Classical field configurations and infrared slavery

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(Received 4 February 1987)

The problem of determining the energy of two spinor particles interacting through massless-particle exchange is analyzed using the path-integral method. A form for the long-range interaction energy is obtained by analyzing an abridged vertex derived from the parent theory. This abridged vertex describes the radiation of zero-momentum particles by pointlike sources. A path-integral formalism for calculating the energy of the radiation field associated with this abridged vertex is developed and applications are made to determine the energy necessary for adiabatic separation of two sources in quantum electrodynamics and for an SU(2) Yang-Mills theory. The latter theory is shown to be consistent with confinement via infrared slavery.

I. INTRODUCTION

It is the foundation of classical electrodynamics that the energy required to separate two static electric charges is inversely proportional to the distance between them. The separation must take place adiabatically even in the classical theory to avoid the complication of radiated energy. The version of electrodynamics consistent with quantum effects is quantum electrodynamics. Of course, in the Coulomb-gauge formulation of quantum electrodynamics, the Coulomb interaction is manifestly present in the effective Hamiltonian used to calculate transition elements. However, the presence of the Coulomb interaction is obscured in the more commonly used manifestly covariant formulations of quantum electrodynamics.[1] There the Coulomb interaction is present in the long-distance behavior of the photon propagator, which is the relativistic generalization of the Coulomb potential. The form of the photon propagator depends crucially on the masslessness of the photon.

Quantum electrodynamics is a unique theory in the sense that its classical limit was known before the quantized version was formulated. Most of the currently topical field theories, including quantum chromodynamics, do not enjoy this advantage. As a result, the form of the forces transmitted by the nonlinear particle exchange has remained problematic, and therefore, the spectra of these theories have remained unknown. Of course, great strides have been made through lattice gauge computational methods, but these have not explicitly revealed the mechanism of confinement in the manner an analytical approach would.

The purpose of this paper is to present another approach for extracting the long-distance behavior of a theory when a massless particle is present. The method is a direct descendant of the technique developed by Faddeev and Kulish[2] for removing the infrared divergences from quantum electrodynamics. Their approach was to find a vertex which emulates the standard vertex of quantum electrodynamics only in the long-distance, or low-momentum, limit. This was accomplished by constraining the standard vertex to induce no change in momenta of the spinor particles during a scattering process. When S-matrix elements are calculated with the abridged vertex, the conservation of energy forces all radiated photon lines to carry zero momentum.

The abridged vertex analyzed by Faddeev and Kulish for quantum electrodynamics has the property that it can be exactly diagonalized. The charged eigenstates of this amended Hamiltonian are found to be dressed by off-shell intermediate photons in a manner sufficient to create the correct classical magnetic and electric fields familiar from solving the classical form of Maxwell's equations with constant velocity point sources. The purpose of their analysis, and that of subsequent authors,[3] was to develop a set of basis states, the eigenstates of the abridged Hamiltonian, for perturbative analysis of the full quantum-electrodynamic vertex. It was demonstrated that use of these states renders quantum electrodynamics perturbation theory free of infrared divergences by altering the spinor propagator to possess a cut rather than a simple pole. It was not the object of previous authors to examine the form of long-range forces present in the abridged Hamiltonian. However, it should be apparent that the removal of infrared divergences is intimately related to the presence of the Coulomb interaction, however well hidden, in quantum electrodynamics.

It is the intent of this paper to generalize this method so that the possible structure of long-range forces present in the abridged version of the theory may be ascertained. This generalization is most easily accomplished and is most transparent in application in a functional formulation. However, the motivation for this approach stems from canonical quantization and perturbation theory, although the final result is nonperturbative. The idea is straightforward, and involves constructing a vertex for the theory under consideration which emulates the full vertex in the low-momentum limit. This is accomplished by arbitrarily constraining the massive excitations of the theory, i.e., those excitations which begin with a bare mass in the action, to suffer no deviation in energy-momentum through the vertex, while the massless excitations are allowed to range over all values of momentum. In effect, this vertex describes the radiation of massless particles by a point source moving at constant velocity.

