5}$$

where $|\,s\,\rangle$ denotes a state of the string (including zero modes).

In the light-cone gauge of a second-quantized theory,[24] the coefficient Φ_s characterizing the state of the field $|\,\Phi\rangle$ is a second-quantized field operator acting on the Hilbert space $\mathcal{H}$ of the physical states. The space $\mathcal{H}$ is also a Fock space which is spanned by the action of creation operators $A_s^\dagger$ on the vacuum state. The creation and annihilation operators in such a formalism create and annihilate entire strings and should not be confused with the operators of the oscillator algebra of the single string modes. Statistical averages in the context of canonical quantization of string field theory should involve a trace operation taken among the states of the space $\mathcal{H}$, not those of the space $\mathcal{F}$ for the oscillator modes of the string. As mentioned earlier, however, the light-cone gauge is not a suitable gauge to describe thermodynamics. Consequently, we regard Φ_s in Eq. (3.5) as a c-number field for which second quantization is carried out in the path-integral formalism. Thermodynamics will be introduced in the next section through boundary conditions.

A gauge-fixed BRST-invariant action is, in general, a functional of a field ϕ_s,

$$S=S[\phi_s] \; , \tag{3.6}$$

which is usually expressed as

$$S=S_0[\phi_s]+S_I[\phi_s] \; , \tag{3.7}$$

where S_0 and S_I are the free and interaction actions, respectively, and in which the ghost zero-modes have been integrated over or eliminated by the gauge-fixing procedure (this is why we use ϕ_s instead of Φ_s). Recently, two gauge-fixed actions have been proposed for the open bosonic string field theory and both seem to lead to consistent theories (except for the tachyon problem) reproducing the scattering amplitudes of the covariant first quantized bosonic string theory. While one model is based on Witten's action for open string with a cubic interaction term,[30] the other uses a different representation for vertices and contains a quartic interaction term in addition to the cubic term.[29] Accordingly, the BRST transformation takes a slightly more complicated form in the latter model. In both cases, the nilpotency of the BRST transformation is achieved. While in the first model this is done with the use of the equations of motion, nilpotency is obtained off-shell in the second case, pending the introduction of a new unphysical "string length" parameter α. It has been shown, however, that on-shell physical amplitudes are independent of α to all orders in perturbation theory.[29] A closed-string model with cubic interaction has also been proposed.[29] Since our purpose here is not to discuss the specifics of the gauge-fixing procedure of proposed models but rather to show in general how one can extend such models to finite temperature situations, we shall not dwell further upon their detailed characteristics. It is enough to remark that, in general, the gauge-fixed action (3.7) can be written in the form

$$S_0[\phi_s]=\tfrac{1}{2}\sum_{s_1 s_2}\Delta^{-1}{}_{s_1 s_2}\phi_{s_2}\phi_{s_1} \tag{3.8}$$

for the free part and

$$S_I[\phi_s]=\tfrac{2}{3}g\sum_{s_1 s_2 s_3}V_{s_1 s_2 s_3}\phi_{s_3}\phi_{s_2}\phi_{s_1}\left[+\tfrac{2}{4}g^2\sum_{s_1 s_2 s_3 s_4}V_{s_1 s_2 s_3 s_4}\phi_{s_4}\phi_{s_3}\phi_{s_2}\phi_{s_1}\right] \tag{3.9}$$

for the interaction part and where the large parentheses in (3.9) means that in some models the quartic coupling may be omitted. In the open-string case, internal-symmetry quantum numbers may also be incorporated in the theory by letting the string fields be matrix values in a given representation of the internal- (Yang-Mills) symmetry group. For an orientable string, the internal-symmetry group is restricted to be U(N).

String field correlation functions can be obtained from the path-integral formula[30]

$$\langle\langle\phi_{s_1}\phi_{s_2}\cdots\phi_{s_n}\rangle\rangle \equiv \frac{\int\prod_s d\phi_s(\phi_{s_1}\phi_{s_2}\cdots\phi_{s_n})\exp(iS[\phi_s])}{\int\prod_s d\phi_s\exp(iS[\phi_s])}, \tag{3.10}$$

in which one must assume BRST invariance of the functional integral measure. To obtain the Feynman rules of this field theory, one adds usual source terms $i\sum_s J_s\phi_s$ to the action, express the moments as well as the interaction action in terms of functional derivatives of external currents, and perform the Gaussian functional integrations given by the free part of the action. The result is given by the expression

$$\langle\langle\phi_{s_1}\phi_{s_2}\cdots\phi_{s_n}\rangle\rangle=\frac{i^{-n}\delta^n}{\delta J_{s_n}\cdots\delta J_{s_2}\delta J_{s_1}}Z[J]\bigg|_{J=0}, \tag{3.11}$$

where

$$Z[J]=N^{-1}\exp\left[iS_I\left[\frac{-i\delta}{\delta J_s}\right]\right]Z_0[J], \tag{3.12}$$

in which $Z_0[J]$ has been obtained as

$$Z_0[J]=\exp\left[\frac{i}{2}\sum_{s_1 s_2}J_{s_2}\Delta_{s_2 s_1}J_{s_1}\right]. \tag{3.13}$$

Equations (3.11)–(3.13) yield the Feynman rules of the theory.

Note that the Feynman convergence factor must be included in the propagator $\Delta_{s_2 s_1}$. This is a necessary prescription if one wants to perform the Gaussian integrations explicitly.

