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Particles, superparticles, and twistors

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The covariant Green-Schwarz action for a superstring has never been quantized covariantly. The physics behind this is discussed. We then consider the corresponding point-particle action in four dimensions, and write down a master action from which it can be obtained as a gauge choice: the "space-time gauge." There is also a "twistor gauge," in which covariant quantization is straightforward (as noted previously by Shirafuji).

I. INTRODUCTION

Many authors have expressed the feeling that there should be a way of quantizing the superstring covariantly, while keeping its supersymmetry manifest. The only solid result in this direction seems to be the Green-Schwarz action.[1] However, all conventional methods of covariant quantization fail when applied to this action because of a problem in its massless sector[2] (the authors of Ref. 3 disagree with us). It appears to be widely assumed that this has to do with superspace technicalities, but it seems to us that the heart of the matter lies elsewhere: namely, in the relation

$$P^0>0\ , \tag{1}$$

which is characteristic of supersymmetric theories. In the present context, this means that the spectrum contains particles only, since antiparticles have negative energy and propagate backwards in time, in any first-quantized theory.[4] Now, it is well known that it is impossible to define a covariant position operator for spinning particles[5] unless negative-energy states are admitted; the best that can be done covariantly is to define position operators that do not commute, and therefore do not yield a sensible space-time description. Although we have found no explicit statement to this effect in the literature, we believe that even this latter option is excluded for massless particles. It is basically this problem which plagues the Green-Schwarz action. Presumably, any first-quantized action which leads to a phase space in which condition (1) holds everywhere would have a similar problem. Note, however, that the problem would be avoided in a light-front formulation, where the situation is a bit peculiar, and the above-mentioned properties of antiparticles do not hold;[6] manifest supersymmetry does not cause any difficulties on the light front.

The problem can be studied in the simpler setting of the superparticle action, given in Ref. 7. Technically, what happens is that the massive superparticle contains a spinor of second-class constraints. This ensures that the covariant coordinates do not commute—that this in itself is an unsatisfactory situation was stressed in Ref. 8. Since the number of degrees of freedom in a massless supermultiplet is only one-half that of a massive multiplet, one-half of these constraints become first class in the massless limit.[9,10] However, it is quite obvious in $D=10$ dimensions, and almost as obvious in $D=4$, that the two types of constraints cannot be separated from each other covariantly, and consequently, covariant methods of quantization fail to be covariant.[2]

It is not clear what the most efficient response to this situation is. One could try to include the negative-energy states from the beginning, and then proceed along the lines of Ref. 11, say; certainly a difficult task in the string case. It is conceivable that one would end up by deriving the Neveu-Schwarz-Ramond (NSR) formalism in this way. Another possibility, at least for massless particles, is to give up any direct space-time description and quantize in twistor space.[12] It has been shown by Shirafuji that the massless superparticle in $D=4$ can be covariantly quantized in this way.[9] It is not clear how far Shirafuji's treatment would carry in the case of superstrings in $D=10$, but there is some scattered evidence that twistor methods could be useful for strings also.[13,14] There have also been attempts to include extra degrees of freedom in the action;[15] we have not studied them, nor their physical interpretation.

The purpose of this paper is to provide the logical steps connecting Shirafuji's treatment with the conventional formulation; there is a problem here, since Shirafuji essentially uses the formulas of Ref. 16 to translate between twistor space and space-time. However, the x of Ref. 16 is by assumption a space-time coordinate, whereas the x that occurs in the superparticle action is endowed with dynamical properties inconsistent with such a role. It is clear that the connection between the two formulations is slightly indirect; quantum mechanically, we are dealing with operators acting in two distinct Hilbert spaces. We address the problem by writing down a kind of master action, from which both formulations can be obtained by choosing appropriate gauges, or, more precisely, by using Dirac's procedure[17] to consistently remove unwanted degrees of freedom from phase space. Since the reader may be unfamiliar with twistors, we start out with a discussion of the spinless case in Secs. II and III before dealing with the more difficult case of the superparticle in Sec. IV. Our notation for two-component spinors is that of Ref. 18.