If this vertex has a trivial S matrix for the massive excitations, a fact which may be determined perturbatively, then it will be diagonalized, at least approximately. The massive eigenstates found from the diagonalization process are used to determine the energy of the radiation field which is dressing these states. It will be seen that this energy is a function of time, and this allows the relation of energy to distance of separation and, therefore, the calculation of the energy of interaction. The diagonalization will be performed using functional techniques, since these most illuminate the relation of this approach to classical solutions of the equations of motion in the presence of point sources. By looking at adiabatic processes, i.e., those involving only very-low-velocity point sources, retardation effects may be ignored and static Green's function techniques may be used to evaluate the path integral. The importance of these solutions to problems such as confinement has been discussed previously by Adler and Piran[4] and Baker, Ball, and Zachariasen.[5]

The outline of the paper is as follows. In Sec. II the method for constructing the abridged vertex is presented, and the motivation is given within an operator formalism. The path-integral form for calculating the energy of the particle states is then developed. In Sec. III manifestly covariant quantum electrodynamics is evaluated to demonstrate that the technique yields the familiar Coulomb interaction. In Sec. IV an SU(2)-invariant Yang-Mills theory is evaluated by the non-Abelian extension of the quantum electrodynamic vertex, and it is shown that a Coulombic potential is not allowed. Instead, to the order of approximate diagonalization presented in this paper, the theory is consistent with the formation of flux tubes between "color" singlets, and this gives the radiation field an energy of interaction consistent with confinement via infrared slavery.

II. METHODOLOGY

Throughout this paper consideration is limited to theories with massive bisponor field(s) interacting through massless vector fields. In Sec. III quantum electrodynamics will be analyzed, while in Sec. IV the generalization of these results to non-Abelian fields will be presented.

The motivation for the technique used in this paper is most readily given by a review of the Faddeev-Kulish analysis of quantum electrodynamics. When written in configuration space the standard vertex of quantum electrodynamics takes the form

$$H_I=\int d^3x\, e A_\mu \bar{\Psi}\gamma^\mu \Psi \;. \tag{2.1}$$

This theory is usually evaluated by invoking an interaction-picture representation. In effect, the interacting fields of (2.1) are replaced by free fields (modulo a choice of gauge) and a perturbation series is defined to calculate S-matrix elements. In the Feynman gauge the free fields have the momentum-space forms[6]

$$\Psi(x)=\int \frac{d^3k}{(2\pi)^3}\left[\frac{m}{\epsilon_k}\right]^{1/2}\sum_{s=1}^{2}[b_s(\mathbf{k})u(\mathbf{k},s)e^{-ikx}+d_s^\dagger(\mathbf{k})v(\mathbf{k},s)e^{ikx}]\;,$$
$$\epsilon_k{}^2=\mathbf{k}^2+m^2\;, \tag{2.2a}$$

and

$$A_\mu(x)=\int \frac{d^3k}{(2\pi)^{3/2}}(2\omega_k)^{-1/2}[a_\mu(\mathbf{k})e^{-ikx}+a_\mu^\dagger(\mathbf{k})e^{ikx}]\;,$$
$$\omega_k{}^2=\mathbf{k}^2\;, \tag{2.2b}$$

where b_s and d_s are electron-positron operators and a_μ is the free photon operator. From (2.1) the infrared vertex first discussed by Faddeev and Kulish can be obtained by inserting the forms (2.2) into (2.1) and constraining the spinor fields to undergo no deviation in momentum as a result of the interaction. For such a constraint the vertex becomes

$$H_1(t)=\int \frac{d^3p}{(2\pi)^{3/2}}\frac{d^3k}{(2\pi)^3}(2\omega_k)^{-1/2}\frac{p^\mu}{\epsilon_p}\left[a_\mu^\dagger(\mathbf{k})\exp\left[\frac{ikp}{\epsilon_p}(t-t_-)\right]+a_\mu(\mathbf{k})\exp\left[-\frac{ikp}{\epsilon_p}(t-t_-)\right]\right]\rho(\mathbf{p})\;, \tag{2.3}$$

where

$$\rho(\mathbf{p})=e\sum_{s=1}^{2}[b_s^\dagger(\mathbf{p})b_s(\mathbf{p})-d_s^\dagger(\mathbf{p})d_s(\mathbf{p})]\;,$$
$$\frac{kp}{\epsilon_p}=\omega_k-\frac{\mathbf{k}\cdot\mathbf{p}}{\epsilon_p}\;. \tag{2.4}$$

The presence of the time t_- is merely an artifice for placing all point charges at the same position at the time t_-. This will become clearer when (2.3) is given a configuration-space representation.