A useful expression for the free string field propagator is

$$\Delta_{s_1 s_2}=\alpha'\left\langle z_1\left|\frac{1}{L_0-\alpha(0)-i\alpha'\delta}\right|z_2\right\rangle(2\pi)^D\delta(k_1+k_2), \tag{3.14}$$

where

$$L_0-\alpha(0)\equiv\alpha'(k_1{}^2+M^2), \tag{3.15}$$

and in which the mass operator is given as

$$M^2=\frac{1}{\alpha'}\left[\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n+\sum_{n=1}^{\infty}n(c_{-n}b_n+b_{-n}c_n)\right]-\frac{\alpha(0)}{\alpha'}, \tag{3.16}$$

where α' is the Regge slope and $\alpha(0)$ is the intercept of a given trajectory.

In the expression (3.14), the states $|z_1\rangle$ and $|z_2\rangle$ denote the states of the string modes in the oscillator basis with the string zero modes k_1^μ and k_2^μ excluded. Contact with earlier notation is established through the relations

$$\phi[s]=\phi_s=\langle s\,|\,\phi\rangle=\langle z\,|\,\phi(k)\rangle=\phi_z(k). \tag{3.17}$$

Therefore one has

$$\langle\langle\phi_{s_1}\phi_{s_2}\rangle\rangle\equiv\langle z_1\,|\,\langle\langle\,|\phi(k_1)\rangle\langle\phi(k_2)|\,\rangle\rangle\,|z_2\rangle. \tag{3.18}$$

In view of (3.14) one can now define the following propagator in an operator form:

$$\Delta(k_1,k_2)\equiv\langle\langle\,|\phi(k_1)\rangle\langle\phi(k_2)|\,\rangle\rangle =\alpha'\Delta(k_1)(2\pi)^D\delta(k_1+k_2), \tag{3.19}$$

where

$$\Delta(k)\equiv\frac{1}{L_0-\alpha(0)-i\alpha'\delta}. \tag{3.20}$$

The propagator (3.20) is a very useful operator and appears regularly in the computation of string scattering amplitudes in the covariant formulation of the first-quantized string theory (BRST formalism).

One is now ready to introduce statistical mechanics in our second quantized string theory.

IV. STRING FIELD THEORY AT FINITE TEMPERATURE AND KMS CONDITION

In this section we shall formulate the string field theory at finite temperature in the path-integral quantization outlined in the previous section and with help from the real-time formalism (TFD) presented in Sec. II. In order to appreciate the similarities between string field theory and ordinary quantum field theory, let us write down the following expression for the generating functional of the free string field theory:

$$Z_0[J]=N^{-1}\int \prod_s d\phi_s \exp(iS_0[\phi_s]+iJ_s\phi_s)\ . \tag{4.1}$$

Recalling the decomposition (3.17) for the zero-mode and realizing that the functional integral measure in (4.1) involves an infinite product over string modes z and zero-modes k (or x), that is,

$$\prod_s d\phi_s=\prod_z\prod_k d\phi_z(k)\ , \tag{4.2}$$

and realizing that the free string propagator is diagonal, one can then rewrite the generating functional (4.1) as a product over z of the functionals

$$Z_0[J]=\prod_z Z_0[J;z]\ , \tag{4.3}$$

where

$$Z_0[J;z]\equiv N_z{}^{-1}\int \prod_k d\phi_z(k)\exp\left[i\sum_k \phi_z(k)\Delta^{-1}(k;z)\phi_z(-k)+i\sum_k J_z(k)\phi_z(-k)\right]\ . \tag{4.4}$$

One easily recognizes that $Z_0[J;z]$ coincides with the generating functional of an ordinary field theory describing a free particle of mass $M\{z\}$ labeled by the string mode z. The statistical mechanics for such a case has been described in Sec. II. The prescription is now clear. We first introduce unphysical fields $\tilde{\phi}_s$ and currents $\tilde{J}_s$ in the theory and define, following (2.17), the total action $\hat{S}_J$ as

$$\hat{S}_J\equiv S[\phi_s]-S^*[\tilde{\phi}_s]+J_s\phi_s-\tilde{J}_s\tilde{\phi}_s\ . \tag{4.5}$$

Temperature now formally enters the picture through the Feynman boundary term. Since it takes similar form for all the string modes z [see Eq. (2.44)] it is introduced in a universal way through the operator form (3.20) of the free propagator. Introducing thermal doublet notations

$$\phi_s^\alpha=\begin{bmatrix}\phi_s\\ \tilde{\phi}_s\end{bmatrix},\quad J_s^\alpha=\begin{bmatrix}J_s\\ \tilde{J}_s\end{bmatrix}\ , \tag{4.6}$$

one then has

$$\langle\!\langle\ |\phi^\alpha(k_1)\rangle\langle\phi^\beta(k_2)|\ \rangle\!\rangle=\alpha'(2\pi)^D\delta(k_1+k_2)\Delta^{\alpha\beta}(k_1)\ , \tag{4.7}$$

where

$$\Delta^{\alpha\beta}(k)\equiv\left[U_B(|k_0|)\frac{\tau}{L_0-\alpha(0)-i\alpha'\tau\delta}U_B(|k_0|)\right]^{\alpha\beta}\ , \tag{4.8}$$

in which the Bogoliubov transformation matrix $U_B(\omega)$ is given by Eq. (2.33).