II. THE SPINLESS PARTICLE

The description of a massless particle, which we refer to as the conventional one, is defined by the action[19]

$$S=\int d\tau(-\dot{x}^{\dot{\alpha}\alpha}P_{\alpha\dot{\alpha}}+\tfrac{1}{2}V_1P_{\alpha\dot{\alpha}}P^{\dot{\alpha}\alpha}) . \quad (2)$$

Since we are in $D=4$, we use the two-component formalism throughout. Our notation combines that of Bagger and Wess and that of Newton, so that an overdot is used both to distinguish between spinor indices and to denote a derivative with respect to τ. We hope that this will not confuse the reader. In the Hamiltonian analysis of this action one finds the constraint

$$P^2=0 . \quad (3)$$

Note that this is a Casimir operator of the Poincaré group—the physical subspace is defined by a mass-shell condition. The twistor description[12] is given by the action

$$S=\int d\tau[\dot{\omega}^{\alpha}\bar{\psi}_{\alpha}+\dot{\bar{\omega}}^{\dot{\alpha}}\psi_{\dot{\alpha}}+iV_2(\omega^{\alpha}\bar{\psi}_{\alpha}-\bar{\omega}^{\dot{\alpha}}\psi_{\dot{\alpha}})] , \quad (4)$$

which leads to the constraint

$$\frac{i}{2}(\omega^{\alpha}\bar{\psi}_{\alpha}-\bar{\omega}^{\dot{\alpha}}\psi_{\dot{\alpha}})=0 . \quad (5)$$

This is actually, although not obviously, the other Casimir operator, viz., the helicity. The physical subspace is now given by a spin shell condition. The translation between the two pictures is effected by the formulas

$$P_{\alpha\dot{\alpha}}=\bar{\psi}_{\alpha}\psi_{\dot{\alpha}},\quad \omega^{\alpha}=\psi_{\dot{\alpha}}x^{\dot{\alpha}\alpha},\quad \bar{\omega}^{\dot{\alpha}}=x^{\dot{\alpha}\alpha}\psi_{\alpha} , \quad (6)$$

where ψ is a commuting spinor. A four-dimensional lightlike vector, such as the momentum of a massless particle, can always be written in the above way, and that is, in a way, the basic fact on which the twistor formalism rests.

Equations (6) can hold only on shell, since they imply the on-shell conditions (3) and (5). Also they imply that P is future directed—the opposite case could be taken care of by the insertion of a few signs. Now consider the action

$$S=\int d\tau[-\dot{x}^{\dot{\alpha}\alpha}P_{\alpha\dot{\alpha}}+\dot{\omega}^{\alpha}\bar{\psi}_{\alpha}+\dot{\bar{\omega}}^{\dot{\alpha}}\psi_{\dot{\alpha}}+\tfrac{1}{2}V_1P_{\alpha\dot{\alpha}}P^{\dot{\alpha}\alpha}+iV_2(\omega^{\alpha}\bar{\psi}_{\alpha}-\bar{\omega}^{\dot{\alpha}}\psi_{\dot{\alpha}})+\pi^{\dot{\alpha}\alpha}(P_{\alpha\dot{\alpha}}-\bar{\psi}_{\alpha}\psi_{\dot{\alpha}})$$
$$+\bar{\Lambda}_{\alpha}(\omega^{\alpha}-\psi_{\dot{\alpha}}x^{\dot{\alpha}\alpha})+\Lambda_{\dot{\alpha}}(\bar{\omega}^{\dot{\alpha}}-x^{\dot{\alpha}\alpha}\bar{\psi}_{\alpha})] . \quad (7)$$

Although this action contains an explicit x, it is translation invariant since ω, being not a spinor but the second component of a twistor $Z^{\alpha}=(\omega^{\alpha},\psi_{\dot{\alpha}})$, transforms nontrivially under translations.

Varying with respect to the Lagrange multipliers we obtain the constraints

$$\begin{aligned}
&P^2=-P_{\alpha\dot{\alpha}}P^{\dot{\alpha}\alpha}=0 ,\\
&T_{\alpha\dot{\alpha}}=P_{\alpha\dot{\alpha}}-\bar{\psi}_{\alpha}\psi_{\dot{\alpha}}=0 ,\\
&g^{\alpha}=\omega^{\alpha}-\psi_{\dot{\alpha}}x^{\dot{\alpha}\alpha}=0 ,\\
&\bar{g}^{\dot{\alpha}}=\bar{\omega}^{\dot{\alpha}}-x^{\dot{\alpha}\alpha}\bar{\psi}_{\alpha}=0 ,\\
&h=i\omega^{\alpha}\bar{\psi}_{\alpha}-i\bar{\omega}^{\dot{\alpha}}\psi_{\dot{\alpha}}=0 .
\end{aligned} \quad (8)$$

Incidentally, g^{α} is called the associated spinor field of the twistor Z^{α}. Using the naive Poisson brackets, it is straightforward to show that these constraints form a closed algebra, i.e., they are all first class, and the fact that they are inconsistent with the brackets does not matter. In the gauge $V_1=V_2=1$, $\pi=\Lambda=\bar{\Lambda}=0$, the Hamiltonian becomes

$$H=-\tfrac{1}{2}P_{\alpha\dot{\alpha}}P^{\dot{\alpha}\alpha}+i(\omega^{\alpha}\bar{\psi}_{\alpha}-\bar{\omega}^{\dot{\alpha}}\psi_{\dot{\alpha}}) . \quad (9)$$