The vertex (2.3) does not commute with the total momentum, and so the vertices do not conserve total momentum. However, it is not difficult to see that any S-matrix element calculated using the vertex (2.3) will be trivial in the spinor sector. The constraint on the spinor-field momentum results in an absence of scattering for asymptotic on-shell fermions, and the vertex describes spinor particles traveling at constant velocity. However, these same constant velocity spinors may still radiate through the vertex (2.3). It follows from the form of (2.3) that any radiation matrix element for a photon with three-momentum $\mathbf{k}$ from a spinor with three-momentum $\mathbf{p}$ will be prefaced by a δ function of the form $\delta(kp/\epsilon_p)$. It is clear that this δ function leads to the constraint on the three-momentum of radiated photons of $\mathbf{k}=0$. It follows that an asymptotic initial state with no photons can overlap only with outgoing states containing additional zero-momentum photons. It is tedious, but straightforward, to show that the vertex (2.3) cannot defeat this limitation to zero-momentum asymptotic photons in higher orders. In effect, all radiated asymptotic photons are accompanied by a δ function of the sort mentioned. Thus, the vertex (2.3) describes the radiation of zero-momentum photons by constant velocity spinors, but in a manner similar to the original vertex (2.1). Diagonalizing such a vertex provides one possible way to study the infrared or long-distance behavior of the original theory. While this variant of the conservation of energy allows only zero-momentum photons to escape the "scattering" region into the asymptotic region, intermediate off-shell photons are not similarly constrained, and thus the time dependence of the vertex is not moot. In effect, charged eigenstates will have a cloud of virtual off-shell photons possessing a net energy, and this will be reflected in the fact that the expectation of the photon field, in the presence of spinor fields, will develop a nontrivial form at finite times.

When expressed in configuration space the vertex (2.3) takes the form

$$H_I(t)=\int d^3x\, d^3y\, d^3z\, \Psi_a^\dagger(\mathbf{x},t)V_{ab}(\mathbf{x},\mathbf{y},\mathbf{z},t)\Psi_b(\mathbf{y},t)\ , \tag{2.5}$$

where the function $V_{ab}(\mathbf{x},\mathbf{y},\mathbf{z},t)$ is given by

$$V_{ab}(\mathbf{x},\mathbf{y},\mathbf{z},t)=\int \frac{d^3p}{(2\pi)^6}\frac{mp^\mu}{\epsilon_p{}^2}A_\mu(\mathbf{z},t)\delta^3\left[\mathbf{z}-\frac{\mathbf{p}}{\epsilon_p}(t-t_-)\right][\Lambda_{ac}^+(p)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+\Lambda_{ac}^-(p)e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}]\gamma_{cb}^0\ . \tag{2.6}$$

The $\Lambda^\pm$ are the standard spinor projection operators.[6]

Form (2.5) makes it apparent that the Hamiltonian for the system described by (2.3) is time dependent since V_{ab} itself is time dependent. A simple physical argument reveals why this must be so. Since the spinor particles are constrained to move at a constant velocity their kinetic energy remains the same. However, the energy of the radiation field changes to reflect the fact that the vertex does not conserve total momentum. The change in energy can be related to the spinors' relative position. The vertex (2.3) cannot therefore describe a closed system, and the Hamiltonian must be manifestly time dependent. In passing, it is to be noted that the eigenstates of this Hamiltonian can serve as asymptotic states only if the change in energy of these eigenstates tends to zero for sufficiently asymptotic times, a condition which will be shown to be satisfied for the infrared quantum electrodynamic vertex in the next section. However, for the purposes of this paper, this time dependence of the energy does not present a difficulty since the object is precisely to calculate the change in energy necessary to separate the spinor particles. Upon examination of (2.6) it becomes apparent that the vertex in configuration space describes a point source moving in such a way that it is at the origin at $t=t_-$. It follows that all fields whose expectation values are calculated will describe particles which have this property since their time development is given by the same Hamiltonian.