Finite-temperature string field correlation functions are now readily expressed as

$$\langle\langle\phi_{s_1}^{\alpha_1}\cdots\phi_{s_n}^{\alpha_n}\rangle\rangle_\beta\equiv\frac{i^{-n}\epsilon^{\alpha_1}\cdots\epsilon^{\alpha_n}\delta^n}{\delta J_{s_n}^{\alpha_n}\cdots\delta J_{s_1}^{\alpha_1}}\hat{Z}[J]\bigg|_{J=0}\ , \tag{4.9}$$

where

$$\hat{Z}[J]=\hat{N}^{-1}\exp\left\{iS_I\left[\frac{-i\delta}{\delta J_s}\right]-S_I^*\left[\frac{i\delta}{\delta\tilde{J}_s}\right]\right\}\hat{Z}_0[J]\ , \tag{4.10}$$

in which $\hat{Z}_0[J]$ is given explicitly by

$$\hat{Z}_0[J]=\exp\left[\frac{i}{2}\sum_{s_1s_2,\alpha}(J_{s_1}\tau\Delta_{s_2s_1}\tau J_{s_1})^{\alpha\alpha}\right]\ . \tag{4.11}$$

The form of the interacting action in (4.10) is motivated by tilde conjugation rules (2.9)–(2.11).

Note that $\tau^{\alpha\beta}$ has been defined by Eq. (2.23) and ϵ^α is $+1$ (-1) for $\alpha=1$ (2). The results (4.9)–(4.11) are obtained directly from the following functional integral form for $\hat{Z}[J]$:

$$\hat{Z}[J]=\hat{N}^{-1}\int\prod_{\alpha=1}^{2}\prod_s d\phi_s^\alpha\exp(iS[\phi]-iS^*[\tilde{\phi}]+iJ_s\phi_s-i\tilde{J}_s\tilde{\phi}_s+\text{boundary term})\ . \tag{4.12}$$

It is interesting to note that, in the zero-slope limit ($\alpha'\to0$), bosonic string field theory is known to reproduce (in the open-string case) the $\lambda\varphi^3$ theory or the Yang-Mills theory at zero temperature. That it is so is due to the fact that all massive string modes become infinitely massive in such a limit and decouple from the dynamics leaving an effective theory of self-interacting massless particles. At finite temperature, since thermal effects are introduced only through boundary conditions, one can see that our formalism reduces to the statistical mechanics of a gas of self-interacting scalar bosons or non-Abelian gauge vector particles in the limit $\alpha'\to0$.

The Feynman rules of our string field theory are therefore given by Eqs. (4.7)–(4.11). The rules are similar to the zero-temperature case.

Before closing this section we would like to comment on the choice of the boundary term (2.44) for the free thermal propagator. Rewriting string field correlation functions in the space representation of the zero-mode, one defines the function

$$\Delta^{>}_{z_1 z_2}(t) \equiv \langle \langle \phi_{z_1}(t) \phi_{z_2}(0) \rangle \rangle \tag{4.13}$$

and

$$\Delta^{<}_{z_1 z_2}(t) \equiv \langle \langle \phi_{z_1}(0) \phi_{z_2}(t) \rangle \rangle , \tag{4.14}$$

where t is the string zero mode x^0 (quantum-mechanical time).

The KMS condition states that, in order to describe equilibrium thermodynamics, the following relations should be satisfied:

$$\Delta^{>}_{z_1 z_2}(t) = \Delta^{<}_{z_1 z_2}(t - i\beta) \tag{4.15}$$

and

$$\Delta^{<}_{z_1 z_2}(t) = \Delta^{>}_{z_1 z_2}(t + i\beta) , \tag{4.16}$$

in which β is the c-number inverse temperature. It is important to note that the c-number (imaginary) shift of the time zero mode cannot be obtained from reparametrization, that is, it is not generated by the string Hamiltonian

$$H = L_0 + \text{const.} \tag{4.17}$$

The generator of c-number space-time translation is the D-momentum zero mode:

$$P_\mu = \int_0^\pi \frac{d\sigma}{\pi} P_\mu(\sigma) = p_\mu . \tag{4.18}$$

This is why the zero mode $|p_0|$ appears in the Bogoliubov transformation matrices in (4.8) instead of the generator L_0 of reparametrization, which one might have naively chosen to describe the density matrix (Boltzmann factor) $e^{-\beta H}$. This justifies the boundary term (2.44) for our case. The form (4.8) for the propagator has been shown[11] to satisfy the KMS conditions (4.15) and (4.16). Moreover, the Hamiltonian (4.17) is a Lorentz-invariant operator, a quality known to be broken at finite-temperature in ordinary field theory (our β is not an invariant temperature). The "choice" $|p_0|$ also ensures that one recovers correct dispersion relations for the relativistic particles which emerge from the string theory in the zero-slope limit.

In the next section we apply the finite-temperature Feynman rules obtained in this section to explicit computations of string amplitudes.

V. COMPUTATION OF STRING AMPLITUDES AT FINITE TEMPERATURE

In this section we apply the finite-temperature formalism developed in the previous section to practical problems such as the computation of string scattering amplitudes. As an explicit example we calculate the four-tachyon scattering amplitude for the open bosonic string in the tree approximation following the steps of a similar zero-temperature computation by Thorn.[30] We then indicate how the finite-temperature Feynman rules carry over to the first-quantized (BRST-)covariant formalism and then check the equivalence of the results with the second-quantized formalism. We also compute the one-loop correction to the four-tachyon amplitude.