Now it is often said that a first-class constraint can be used to reduce the number of physical degrees of freedom by two. It is clear that if this were true without qualifications, we would be in trouble. To convince the reader that we are in fact not in trouble, we pause to discuss this point. The general statement about gauge conditions is that whenever a condition is added to the theory, two things should be checked: that the condition is consistent with the equations of motion and that it does not entail any unphysical restriction on the set of their solutions. When, in the treatment of the action (2), the gauge $V_1=1$ is chosen for the Lagrange multipliers, these conditions are satisfied, since in fact the equations of motion leave V_1 undetermined. The so-called proper-time gauge (or the light-front gauge) is quite a different matter. It is a condition which enables one to treat the constraint $P^2=0$ as strongly valid, in the sense of Dirac. The condition is

$$x^0=P^0\tau . \quad (10)$$

However, this is not gauge fixing in the usual sense; rather, it means that one has found essentially the most general solution to the equation of motion for x^0 and replaced x^0, wherever it appears, by this explicit solution. Then the Hamiltonian formalism is no longer needed as far as x^0 is concerned, and this variable is quietly dropped from phase space, without having been assigned any fixed value. It is a gauge choice in the sense that it enables Dirac brackets compatible with itself and $P^2=0$ to be computed, though, and we will stick to this terminology. Note that it would be consistent to set $x^0=0$ instead; it would restrict the theory rather severely, though, since the equations of motion would then imply that the x^i's are constant.

Having, we hope, clarified matters to some extent, we turn to the task of reducing our phase space in a way consistent with the constraints (8). Suppose, to begin with, that we want to turn $g=\bar{g}=0$ into strong equalities. The appropriate gauge conditions are

$$\bar{\psi}_\alpha=\bar{\psi}_{\alpha f}, \quad \psi_{\dot\alpha}=\psi_{\dot\alpha f} , \tag{11}$$

where $\bar{\psi}_{\alpha f}$ is the general solution to the Hamiltonian equations of motion for $\bar{\psi}_\alpha$ (its explicit form will not be needed). Then

$$\{\bar{\psi}_\alpha-\bar{\psi}_{\alpha f}, g^\beta\}=-\delta_\alpha{}^\beta . \tag{12}$$

Since this is an invertible matrix, Dirac brackets compatible with $g=0$ can now be computed. It is clear that the brackets involving x and P only will be unaffected by this. It is also clear that the constraint $T_{\alpha\dot\alpha}=0$ entails no restriction on $P_{\alpha\dot\alpha}$, beyond what we already know from the fact that $P^2=0$. Since $h=0$ is a consequence of $g=\bar{g}=0$, we see that we end up with precisely the same Hamiltonian system for x and P as that following from the conventional action (2). In addition, we have been reminded of the fact that the twistor functions ψ and ω can always be defined.

This shows how the conventional phase space can be consistently recovered from our "master action." For obvious reasons, we call the conditions (11) "the space-time gauge." The "twistor gauge," on the other hand, is defined by

$$x_{\alpha\dot\alpha}=x_{\alpha\dot\alpha f} , \tag{13}$$

where $x_{\alpha\dot\alpha f}$ is a general solution of the equations of motion for $x_{\alpha\dot\alpha}$, whose explicit form will not be needed. Since

$$\{x^{\dot\alpha\alpha}-x^{\dot\alpha\alpha}{}_f, T_{\beta\dot\beta}\}=-\delta^\alpha{}_\beta\delta^{\dot\alpha}{}_{\dot\beta} \tag{14}$$

is an invertible matrix, we can now compute Dirac brackets that enable us to treat $T_{\alpha\dot\alpha}=0$ as a strong equation. The brackets among the ψ's and ω's are unaffected by this, and we end up with the Hamiltonian system that one obtains from the action (4), having been reminded that the functions x and P can always be defined (should this have been forgotten by the twistor devotee).

Thus we have obtained a kind of bird's-eye view of the two formalisms. It may seem as if we have been inventing a complicated procedure for doing the obvious, but we will see that the procedure is in fact useful in the superparticle case, since it helps in getting the physical interpretation right—and this is not obvious for the superparticle.