It is assumed that the form of the vertex remains unchanged when written in terms of the interacting fields. This will involve both wave-function renormalization and mass renormalization of the spinor field in the case of quantum electrodynamics, and wave-function renormalization for the gauge fields in the non-Abelian case (see Sec. IV). The full Hamiltonian $H(t)$ for quantum electrodynamics, to be analyzed in Sec. III, is written

$$H(t)=H_0[A_\mu,\Psi]+H_I(t)+\int d^3x\,\delta m\,\overline{\Psi}\Psi\ , \tag{2.7}$$

where H_0 is the free Feynman-gauge Hamiltonian. The Feynman gauge is chosen so that the method being developed may be demonstrated on a form of quantum electrodynamics which does not manifestly exhibit the Coulomb interaction.

Since the spinor particles are being modeled as point sources it will be assumed that the spinor states can be described by a Fock space. That this is possible must be determined self-consistently. No assumption will be made regarding the radiation field states. While the form of the interaction (2.5) was motivated by a Fock decomposition of the radiation field, it will be assumed that the form of the interaction is unchanged even if the radiation field were to obey a nonstandard equation of motion or to possess unusual boundary conditions. Such an assumption is usually made in the opposite sense in field-theoretical analyses; the nonperturbative aspects of a theory are assumed to be retrieved by a summation of infinite numbers of Feynman diagrams. Here it is assumed that a nonperturbative solution subsumes the perturbation series which motivated it. However, the reader should bear in mind that the argument regarding the limitation to radiation of zero-momentum particles may break down if non-Fock decompositions of the radiation field are made.

In the interaction picture the spinor eigenstates $|\,\mathbf{p},t\,\rangle$ satisfy the eigenvalue equation,[7] suppressing spin arguments,

$$H(t)\,|\,\mathbf{p},t\,\rangle=E_p(t)\,|\,\mathbf{p},t\,\rangle\ , \tag{2.8a}$$

where

$$|\,\mathbf{p},t\,\rangle=N(t)U(t,t_-)\,|\,\mathbf{p}\rangle,\quad N(t)=\langle\,\mathbf{p}\,|\,U(t,t_-)\,|\,\mathbf{p}\,\rangle^{-1}\ . \tag{2.8b}$$

In (2.8a) $H(t)$ is the full Hamiltonian written in terms of free interaction-picture fields evaluated at the time t, while in (2.8b) the state $|\,\mathbf{p}\rangle$ is in the Heisenberg picture, which has been chosen to coincide with the interaction picture at $t=t_-$. $U(t,t_-)$ is the evolution operator in the interaction picture, and has the representation

$$U(t,t_-)=T\left[\exp\left[-i\int_{t_-}^{t} d\tau H_I(\tau)\right]\right], \tag{2.9}$$

where H_I is expressed in terms of the interaction picture fields. The Heisenberg picture spinor states will be given a δ-function normalization, as opposed to the normalization of Ref. 6. If $u_p(x)$ is the normalized free bispinor solutions of the massive Dirac equation, then, in terms of the interaction picture fields,

$$\begin{aligned}\langle \mathbf{p}\,|\,\mathbf{p}\rangle &= \int d^3x_1 d^3x_2 u_p^\dagger(\mathbf{x}_1,t_-)u_p(\mathbf{x}_2,t_-)\\ &\quad\times\langle 0\,|\,\Psi(\mathbf{x}_1,t_-)\Psi^\dagger(\mathbf{x}_2,t_-)\,|\,0\rangle\\ &= \int d^3x\, u_p^\dagger(\mathbf{x},t_-)u_p(\mathbf{x},t_-)=\delta^3(0)\ .\end{aligned} \tag{2.10}$$

The energy $E_p(t)$ of the state (2.8) at time t is then given by

$$E_p(t)=N(t)\langle \mathbf{p}\,|\,H(t)U(t,t_-)\,|\,\mathbf{p}\rangle\ . \tag{2.11}$$