Following Thorn,[30] the four-tachyon tree amplitude for the s-t diagram $[s \equiv -(k_1+k_2)^2;\ t \equiv -(k_2+k_3)^2]$ is given as (modulo a symmetry factor)

$$A_4(s,t) = g^2 V_{k_2 a k_1} \Delta_{ba} V_{k_4 b k_3} , \tag{5.1}$$

where $V_{k_j a k_i}$ is the vertex function describing the coupling of two on-shell tachyons to intermediate string states $|a\rangle$, and Δ_{ba} is the propagator for such intermediate states. A summation over the states $|b\rangle$ and $|a\rangle$ is implicit in Eq. (5.1). The list of states which contribute to the first three poles of the amplitude (5.1) is given as ($\alpha'=1$ in Thorn's convention)

$$\begin{aligned} &|0;p\rangle,\quad M^2=-1 ,\\ &a^\mu_{-1}|0;p\rangle,\quad M^2=0 ,\\ &a^\mu_{-1}a^\nu_{-1}|0;p\rangle,\quad a^\mu_{-2}|0;p\rangle, c_{-1}b_{-1}|0;p\rangle,\quad M^2=1 . \end{aligned} \tag{5.2}$$

Corresponding propagators and vertices have been worked out and are given by

$$\Delta_{ba}(p,p') = \frac{1}{p^2-1}\delta(p+p')(2\pi)^D,\quad M^2=-1 , \tag{5.3}$$

$$\Delta_{ba}(p,p') = \frac{g^{\mu\nu}}{p^2}\delta(p+p')(2\pi)^D,\quad M^2=0 , \tag{5.4}$$

$$\Delta_{ba}(p,p') = \frac{g^{\rho\mu}g^{\sigma\nu}+g^{\rho\nu}g^{\sigma\mu}}{4(p^2+1)}(2\pi)^D\delta(p+p'),\quad \frac{-g^{\mu\nu}}{2(p^2+1)}\delta(p+p')(2\pi)^D,\quad \frac{-1}{p^2+1}\delta(p+p')(2\pi)^D,\quad M^2=1 , \tag{5.5}$$

and

$$V_{k_2 p k_1} = \left[\frac{4}{3\sqrt{3}}\right]^{p^2-1}\delta(p+k_1+k_2),\quad M^2=-1 , \tag{5.6}$$

$$V_{k_2 p k_1} = \left[\frac{4}{3\sqrt{3}}\right]^{p^2}\frac{1}{\sqrt{2}}(k_2-k_1)^\mu\delta(p+k_1+k_2),\quad M^2=0 , \tag{5.7}$$

$$V_{k_2pk_1}=\begin{cases}\left[\dfrac{4}{3\sqrt{3}}\right]^{p^2+1}[\frac{1}{2}(k_2-k_1)^\mu(k_2-k_1)^\nu-\frac{5}{16}g^{\mu\nu}]\delta(p+k_1+k_2),\\ \left[\dfrac{4}{3\sqrt{3}}\right]^{p^2+1}\left[-\dfrac{3\sqrt{2}}{8}k_2^\mu\right]\delta(p+k_1+k_2), \qquad M^2=1\ ,\\ \left[\dfrac{4}{3\sqrt{3}}\right]^{p^2+1}[\frac{11}{16}]\delta(p+k_1+k_2)\ .\end{cases} \tag{5.8}$$

Insertion of the propagators and vertices (5.3)–(5.8) into Eq. (5.1) yields[30] [dropping $(2\pi)^D$ factors and δ functions for energy-momentum conservation]

$$A_4(s,t)=g^2(\tfrac{16}{27})^{-s}\left[\frac{27}{16}\frac{1}{-s-1}+(2+t+\tfrac{1}{2}s)\frac{1}{-s}+\frac{16}{27}\left[\frac{t^2}{2}+2t+\frac{st}{2}+\frac{s^2}{8}+\frac{45}{64}s+\frac{139}{64}\right]\frac{1}{-s+1}\right]+\cdots\ , \tag{5.9}$$

where the dots stand for contributions from higher-order massive states and where $k_i^2=1$ in the units for which $\alpha'=1$.

At finite temperature, according to the rules (4.7)–(4.11), vertices and propagators now carry thermal indices. Therefore, $V_{s_1s_2s_3}$ and $\Delta_{s_1s_2}$ are now replaced by

$$gV_{s_1s_2s_3}\to gV^{\alpha\beta\gamma}_{s_1s_2s_3}=\epsilon^\alpha\delta^{\alpha\beta\gamma}gV_{s_1s_2s_3} \tag{5.10}$$

and

$$\Delta_{z_1z_2}(p,p')\to\Delta^{\alpha\beta}_{z_1z_2}(p,p')=\left[U_B(|p_0|)\frac{\tau}{p^2+M_{z_1}{}^2-i\tau\delta}U_B(|p_0|)\right]^{\alpha\beta}\delta(p+p')\delta_{z_1z_2}(2\pi)^D\ , \tag{5.11}$$

in which $\delta^{\alpha\beta\gamma}$ is 1 for $\alpha=\beta=\gamma$ and zero otherwise. The amplitude (5.1) now becomes

$$A_4^{\alpha_1\alpha_2\alpha_3\alpha_4}(s,t)=g^2V^{\alpha_2\beta\alpha_1}_{k_2ak_1}\Delta^{\beta\gamma}_{ba}V^{\alpha_4\gamma\alpha_3}_{k_4bk_3}\ , \tag{5.12}$$

where one should sum over repeated thermal and state indices. Since the physical amplitude is given by $\alpha_i=1$ for all i, one gets

$$A_4^{1111}(s,t)=g^2V^{111}_{k_2ak_1}\Delta^{11}_{ba}V^{111}_{k_4bk_3}\ . \tag{5.13}$$