III. SYMMETRIES OF THE MASTER ACTION

The Noether charges for the Poincaré algebra which follows from our action (7) are

$$T_n , \quad J_{mn}=x_mP_n-x_nP_m-\omega\sigma_{mn}\bar{\psi}+\psi\bar{\sigma}_{mn}\bar{\omega} . \tag{15}$$

It is clear that they give a Poisson-brackets realization of the Poincaré algebra when the naive Poisson brackets are used. It is, however, at first sight a little disconcerting that they all vanish weakly; in fact it is readily shown that

$$J_{mn}=x_mT_n-x_nT_m-g\sigma_{mn}\bar{\psi}+\psi\bar{\sigma}_{mn}\bar{g} . \tag{16}$$

We are used to Hamiltonians that vanish weakly, but Poincaré generators that do the same thing are less usual. Moreover, the Poincaré generators in the space-time gauge certainly do not vanish. In order to convince the reader that we do not have a problem on our hands, we digress a little bit more to point out another subtlety of the kind of gauge fixing that is exemplified by the proper-time gauge (similar issues arise in ordinary gravity,[20] but in that case, the problem is somewhat obscured by the complexity of the surrounding calculations). Before the proper-time gauge condition was added to the theory, the Hamiltonian was P^2, and this was weakly zero. Afterwards, the Hamiltonian is

$$H=P^0 , \tag{17}$$

which is certainly nonzero. The way one convinces oneself that no inconsistency is at hand is to check that the new Hamiltonian generates, via the Dirac brackets, the same Hamiltonian equations of motion for the variables x and P, as the original Hamiltonian did via the naive brackets:

$$\{f(x^i,P^i),P^0\}^*=-\{f,\tfrac{1}{2}P^2\}C^{-1}\{x^0,P^0\}$$
$$=\frac{1}{P^0}\{f(x^i,P^i),\tfrac{1}{2}P^2\} . \tag{18}$$

(There is a rescaling of the evolution parameter.) Since the whole point about gauge fixing in the present sense is that it should not change the set of solutions of the equations of motion of the theory in any way, this is what one has to show.

In our case, an even simpler calculation will convince us that the conventional, nonzero, space-time gauge Lorentz generator

$$L_{mn}=x_mP_n-x_nP_m \tag{19}$$

will, via the Dirac brackets, generate the same transformations of the variables x and P as the generator J_{mn} did originally. Again, this is precisely what one needs to show. In the twistor gauge, we similarly require that the transformation properties of ψ and ω remain unchanged.

So the conclusion is that the Poincaré generators are included in the rigid gauge algebra, when field-dependent coefficients are admitted. However, this property of the Poincaré generators is of no great importance in the present context. It is sufficiently amusing to merit some further investigation, though. To this end we study the local gauge symmetries of the action (7). Although these do not play any role as far as the canonical structure at the Hamiltonian level is concerned—apart from enabling it to be derived—they may be suggestive. What one finds is that the action is invariant under the following transformations, where the parameters are τ dependent:

$$\begin{aligned}
&\delta x^{\dot\alpha\alpha}=\xi\dot x^{\dot\alpha\alpha}+t^{\dot\alpha\alpha}\ ,\\
&\delta P_{\alpha\dot\alpha}=\xi\dot P_{\alpha\dot\alpha}+\bar\lambda_\alpha\psi_{\dot\alpha}+\bar\psi_\alpha\lambda_{\dot\alpha}\ ,\\
&\delta\bar\psi_\alpha=\xi\dot{\bar\psi}_\alpha+\bar\lambda_\alpha+i\phi\bar\psi_\alpha\ ,\\
&\delta\psi_{\dot\alpha}=\xi\dot\psi_{\dot\alpha}+\lambda_{\dot\alpha}-i\phi\psi_{\dot\alpha}\ ,\\
&\delta\omega^\alpha=\xi\dot\omega^\alpha=\psi_{\dot\alpha}t^{\dot\alpha\alpha}+\lambda_{\dot\alpha}x^{\dot\alpha\alpha}-i\phi\omega^\alpha\ ,\\
&\delta\bar\omega^{\dot\alpha}=\xi\dot{\bar\omega}^{\dot\alpha}+t^{\dot\alpha\alpha}\bar\psi_\alpha+x^{\dot\alpha\alpha}\lambda_\alpha+i\phi\bar\omega^{\dot\alpha}\ ,\\
&\delta V_1=(\xi\dot V_1)\ ,\\
&\delta V_2=(\xi\dot V_2)+\dot\phi\ ,\\
&\delta\pi^{\dot\alpha\alpha}=(\xi\dot\pi^{\dot\alpha\alpha})+\dot t^{\dot\alpha\alpha}-V_1\bar\lambda^\alpha\psi^{\dot\alpha}-V_1\psi^\alpha\bar\lambda^{\dot\alpha}\ ,\\
&\delta\bar\Lambda_\alpha=(\xi\dot{\bar\Lambda}_\alpha)+\dot{\bar\lambda}^\alpha-iV_2\bar\lambda_\alpha+i\phi\bar\Lambda_\alpha\ ,\\
&\delta\Lambda_{\dot\alpha}=(\xi\dot\Lambda_{\dot\alpha})+\dot\lambda_{\dot\alpha}+iV_2\lambda_{\dot\alpha}-i\phi\Lambda_{\dot\alpha}\ ,\\
&\delta\mathcal{L}=\partial(\xi\mathcal{L}+\psi_{\dot\alpha}t^{\dot\alpha\alpha}\bar\psi_\alpha+\omega^\alpha\bar\lambda_\alpha+\bar\omega^{\dot\alpha}\lambda_{\dot\alpha})\ .
\end{aligned}\tag{20}$$