For maximum clarity expression (2.11) will be converted into a path-integral form. In order to do this (2.11) must be written completely in terms of time-ordered products of operators since the path integral generates time-ordered products. However, this must be done in such a manner that the act of time ordering simply reproduces the sequence of operators which appears in (2.11). Taking advantage of the form of the functions $u_p(x)$ and the fact that the interaction picture field time development is the same as the free field, it is straightforward to show that the time-ordered product

$$E_p(t)=N(t)\int d^3x_1d^3x_2u_p^\dagger(\mathbf{x}_1,t)u_p(\mathbf{x}_2,t_-)\langle 0\,|\,T[\Psi(\mathbf{x}_1,t)H(t)\Psi^\dagger(\mathbf{x}_2,t_-)U(t,t_-)]\,|\,0\rangle \tag{2.12}$$

reproduces the operator sequence originally present in (2.11). Of course, a similar form may be given to the normalizing factor $N(t)$. The two-particle states may be evaluated as well, differing only by the insertion of extra fields at the times t and t_-, respectively. Since the separation of the particles in the two-particle state begins at t_-, $(t-t_-)$ can be related to the distance of separation of the particles through the relative velocity of the particles.

The methods for converting the matrix element of the time-ordered product appearing in (2.12) into a functional integral are well known.[8] The most straightforward technique is to use coherent states,[9] written in terms of the interaction picture fields, to evaluate matrix elements. Such an approach uses a functional projection operator of the form

$$\int [dA_\mu d\Psi^\dagger d\Psi]\,|\,A_\mu,\Psi^\dagger,\Psi,t\rangle\langle A_\mu,\Psi^\dagger,\Psi,t\,|=1\ , \tag{2.13}$$

where $[dA_\mu d\Psi^\dagger d\Psi]$ represents an integration over the modes of the fields. Whether this measure can be made sensible or even exists for a given theory is a difficult question which continues to draw attention.[10] Throughout the remainder of this paper it will be assumed that this measure is defined in configuration space by partitioning the space into small cells and treating the field strengths over the cell at the time t as a variable of integration. There is no known demonstration that this is a more general measure than integrating over all possible Fock modes, although the two are equivalent when a continuum of states is available to the theory.[8]

The matrix element is evaluated by partitioning the time interval $(t-t_-)$ into arbitrarily small elements of duration Δt and inserting the coherent-state projection operator at the respective times. It follows that, for the one-particle state of momentum $\mathbf{p}$,

$$\begin{aligned}N^{-1}(t)E_p(t)&=\int d^3x_1d^3x_2u_p^\dagger(\mathbf{x}_1,t)u_p(\mathbf{x}_2,t_-)\\ &\quad\times\int[dA_\mu d\Psi^\dagger d\Psi]\langle 0\,|\,\Psi(\mathbf{x}_1,t)H(t)U(t,t-\Delta t)\,|\,A_\mu,\Psi^\dagger,\Psi,t-\Delta t\rangle\cdots\\ &\quad\times\int[dA'_\mu d\Psi^{\dagger\prime}d\Psi']U(t_-+\Delta t,t_-)\Psi^\dagger(\mathbf{x}_2,t_-)\,|\,A'_\mu,\Psi^{\dagger\prime},\Psi',t_-\rangle\langle A'_\mu,\Psi^{\dagger\prime},\Psi',t_-\,|0\rangle\ .\end{aligned} \tag{2.14}$$

A similar expression follows for the two-particle state. It is not difficult to show that, by virtue of Δt being infinitesimal, each individual matrix element takes the form[8]

$$\begin{aligned}&\langle A_\mu,\Psi^\dagger,\Psi,t-N\Delta t\,|\,U(t-N\Delta t,t-(N+1)\Delta t)\,|\,A'_\mu,\Psi^{\dagger\prime},\Psi',t-(N+1)\Delta t\rangle\\ &\qquad=\exp\left[i\Delta t\int d^3x\,L[A_\mu,\Psi^\dagger,\Psi,t-N\Delta t]\right],\end{aligned} \tag{2.15}$$