Recalling Eq. (2.33) for the Bogoliubov transformation matrix $U_B(|p_0|)$, the propagator (5.11) can be written as

$$\Delta^{\alpha\beta}_{z_1,z_2}(p,p')=\left[\frac{\tau}{p^2+M_{z_1}{}^2-i\tau\delta}+\frac{2\pi i\delta(p^2+M_{z_1}{}^2)}{e^{\beta|p_0|}-1}\begin{bmatrix}1 & e^{\beta|p_0|/2}\\ e^{\beta|p_0|/2} & 1\end{bmatrix}\right]\delta(p+p')(2\pi)^D\delta_{z_1,z_2}\ , \tag{5.14}$$

where $\delta_{z_1z_2}$ may have a complicated tensor structure.

Given the above considerations for the finite-temperature case and recalling the zero-temperature expression (5.9) for the four-tachyon amplitude, one obtains

$$A_4^{1111}(s,t)=g^2\left[\frac{16}{27}\right]^{-s}\left[\frac{27}{16}\left[\frac{1}{-s-1}+\frac{2\pi i\delta(s+1)}{e^{\beta|k_{0_1}+k_{0_2}|}-1}\right]+\left[2+t+\frac{s}{2}\right]\left[\frac{1}{-s}+\frac{2\pi i\delta(s)}{e^{\beta|k_{0_1}+k_{0_2}|}-1}\right]\right.$$
$$\left.+\frac{16}{27}\left[\frac{t^2}{2}+2t+\frac{st}{2}+\frac{s^2}{8}+\frac{45}{64}s+\frac{139}{64}\right]\left[\frac{1}{-s+1}+\frac{2\pi i\delta(s-1)}{e^{\beta|k_{0_1}+k_{0_2}|}-1}\right]\right]+\cdots\ . \tag{5.15}$$

Therefore,

$$A_4^{1111}(s,t)=A_4(s,t)+g^2\left[\frac{2\pi i}{e^{\beta|k_{0_1}+k_{0_2}|}-1}\{\delta(\alpha(s))+[\alpha(t)+1]\delta(\alpha(s)-1)+\tfrac{1}{2}[\alpha(t)+1][\alpha(t)+2]\delta(\alpha(s)-2)+\cdots\}\right], \tag{5.16}$$

where

$$\alpha(s)\equiv\alpha's+\alpha(0)\ , \tag{5.17}$$

in which $\alpha'=\alpha(0)=1$.

It is easy to see that the expression (5.16) together with (5.9) reproduces the first three poles of the amplitude

$$A_4^{1111}(s,t)=g^2\left[B(-\alpha(s),-\alpha(t))+\frac{2\pi i}{e^{\beta|k_{0_1}+k_{0_2}|}-1}\sum_{n=0}^{\infty}\frac{1}{n!}[\alpha(t)+1][\alpha(t)+2]\cdots[\alpha(t)+n]\delta(\alpha(s)-n)\right], \quad (5.18)$$

where $B(a,b)$ is the Euler beta function:

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} . \quad (5.19)$$

We now show that the result (5.18) can be obtained through the first-quantized (BRST-)covariant string formalism by letting the finite-temperature Feynman rules of the previous section be carried over to the first-quantized operator formalism. This in turn enables us to obtain finite-temperature expressions for various string theories (such as fermionic strings and superstrings) for which a field theory has not yet been constructed.

In the operator formalism of string theory,[1] vertices and propagators are operators acting on the Fock space of the string modes. Correspondence with the rules (4.7)–(4.11) suggests the following:

$$V(k)\to V^{\alpha\beta\gamma}(k)=\begin{cases}V(k), & \alpha=\beta=\delta=1 ,\\ -V^*(k), & \alpha=\beta=\delta=2 ,\\ 0 & \text{otherwise}\end{cases} \quad (5.20)$$

and

$$\Delta(p)\to\Delta^{\alpha\beta}(p)=\left[U_B(|p_0|)\frac{\tau}{L_0-1-i\alpha'\tau\delta}U_B(|p_0|)\right]^{\alpha\beta} , \quad (5.21)$$

where L_0 is given by (3.15) and (3.16). The physical on-shell four-tachyon amplitude in the first-quantized operator formalism is then written as

$$A_4^{\text{FT}}(s,t)=g^2\langle 0;k_1|V^{111}(k_2)\Delta^{11}V^{111}(k_3)|0;k_4\rangle , \quad (5.22)$$

where the tachyon vertex $V(k)$ is given as

$$V(k)=:e^{ik\cdot X(1)}: , \quad (5.23)$$

in which $X^\mu(z)$ is defined as

$$X^\mu(z)=x^\mu-ip^\mu\ln z+i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_n^\mu z^{-n}-\alpha_{-n}^\mu z^n) . \quad (5.24)$$

Inserting (5.24) into (5.23) and taking into account normal ordering, the vertex $V(k)$ is obtained as

$$V(k)=Z(k)W(k) , \quad (5.25)$$

where

$$Z(k)=e^{ik\cdot x} \quad (5.26)$$

and

$$W(k)=\exp\left[k\cdot\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\right]\exp\left[-k\cdot\sum_{n=1}^{\infty}\frac{1}{n}\alpha_n\right] . \quad (5.27)$$