In particular, we recognize the ordinary reparametrization invariance (parameter ξ) and the phase invariance of the twistor formulation (parameter ϕ). There is also a kind of translational gauge symmetry in the x and ψ directions. At first sight, it looks as if x and ψ could be "gauged away," but although this is true, it is beside the point, since if $x_{\alpha\dot\alpha}$ is transformed to zero, a contribution from $\pi_{\alpha\dot\alpha}$ steps in to take its place. Examination of the equations of motion reveal that what really matters are gauge-invariant quantities such as $\dot x_{\alpha\dot\alpha}-\pi_{\alpha\dot\alpha}$, so that there is no problem here. The algebra of these transformations is rather trivial; except the part involving the reparametrizations, the only nonzero commutators are (the top index refers to the type of transformation, the lower to the parameter)

$$[\delta^\lambda_\lambda,\delta^\phi_\phi]=\delta^\lambda_{i\phi\lambda},\quad[\delta^{\bar\lambda}_{\bar\lambda},\delta_\phi]=-\delta^{\bar\lambda}_{i\phi\bar\lambda}\ .\tag{21}$$

Finally, a word on the gauge choices for the Lagrange multipliers that we have been using, since this is a matter where there is some controversy.[21] The point is that a gauge choice such as $V_1=1$, or the orthonormal (ON) gauge for a string, is impossible to implement if one insists that the action should be invariant even when terms at the boundary $\tau=\pm\infty$ are taken into account. However, this simply does not matter as far as the equations of motion are concerned, and quantum mechanically we know that a consistent theory can be set up based on the gauge $V_1=1$ (or the ON gauge for the free string). If the reader nevertheless insists on using more sophisticated gauges, there should be no problems involved in modifying our analysis.

IV. THE SUPERPARTICLE

The treatment of the superparticle is considerably more involved, not because of any difficulties in twistor space, but because of the difficulties in obtaining a space-time picture of a positive-energy massless spinning particle. This will make the translation process somewhat cumbersome, and noncovariant, even though the end result[9] is exceedingly simple. Fortunately it is an overstatement to say that no space-time picture whatsoever exists for such entities;[5] as we will see, a light-front formulation is in fact possible.

We start out from the action

$$\begin{aligned}S=\int d\tau\{&-(x^{\dot\alpha\alpha}-i\dot{\bar\theta}^\alpha\theta^{\dot\alpha}-i\dot\theta^{\dot\alpha}\theta^\alpha)P_{\alpha\dot\alpha}+\dot\omega^\alpha\bar\psi_\alpha+\dot{\bar\omega}^{\dot\alpha}\psi_{\dot\alpha}+\tfrac12V_1P_{\alpha\dot\alpha}P^{\dot\alpha\alpha}+\tfrac12V_2(i\omega^\alpha\bar\psi_\alpha-i\bar\omega^{\dot\alpha}\psi_{\dot\alpha}+2\psi_{\dot\alpha}\theta^{\dot\alpha}\bar\psi_\alpha\bar\theta^\alpha)\\
&+\pi^{\dot\alpha\alpha}(P_{\alpha\dot\alpha}-\bar\psi_\alpha\psi_{\dot\alpha})+\bar\Lambda_\alpha[\omega^\alpha-\psi_{\dot\alpha}(x^{\dot\alpha\alpha}+i\theta^{\dot\alpha}\bar\theta^\alpha)]+\Lambda_{\dot\alpha}[\bar\omega^{\dot\alpha}-(x^{\dot\alpha\alpha}-i\theta^{\dot\alpha}\bar\theta^\alpha)\bar\psi_\alpha]\}\ .\end{aligned}\tag{22}$$

Here θ and $\bar\theta$ are composed of odd Grassmann numbers, and the modification of the action, as compared to the spinless case, is the minimal one consistent with Ref. 7, and the requirement that we shall end up with a first-class algebra.