where L is the Lagrangian density of the theory. Therefore, in the limit, the Hamiltonian matrix element for the two-particle state of momenta $\mathbf{p}$ and $\mathbf{q}$ is given by

$$\begin{aligned}N^{-1}(t)E(t)&=\int d^3x_1\cdots d^3x_4u_p^\dagger(\mathbf{x}_1,t)\cdots u_q(\mathbf{x}_4,t_-)\\ &\quad\times\int[dA_\mu d\Psi^\dagger d\Psi]\Psi(\mathbf{x}_1,t)\Psi(\mathbf{x}_2,t)H(t)\Psi^\dagger(\mathbf{x}_3,t_-)\Psi^\dagger(\mathbf{x}_4,t_-)\exp\left[i\int_{t_-}^{t}dt'\int d^3x\,L(\mathbf{x},t')\right],\end{aligned} \tag{2.16}$$

where the measure now runs over all space-time elements between t_- and t. Clearly, the normalization can be given a path-integral form as well, and would differ from the right-hand side of (2.16) solely by the absence of the Hamiltonian. Result (2.16) is the starting point for the determination of the energy of interaction between two spinor particles and will be evaluated for several theories in Secs. III and IV.

III. QUANTUM ELECTRODYNAMICS

In both this section and the following, several applications of the method developed in the previous section will be made. Consideration will be limited to calculating the energy of states with two charged particles. One will be chosen to be at rest, while the other will possess an infinitesimal momentum $\mathbf{p}$. The charges will be arbitrary, and this is reflected by allowing either particles or antiparticles in the eigenstate. While the formula (2.16) is expressed for particle states, a similar expression for antiparticles is obtained by replacing $\Psi^\dagger$ by Ψ and u by v.

The vertex (2.5) has already been developed for quantum electrodynamics. For the case of two particles the Hamiltonian matrix element is given by (2.16). The first step in evaluation of (2.16) is to decouple the spinors from the gauge field via the transformation[11]

$$\Psi(\mathbf{x},t)=\int d^3x'\frac{d^3p}{(2\pi)^3}\left[\frac{m}{\epsilon_p}\right][\Lambda^+(p)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}+\Lambda^-(p)e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}]\times\gamma_0 e^{-iC_p(t)}\Phi(\mathbf{x}',t)\ , \tag{3.1}$$

where the phase $C_p(t)$ is given by

$$C_p(t)=\int d^3z\int_{t_-}^{t}dt'e\left[\frac{p^\mu}{\epsilon_p}\right]A_\mu(\mathbf{z},t')S(t-t')\times\delta^3\left[\mathbf{z}-\frac{\mathbf{p}}{\epsilon_p}(t'-t_-)\right]\ , \tag{3.2}$$

and $S(t-t')$ is the standard step function given by

$$S(t-t')=\begin{cases}1 \text{ if } t>t',\\ 0 \text{ if } t<t'\ .\end{cases} \tag{3.3}$$

It is not difficult to show that both the measure of the path integral and the vertex are form invariant under the change of variables (3.2). However, this transformation decouples the spinor field from the gauge field, leaving the spinor fields with an effectively free action. Setting the momentum $\mathbf{q}=0$, using the fact that the phase $C_p(t)$ vanishes at $t=t_-$, and the properties of the $\Lambda^\pm$, it follows that expression (2.16) becomes

$$N^{-1}E(t)=\int d^3x_1\cdots d^3x_4u_p^\dagger(\mathbf{x}_1,t)\cdots u_0(\mathbf{x}_4,t_-)\int[dA_\mu d\Phi^\dagger d\Phi]\Phi(\mathbf{x}_1,t)\Phi(\mathbf{x}_2,t)H[A_\mu,\Phi,t]\times\Phi^\dagger(\mathbf{x}_3,t_-)\Phi^\dagger(\mathbf{x}_4,t_-)e^{-i[C_p(t)+C_0(t)]}\exp\left[i\int_{t_-}^{t}dt'\int d^3x\,L_0\right]\ , \tag{3.4}$$

where L_0 is the free action in the Feynman gauge for both the spinor field Φ and the gauge field A_μ. As a result of the transformation it is trivial to integrate the fermion variables completely out of the path integral. In order to demonstrate the result it is convenient to split the Hamiltonian appearing in (3.4) into three pieces. In terms of the new variables the Hamiltonian is written