Note that we chose the convention $\alpha'=\frac{1}{2}$ and set $\alpha(0)=1$ in the expression for propagators and vertices of the first-quantized operator formalism. From now on we shall stick to this convention. Rewriting the 1-1 matrix element of the propagator (5.21) as

$$\Delta^{11}(p)=\frac{e^{\beta|p_0|}}{e^{\beta|p_0|}-1}\frac{1}{L_0-1-i\alpha'\delta}-\frac{1}{e^{\beta|p_0|}-1}\frac{1}{L_0-1+i\alpha'\delta} , \quad (5.28)$$

which can also be expressed in the parametrized form

$$\Delta^{11}(p)=\int_0^1 dx\left[\frac{e^{\beta|p_0|}}{e^{\beta|p_0|}-1}x^{L_0-2-i\alpha'\delta}-\frac{1}{e^{\beta|p_0|}-1}x^{L_0-2+i\alpha'\delta}\right] , \quad (5.29)$$

the amplitude (5.22) is now rewritten as

$$\begin{aligned}A_4^{\text{FT}}(s,t)&=g^2\left\langle 0\left|\exp\left[k_2\cdot\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\right]\exp\left[-k_2\cdot\sum_{n=1}^{\infty}\frac{1}{n}\alpha_n\right]\Delta^{11}(k_1+k_2)\exp\left[k_3\cdot\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\right]\exp\left[-k_3\cdot\sum_{n=1}^{\infty}\frac{1}{n}\alpha_n\right]\right|0\right\rangle\\&=g^2\int_0^1 dx\left[\left[\frac{e^{\beta|k_{0_1}+k_{0_2}|}}{e^{\beta|k_{0_1}+k_{0_2}|}-1}x^{\alpha'(k_1+k_2)^2-2-i\alpha'\delta}-\frac{1}{e^{\beta|k_{0_1}+k_{0_2}|}-1}x^{\alpha'(k_1+k_2)^2-2+i\alpha'\delta}\right]\right.\\&\qquad\left.\times\prod_{n=1}^{\infty}\prod_{\mu=0}^{D-1}\left\langle 0\left|\exp\left[-k_{2\mu}\frac{\alpha_n^\mu}{n}\right]x^{\alpha_{-n\mu}\alpha_n^\mu}\exp\left[k_3^\mu\frac{\alpha_{-n\mu}}{n}\right]\right|0\right\rangle\right] ,\end{aligned} \quad (5.30)$$

where we used the commutator (3.2) for the zero modes together with the Baker-Campbell-Hausdorff formula as well as the energy-momentum-conservation law $\sum_i k_i^\mu=0$. Applying well-known coherent states techniques, the amplitude is obtained as

$$A_4^{\text{FT}}(s,t)=g^2\int_0^1 dx\left[\left[\frac{e^{\beta|k_{0_1}+k_{0_2}|}}{e^{\beta|k_{0_1}+k_{0_2}|}-1}x^{-\alpha(s)-i\alpha'\delta-1}-\frac{1}{e^{\beta|k_{0_1}+k_{0_2}|}-1}x^{-\alpha(s)+i\alpha'\delta-1}\right](1-x)^{-\alpha(t)-1}\right] . \quad (5.31)$$

Recalling that

$$B(a,b)=\int_0^1 dx\, x^{a-1}(1-x)^{b-1}\ , \tag{5.32}$$

one gets immediately

$$A_4^{\mathrm{FT}}(s,t)=g^2\left[\frac{e^{\beta|k_{0_1}+k_{0_2}|}}{e^{\beta|k_{0_1}+k_{0_2}|}-1}B(-\alpha(s)-i\alpha'\delta,-\alpha(t))-\frac{1}{e^{\beta|k_{0_1}+k_{0_2}|}-1}B(-\alpha(s)+i\alpha'\delta,-\alpha(t))\right]\ . \tag{5.33}$$

Expressing the Euler beta function as an expansion about the poles in the s channel and making use of the formula

$$2\pi i\delta(y)=\frac{1}{y-i\delta}-\frac{1}{y+i\delta}\ , \tag{5.34}$$

one obtains finally

$$A_4^{\mathrm{FT}}(s,t)=g^2\left[B(-\alpha(s),-\alpha(t))+\frac{2\pi i}{e^{\beta|k_{0_1}+k_{0_2})}-1}\sum_{n=0}^{\infty}\frac{1}{n!}(\alpha(t)+1)(\alpha(t)+2)\cdots(\alpha(t)+n)\delta(\alpha(s)-n)\right]\ , \tag{5.35}$$

a result which agrees with the tachyon amplitude (5.16)–(5.18) of string field theory (except, of course, for the different choice of the slope parameter α').