Choosing the Lagrange multipliers as in the spinless case, gives us the Hamiltonian

$$H=-\tfrac12P_{\alpha\dot\alpha}P^{\dot\alpha\alpha}-\frac{i}{2}(\omega^\alpha\bar\psi_\alpha-\bar\omega^{\dot\alpha}\psi_{\dot\alpha})-\psi_{\dot\alpha}\theta^{\dot\alpha}\bar\psi_\alpha\bar\theta^\alpha\ ,\tag{23}$$

and the constraints

$$\begin{aligned}
&P^2=-P_{\alpha\dot\alpha}P^{\dot\alpha\alpha}=0\ ,\\
&T_{\alpha\dot\alpha}=P_{\alpha\dot\alpha}-\bar\psi_\alpha\psi_{\dot\alpha}=0\ ,\\
&g^\alpha=\omega^\alpha-\psi_{\dot\alpha}(x^{\dot\alpha\alpha}+i\theta^{\dot\alpha}\bar\theta^\alpha)=0\ ,\\
&\bar g^{\dot\alpha}=\bar\omega^{\dot\alpha}-(x^{\dot\alpha\alpha}-i\theta^{\dot\alpha}\bar\theta^\alpha)\bar\psi_\alpha=0\ ,\\
&h=-\frac{i}{2}(\omega^\alpha\bar\psi_\alpha-\bar\omega^{\dot\alpha}\psi_{\dot\alpha})-\psi_{\dot\alpha}\theta^{\dot\alpha}\bar\psi_\alpha\bar\theta^\alpha=0\ ,\\
&\phi_\alpha=P_\alpha+iP_{\alpha\dot\alpha}\theta^{\dot\alpha}=0\ ,\\
&\bar\phi_{\dot\alpha}=\bar P_{\dot\alpha}+i\bar\theta^\alpha P_{\alpha\dot\alpha}=0
\end{aligned}\tag{24}$$

(where P_α and $\bar P_{\dot\alpha}$ are canonical momenta corresponding to $\bar\theta^\alpha$ and $\theta^{\dot\alpha}$). These constraints do not form a closed algebra. The troublesome part of the constraint matrix is the same as that which occurs in the conventional superparticle,[2] viz.,

$$\{\phi_\alpha,\bar\phi_{\dot\alpha}\}=2iP_{\alpha\dot\alpha}\ .\tag{25}$$

Clearly some of the ϕ and $\bar\phi$ are second class, but since the determinant of the right-hand side vanishes, there are some first-class constraints as well. If we choose ϕ_1 and $\bar\phi_{\dot 1}$ as second-class constraints, we find that all the remaining constraints turn first class when the appropriate Dirac brackets are used. However, and inevitably, this means that we have given up covariance. This is the technical reason why the space-time picture of the superparticle is complicated. We will adopt this choice of second-class constraints from now on, and it should be noticed that this means that we have committed ourselves to a light-front description, since there will now occur quantities that are singular when $P_{1\dot{1}}$, that is, P^+, in conventional notation, is zero. Such a singularity makes sense only on the light front (and even there requires considerable care,[22] although we will not discuss this). We suppose that it is necessary to use the light-front frame in order to define commuting coordinates for massless particles.

Having dealt with the second-class constraints, we can now consider going to the "space-time gauge"; the appropriate gauge choice is

$$\bar{\psi}_\alpha=\bar{\psi}_{\alpha f},\quad \psi_{\dot\alpha}=\psi_{\dot\alpha f}\ . \tag{26}$$

Then the conventional superparticle is recovered.

It requires more thought to impose the "twistor gauge." Clearly we expect that the space-time coordinates that will occur in the translation formulas should have a clear geometrical meaning; a minimum requirement is that they can be used to label space-time events. This is not the case with the x's as they stand, however, since they do not commute. In fact their Dirac brackets are

$$\begin{aligned}\{x^{\dot{1}1},x^{\dot{1}2}\}^*&=\frac{i}{2P_{1\dot{1}}}\theta^{\dot{1}}\bar\theta^{2}\ ,\\ \{x^{\dot{1}1},x^{\dot{2}1}\}^*&=\frac{i}{2P_{1\dot{1}}}\bar\theta^{1}\theta^{\dot{2}}\ ,\\ \{x^{\dot{1}2},x^{\dot{2}1}\}^*&=\frac{i}{2P_{1\dot{1}}}\bar\theta^{2}\theta^{\dot{2}}\ .\end{aligned} \tag{27}$$