$$H=H_0[\Phi]+H_0[A_\mu]+H_I[A_\mu,\Phi]\ , \tag{3.5}$$

where

$$H_0[\Phi]=\int d^3x[i\Phi^+\dot{\Phi}+\delta m\bar{\Phi}\Phi]\ . \tag{3.6}$$

The second and third terms in (3.5) are the free Feynman-gauge radiation field Hamiltonian and the abridged interaction Hamiltonian (2.5), respectively. The second term in (3.6) is a mass renormalization present to control divergences in the radiation field energy, for which the necessity will become apparent shortly. The mass renormalization term is also present in the bare mass in the Lagrangian.

The phases $C_p(t)$ of the transformation may now be absorbed into the exponential argument of the path integral where each acts as a point source for the gauge field. The Hamiltonian matrix element then becomes

$$N^{-1}E(t)=\int d^3x_1\cdots d^3x_4u_p^\dagger(\mathbf{x}_1,t)\cdots u_0(\mathbf{x}_4,t_-)\int[dA_\mu d\Phi^\dagger d\Phi]\Phi(\mathbf{x}_1,t)\Phi(\mathbf{x}_2,t)H[A_\mu,\Phi,t]\times\Phi^\dagger(\mathbf{x}_3,t_-)\Phi^\dagger(\mathbf{x}_4,t_-)\exp\left[i\int_{t_-}^{t}dt'd^3x(L_0+j_\mu A^\mu)\right]\ , \tag{3.7}$$

where j_μ is given by

$$j_\mu(\mathbf{x},t,t')=Q_1\frac{p_\mu}{\epsilon_p}S(t-t')\delta^3\left[\mathbf{x}-\frac{\mathbf{p}}{\epsilon_p}(t'-t_-)\right]+Q_2\delta_{\mu0}S(t-t')\delta^3(\mathbf{x})\ , \tag{3.8}$$

with Q_1 and Q_2 the respective charges of the (anti)spinor particles with momentum $\mathbf{p}$ and momentum zero, respectively. From examining (3.8) it is obvious that j_μ acts as two classical point sources: one stationary at the origin, the other moving away from the origin with the velocity $\mathbf{p}/\epsilon_p$.

The remaining path integral, solely a functional of the gauge field, may now be evaluated exactly. The gauge field picks up an expectation value due to the presence of the point charges despite there being no external photons explicitly present in the initial state. This expectation value is precisely the classical field of the point sources and is most easily evaluated in the adiabatic limit in which **p** is infinitesimal. In this approximation the magnetic field, proportional to **p**, will be suppressed. In addition, the time dependence of the radiation field, also proportional to **p** will be suppressed as well. The upshot of the adiabatic limit is to cause A_0 to be the only relevent component of the vector potential, and in that limit it is approximately static, i.e., $\dot{A}_0 \approx 0$. The remaining effective Lagrangian of the gauge field is then given by

$$L_{\text{gauge}} = -\tfrac{1}{2}\partial_i A_0 \partial_i A_0 + j_0 A_0 \ . \tag{3.9}$$

It is obvious that this Lagrangian will cause the path integral to oscillate about the classical solution of the equation

$$\Delta A_0 = -j_0 \ . \tag{3.10}$$

Equation (3.10) possesses the familiar Coulomb potential solution

$$A_0(\mathbf{x},t) = \frac{Q_1}{\left|\mathbf{x} - \dfrac{\mathbf{p}}{\epsilon_p}(t - t_-)\right|} + \frac{Q_2}{|\mathbf{x}|} \ , \tag{3.11}$$

which does indeed satisfy the restrictions of the adiabatic approximation as long as $\mathbf{p} \approx 0$. It is important to note that the boundary conditions placed on A_0 determine its form, and this will in turn determine the form of the spinor interaction energy. The Coulomb solution to (3.10) arises from demanding a solution over all space.