Next we consider one-loop corrections to (5.35) making use of the first-quantized operator formalism extended to the finite-temperature case. The planar one-loop four-tachyon physical amplitude is given as[1]

$$B_4^{\mathrm{FT}}(k_1,k_2,k_3,k_4)=g^4\int d^Dp\ \mathrm{Tr}[\Delta^{11}V(k_1)\Delta^{11}V(k_2)\Delta^{11}V(k_3)\Delta^{11}V(k_4)]\ , \tag{5.36}$$

where the integration over the loop momentum can be seen as the trace operation over the zero mode p^μ. Making use of the Baker-Campbell-Hausdorff formula and the conservation law $\sum_i k_i^\mu=0$, the planar amplitude (5.36) is rewritten as

$$B_4^{\mathrm{FT}}(k_1,k_2,k_3,k_4)=g^4\int d^Dp\ \mathrm{Tr}[\Delta^{11}(p_1)W(k_1)\Delta^{11}(p_2)W(k_2)\Delta^{11}(p_3)W(k_3)\Delta^{11}(p_4)W(k_4)]\ , \tag{5.37}$$

where $W(k_i)$ has been given in (5.27) and the momentum p_i^μ is defined as

$$p_i^\mu\equiv p^\mu-(k_1+k_2+\cdots+k_{i-1})^\mu\ . \tag{5.38}$$

Now, inserting the parametrized form (5.29) for the propagator into Eq. (5.37) and given the fact that the trace over the oscillator modes can be evaluated from the formula

$$\mathrm{Tr}(A)=\int\frac{dz\,dz^*}{\pi}e^{-|z|^2}\langle z\,|\,A\,|\,z\rangle\ , \tag{5.39}$$

where $|z\rangle$ is a coherent state, the amplitude (5.37) is obtained explicitly as

$$\begin{aligned}B_4^{\mathrm{FT}}(k_1,k_2,k_3,k_4)=g^4\int d^Dp\int_0^1\frac{dx_1dx_2dx_3dx_4}{w^2}&\frac{\mathrm{Tr}_{\mathrm{gh}}(w^{n(c_{-n}b_n+b_{-n}c_n)})}{(1-e^{-\beta|p_{0_1}|})(1-e^{-\beta|p_{0_2}|})(1-e^{-\beta|p_{0_3}|})(1-e^{-\beta|p_{0_4}|})}\\ &\times\prod_{n=1}^{\infty}\prod_{\mu=0}^{D-1}\int\frac{dz\,dz^*}{\pi}e^{-|z|^2}\left\langle z\left|\prod_{r=1}^{4}\exp\left[\frac{k_{r\mu}\alpha_{-n}^\mu}{n}\rho_r{}^n\right]\exp\left[\frac{-k_{r\mu}\alpha_n^\mu}{n}\rho_r{}^{-n}\right]\right|w^nz\right\rangle\\ &\times\left[x_1^{\sigma_1}x_2^{\sigma_2}x_3^{\sigma_3}s_4^{\sigma_4}+\left[\sum_{i=1}^{4}e^{-\beta|p_{0_i}|}x_1^{\sigma_1}\cdots x_i^{\sigma_i^*}\cdots x_4^{\sigma_4}\right]\right.\\ &+\left[\sum_{i<j}e^{-\beta|p_{0_i}|}e^{-\beta|p_{0_j}|}x_1^{\sigma_1}\cdots x_i^{\sigma_i^*}\cdots x_j^{\sigma_j^*}\cdots x_4^{\sigma_4}\right]\\ &+\left[\sum_{\substack{i=1\\ i\neq j,k,l}}^{4}e^{-\beta|p_{0_j}|}e^{-\beta|p_{0_k}|}e^{-\beta|p_{0_l}|}x_1^{\sigma_1^*}\cdots x_i^{\sigma_i}\cdots x_4^{\sigma_4^*}\right]\\ &\left.+e^{-\beta|p_{0_1}|}e^{-\beta|p_{0_2}|}e^{-\beta|p_{0_3}|}e^{-\beta|p_{0_4}|}x_1^{\sigma_1^*}x_2^{\sigma_2^*}x_3^{\sigma_3^*}x_4^{\sigma_4^*}\right]\ ,\end{aligned} \tag{5.40}$$

in which

$$\rho_r \equiv x_1 x_2 \cdots x_r , \quad w \equiv \rho_4 , \tag{5.41}$$

and

$$\sigma_i \equiv \alpha' p_i{}^2 - i\alpha' \delta_i . \tag{5.42}$$

In the expression (5.40), the oscillator parts $\alpha_{-n} \cdot \alpha_n$ of the mass operators have been moved to the right to create the coherent state $|w^n z\rangle$.

Making use of the relation

$$e^{-|z|^2} \left\langle z \left| \prod_{r=1}^{4} \exp\left[\frac{k_{r\mu}\alpha^{\mu}_{-n}}{n}\rho_r^n\right] \exp\left[\frac{-k_{r\mu}\alpha^{\mu}_{n}}{n}\rho_r^{-n}\right] \right| w^n z \right\rangle = e^{-(1-w^n)|z|^2} \exp\left[-\sum_{r<s} k_{r\mu}k_s^{\mu}\frac{1}{n}(\rho_s/\rho_r)^n\right] \times \exp\left[\sum_{r=1}^{4} k_{r\mu}\left[\frac{1}{\sqrt{n}}\rho_r{}^n z^* - \frac{1}{\sqrt{n}}(w/\rho_r)^n z\right]\right] , \tag{5.43}$$

which can be obtained from the general properties of coherent states, and recalling the formula for complex Gaussian integration,

$$\int \frac{dz\, dz^*}{\pi} e^{-c|z|^2} e^{(az+bz^*)} = \frac{1}{c} \exp\left[\frac{ab}{c}\right] , \tag{5.44}$$

together with the contribution from the trace over ghost modes,[1]