We observe that if we can use the fermionic gauge symmetry $\phi_2,\bar\phi_{\dot 2}$ to set

$$\bar\theta^{2}=\theta^{\dot 2}=0\ , \tag{28}$$

then the x's will commute, although they are still noncovariant, since the Lorentz generators in this gauge (and forgetting about their twistorial part) are

$$J_{\alpha\dot\alpha\beta\dot\beta}=2(x_{\alpha\dot\alpha}P_{\beta\dot\beta}-x_{\dot\beta\beta}P_{\alpha\dot\alpha})+S_{\alpha\dot\alpha\beta\dot\beta}\ , \tag{29}$$

where the nonzero parts of the spin generator are

$$\begin{aligned}S_{1\dot{2}2\dot{1}}&=-S_{2\dot{1}1\dot{2}}=\bar\theta^{1}P_1-\theta^{\dot 1}\bar P_{\dot 1}=-\xi\bar\xi\ ,\\ S_{2\dot{2}2\dot{1}}&=-S_{2\dot{1}2\dot{2}}=2\bar\theta^{1}P_2=-\dot{1}\frac{P_{2\dot 1}}{P_{1\dot 1}}\xi\bar\xi\ ,\\ S_{2\dot{2}1\dot{2}}&=-S_{1\dot{2}2\dot{2}}=2\theta^{\dot 1}\bar P_{\dot 2}=\dot{1}\frac{P_{1\dot 2}}{P_{1\dot 1}}\xi\bar\xi\end{aligned} \tag{30}$$

(some extra notation, to be introduced in a moment, has been used in the last step).

We will indeed impose condition (28); that this is allowed follows from a study of the solutions to the equations of motion. In fact, there are no equations of motion for $\bar\theta^{2}$ and $\theta^{\dot 2}$, if the coordinate system is chosen in a way consistent with the occurrence of the $P_{1\dot 1}$ singularity.[13] Therefore, they are really on the same footing as the Lagrange multipliers, which shows that they can be set to zero with impunity. In this gauge, then, we do have a set of commuting but noncovariant coordinates, and that is the best that we can do.

In the fermionic sector, an interesting pair of variables are the "quantum superspace" coordinates of Ref. 8:

$$\begin{aligned}\xi&\equiv\left[\frac{2}{P_{1\dot 1}}\right]^{1/2}P_1=-\dot{1}(2P_{1\dot 1})^{1/2}\theta^{\dot 1}\ ,\\ \bar\xi&\equiv\left[\frac{2}{P_{1\dot 1}}\right]^{1/2}\bar P_{\dot 1}=-\dot{1}(2P_{1\dot 1})^{1/2}\bar\theta^{1}\ .\end{aligned} \tag{31}$$

The second equalities hold only when (28) holds. On the other hand, it should be noted that, as defined in terms of p_1 and $\bar p_{\dot 1}$, ξ and $\bar\xi$ are invariant under the fermionic gauge symmetry. Since there are only two physical fermionic degrees of freedom, they provide us with a kind of field-strength formulation of the fermionic sector.

The Dirac brackets among ξ and $\bar\xi$ are at this stage

$$\{\xi,\bar\xi\}^*=-\dot{1},\quad \{\xi,\xi\}^*=\{\bar\xi,\bar\xi\}^*=0\ . \tag{32}$$

Provided that (28) holds, they furthermore "commute" with x. So at this stage they form a pair of dimensionless fermionic creation and annihilation operators, commuting with ψ and ω. It is important to choose the "twistor gauge" in such a way that these properties are maintained. In particular, it requires some care to prevent ξ and $\bar\xi$ from obtaining nonzero brackets with ψ,ω in the last step. This provides a technical reason why the gauge condition

$$x_{\alpha\dot\alpha}=x_{\alpha\dot\alpha f} \tag{33}$$

is unsuitable, unless we also impose (28). If we do impose (28), however, we are done, since then the condition (33) will result in a phase space where only the twistorial variables are left, and they obey the desired relations

$$\begin{aligned}\{\omega^{\alpha},\bar\psi_\beta\}^*&=\delta^{\alpha}{}_{\beta}\ ,\\ \{\bar\omega^{\dot\alpha},\psi_{\dot\beta}\}^*&=\delta^{\dot\alpha}{}_{\dot\beta}\ ,\\ \{\xi,\bar\xi\}^*&=-\dot{1}\ .\end{aligned} \tag{34}$$

Given that the constraint $T_{\alpha\dot\alpha}=0$ now holds strongly, the Hamiltonian can be rewritten

$$\begin{aligned}H&=-\frac{i}{2}(\omega^{\alpha}\bar\psi_\alpha-\bar\omega^{\dot\alpha}\psi_{\dot\alpha})-\psi_{\dot\alpha}\theta^{\dot\alpha}\psi_\alpha\bar\theta^{\alpha}\\ &=-\frac{i}{2}(\omega^{\alpha}\bar\psi_\alpha-\bar\omega^{\dot\alpha}\psi_{\dot\alpha})-\tfrac{1}{2}\xi\bar\xi\ ,\end{aligned} \tag{35}$$

and the "translation constraints" $g=\bar g=0$ take a simple form.

We note that the discussion can be carried through also outside the gauge $\bar\theta^{2}=\theta^{\dot 2}=0$. A set of commuting coordinates that can be used were found in Refs. 23 and 8: namely,

$$q_{\alpha\dot{\alpha}}=x_{\alpha\dot{\alpha}}+\frac{1}{2\eta^{\dot{\alpha}\alpha}p_{\alpha\dot{\alpha}}}S_{\alpha\dot{\alpha}\beta\dot{\beta}}\eta^{\dot{\beta}\beta} , \quad (36)$$

where

$$\eta^{\dot{\alpha}\alpha}=\delta^{\alpha}{}_{1}\delta^{\dot{\alpha}}{}_{\dot{1}} , \quad (37)$$

and $S_{\alpha\dot{\alpha}\beta\dot{\beta}}$ is the spin part of the Lorentz generators in the space-time picture. These coordinates furthermore commute with ξ and $\bar{\xi}$. They were called "gauge-invariant observables" in Ref. 23, although they are not invariant under the fermionic gauge symmetry. Since they commute with ξ and $\bar{\xi}$, the condition

$$q_{\alpha\dot{\alpha}}=q_{\alpha\dot{\alpha}f} \quad (38)$$

can now be used to go to the twistor gauge. Equation (35) still holds, but the drawback of this more general approach is that the constraints $g=\bar{g}=0$ lose their simple form.

The super Poincaré generators in the twistor picture are

$$\begin{aligned} &P_n=\bar{\psi}\sigma_n\psi ,\\ &J_{mn}=-\omega\sigma_{mn}\bar{\psi}+\psi\bar{\sigma}_{mn}\bar{\omega} , \\ &Q_\alpha=\bar{\psi}_\alpha\xi,\quad \bar{Q}_{\dot{\alpha}}=\psi_{\dot{\alpha}}\bar{\xi} . \end{aligned} \quad (39)$$

Since the Pauli-Lubanski spin vector obeys

$$S_m\equiv-\tfrac{1}{2}\epsilon_{mnpq}P^nJ^{pq}=sP_m , \quad (40)$$

where s is the helicity, we can check that the form of the generators, together with the constraint, imply that the helicity operator is

$$s=-\frac{i}{2}(\omega^\alpha\bar{\psi}_\alpha-\bar{\omega}^{\dot{\alpha}}\psi_{\dot{\alpha}})=\tfrac{1}{2}\xi\bar{\xi} . \quad (41)$$

This completes the classical analysis. The notation can be improved, though. If we define the supertwistor[16] $Z^a=(\omega^\alpha,\psi_{\dot{\alpha}},\xi)$ we can rewrite the fundamental Poisson brackets as

$$\{Z^a,\bar{Z}^{\dot{a}}\}^*=g^{a\dot{a}}=\begin{bmatrix}0 & \delta_\alpha{}^{\beta} & 0\\ -\delta_{\dot{\alpha}}{}^{\dot{\beta}} & 0 & 0\\ 0 & 0 & -i\end{bmatrix} , \quad (42)$$

and the "spin shell condition" (35) becomes simply $Z_a g^{a\dot{a}}\bar{Z}_{\dot{a}}=0$. It generates conformal transformations.

Quantization is now entirely straightforward (following Ref. 12, that is). The physical state condition is given by the Hamiltonian (35), i.e., by the superhelicity operator. Shirafuji[9] argues at this point that the model is consistent only if an even number of supersymmetries are present. We disagree with this: when quantizing, it is necessary to order the ψ's and ω's in the Hamiltonian in accordance with Ref. 12, so that the analytical properties of the wave functions are maintained. The oscillators ξ and $\bar{\xi}$, on the other hand, simply span a small superspace and can be ordered at will. Also a constant, multiplied with $\hbar$, can be added to the Hamiltonian in the quantization procedure, so that this gives us a description of an arbitrary $N=1$ supermultiplet. Note that it is really the operator

$$Q_\alpha=\bar{\psi}_\alpha\xi , \quad (43)$$

which takes us from one state in the supermultiplet to the other. As is well known,[12] $\bar{\psi}$ acts as a helicity-raising operator.

V. OUTLOOK

We have seen how a manifestly supersymmetric covariant quantization of a massless superparticle in four dimensions is easily come by, and how the difficulties encountered in the conventional formulation disappear. Supersymmetry is in fact as straightforwardly treated within the twistor formulation as on the light front. The absence of antiparticles causes no problems in these formalisms, and they are easily added in by hand.

The motivation for our study was the hope that there may be a useful formulation of superstring theory in twistor space. This requires two nontrivial generalizations: first, to a ten-dimensional space-time and then to strings. There are four values of space-time dimensions for which a twistorlike formulation seems at all possible: namely, 3, 4, 6, and 10; we think that it may be possible to utilize the connection to the division algebras here.[24] For a discussion of the quantum theory of strings in twistor language, see Ref. 25. We hope that we will be able to return to these questions in future publications.

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