$$\mathrm{Tr}_{\mathrm{gh}}(w^{n(c_{-n}b_n + b_{-n}c_n)}) = \prod_{n=1}^{\infty} (1-w^n)^2 \equiv [f(w)]^2 , \tag{5.45}$$

the planar amplitude (5.40) takes the form

$$\begin{aligned} B_4^{\mathrm{FT}}(k_1,k_2,k_3,k_4) = g^4 \int d^D p \int_0^1 & \frac{dx_1 dx_2 dx_3 dx_4}{w^2} [f(w)]^{2-D} \exp\left[\sum_{r<s} k_r \cdot k_s \ln(\psi'_{rs})\right] \\ & \times (1-e^{-\beta|p_{0_1}|})^{-1}(1-e^{-\beta|p_{0_2}|})^{-1}(1-e^{-\beta|p_{0_3}|})^{-1}(1-e^{-\beta|p_{0_4}|})^{-1} \\ & \times \Bigg[x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} x_4^{\sigma_4} + \left[\sum_{i=1}^{4} e^{-\beta|p_{0_i}|} x_1^{\sigma_1} \cdots x_i^{\sigma_i^*} \cdots x_4^{\sigma_4}\right] \\ & + \left[\sum_{i<j} e^{-\beta|p_{0_i}|} e^{-\beta|p_{0_j}|} x_1^{\sigma_1} \cdots x_i^{\sigma_i^*} \cdots x_j^{\sigma_j^*} \cdots x_4^{\sigma_4}\right] \\ & + \left[\sum_{\substack{i=1 \\ i\neq j,k,l}}^{4} e^{-\beta|p_{0_j}|} e^{-\beta|p_{0_k}|} e^{-\beta|p_{0_l}|} x_1^{\sigma_1^*} \cdots x_i^{\sigma_i} \cdots x_4^{\sigma_4^*}\right] \\ & + e^{-\beta|p_{0_1}|} e^{-\beta|p_{0_2}|} e^{-\beta|p_{0_3}|} e^{-\beta|p_{0_4}|} x_1^{\sigma_1^*} x_2^{\sigma_2^*} x_3^{\sigma_3^*} x_4^{\sigma_4^*} \Bigg] , \end{aligned} \tag{5.46}$$

where we defined

$$\ln(\psi'_{rs}) \equiv -\sum_{n=1}^{\infty} \frac{c_{sr}^n + (w/c_{sr})^n - 2w^n}{n(1-w^n)} , \tag{5.47}$$

in which

$$c_{sr} \equiv \rho_s/\rho_r = x_{r+1} \cdots x_s . \tag{5.48}$$

In the zero-temperature limit, only the $x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} x_4^{\sigma_4}$ term survives in (5.46) and the momentum integration can be performed explicitly (Gaussian integral) to yield known results.[1] At finite temperature, on the other hand, the momentum integration is nontrivial because of the presence of Boltzmann factors $e^{-\beta|p_{0_i}|}$. One should then perform the integrations over the parameters x_i first, leaving the integral a meromorphic function of p_0. One is finally left with integrals over the spatial components of the momentum. A more detailed analysis of the result (5.46) will be given elsewhere.

VI. SUMMARY

In this paper we presented a real-time finite-temperature formalism for covariant bosonic string field theory in the context of the path-integral quantization and obtained the Feynman rules by making use of functional methods. In such a formalism, temperature comes into the theory through boundary conditions imposed after having performed an effective doubling of the string field degrees of freedom. This is a natural generalization of the thermo field dynamics formalism of ordinary field theory as is easily recognized when one recalls that string field theory can be viewed as the field theory of an infinite number of particle species classified by their masses. This is expressed mathematically by the fact that the free string field generating functional (4.3) is an infinite product over generating functionals of ordinary fields. Our formalism is therefore expected to reproduce the finite-temperature quantum field theory of massless particles in the zero-slope limit. We also argued on the basis of the KMS condition that the Boltzmann factors should be constructed from the generator of c-number time translation p_0, rather than the Lorentz-invariant generator of world-sheet reparametrization L_0. This also ensures correct dispersion relations in the zero-slope limit.

We then proceeded to compute the four-tachyon scattering amplitude in the tree approximation making use of the rules obtained earlier and following closely a recent computation by Thorn in the context of Witten's model for bosonic string field theory. We further indicated how the rules developed for covariant string field theory carry over to the first-quantized (BRST-)covariant formalism and reproduce the amplitude computed from the string field theory model. We finally obtained an expression for the planar one-loop four-tachyon amplitude.

Although the lack of a canonical covariant formalism is a setback when trying to build the statistical mechanics of string fields on a more rigorous basis, we believe, nevertheless, that the formalism presented here is consistent and reproduces desirable features such as correct finite-temperature field theories in the zero-slope limit. Also, it shares similar computational efficiency with ordinary thermo field dynamics. That this is so is due to the fact that the Feynman rules at finite temperature are very similar to the zero-temperature case.

With the tools developed in this paper we can now systematically address important physical issues such as the existence of the so-called Hagedorn[2,41,42] or maximum temperature in string systems as well as phase transitions and extension to situations departing from thermodynamical equilibrium. Given the physical relevance of superstrings and their low-energy limits[43] to the problems of (super-) grand-unification, symmetry breaking, and cosmic evolution, it may be very desirable to compute the Gibbs free energy[44] for such models as well as consider the renormalization problem of the string tension[45] from a finite-temperature renormalization-group approach.[10] This is work for the future.

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