After integrating out the spinor variables and using the normalization $N^{-1} = [\delta^3(0)]^2$, it is a textbook exercise[12] to evaluate the radiation field Hamiltonian, which, in the adiabatic limit, is dominated by the static piece

$$\langle H_{\text{gauge}} \rangle = \int d^3x \tfrac{1}{2}\partial_i A_0 \partial_i A_0 \ . \tag{3.12}$$

Inserting the solution (3.11) into (3.12) it follows that

$$\langle H_{\text{rad}} \rangle = \frac{Q_1 Q_2}{\left|\dfrac{\mathbf{p}}{\epsilon_p}(t - t_-)\right|} + (Q_1{}^2 + Q_2{}^2) \int \frac{d^3x}{|\mathbf{x}|^4} \ . \tag{3.13}$$

Clearly the second term is divergent. It may be removed by cutting off the lower limit of integration over the radial coordinate at R_0 (equivalent to giving the charge a classical radius) and removing the Coulomb self-energy with the mass counterterm

$$\delta m = -\frac{e^2}{R_0} \ . \tag{3.14}$$

In expression (3.14) the small-**p** limit of the mass counterterm has been used. For finite **p** the magnetic field contribution to the self-energy must also be canceled, and this is automatically satisfied by the form of the mass counterterm. The final result is that the energy of the radiation field associated with the two-particle state is given by

$$E_{\text{rad}}(t) = \frac{Q_1 Q_2}{\left|\dfrac{\mathbf{p}}{\epsilon_p}(t - t_-)\right|} \ . \tag{3.15}$$

Clearly, $|(\mathbf{p}/\epsilon_p)(t - t_-)|$ is simply the distance of separation of the two charges as a function of time. Thus, the standard result of the Coulomb interaction has been recovered in the adiabatic limit from the abridged infrared vertex of the theory.

IV. SU(2) YANG-MILLS THEORY

The techniques applied to QED in the previous section can be generalized to a non-Abelian gauge theory. The basic idea is to solve, at least partially, the theory defined by the quadratic piece of the Yang-Mills Lagrangian plus a non-Abelian generalization of the spinor-gauge vertex (2.3). Considerable work[13] has been focused on the problem of cancellation of infrared divergences in non-Abelian theories via a Bloch-Nordsieck procedure. It is not the intent of this paper to reexamine this problem. Instead, attention will be placed on the approach to infrared cancellation advocated by Nelson and co-workers,[14] but solely in the context of understanding the classical field configurations associated with the generalization of the vertex (2.3).

The process of decoupling the spinors from the gauge fields consists of a series of nonlocal SU(2) gauge transformations on the spinors, and this process generates phases for the spinor fields which are nonlinear in the gauge fields. In turn, these phases may be absorbed into the action of the path integral to create an effective gauge field action, as was done in the previous section for QED.

It will be shown that the equations of motion for the gauge fields derived from this effective action do not allow a Coulombic solution over all space as in the case of QED. Instead, to the order of approximation evaluated in this paper, these equations of motion are consistent with an idealized (infinitely thin) flux-tube solution for the gauge fields, and this flux-tube solution possesses finite energy only in the presence of SU(2)-"color" singlets, and then is proportional to the distance of separation.

These results are obtained without consideration of the nonlinear terms in the gauge fields which are already present in the Yang-Mills Lagrangian. It will be seen that their neglect is consistent with the adiabatic limit, but this neglect is conceptually troublesome since their contribution is so critical to the renormalization-group demonstration of asymptotic freedom. Of course, asymptotic freedom is at best only an indication that infrared effects may cause confinement, since its demonstration depends on perturbation theory, which in turn will be valid only in the high-momentum or nonadiabatic limit. Nevertheless, it is generally accepted that the terms nonlinear in the gauge fields are the source of confinement. There are several plausible but unverified mechanisms which may allow the approach of this paper, and its neglect of the original nonlinear terms, to be reconciled with the renormalization-group arguments. It is possible that the nonlinear terms which are generated by decoupling the spinors mimic, in a low-energy sense, the dynamical effect which the original nonlinear terms bring to the renormalization-group equations. On the other hand, it may be that a realistic flux tube, i.e., one with spatial extension and dependence, would arise from the inclusion of the original nonlinear terms, or some part of them, in the action which serves to define the basis states. This brief discussion serves only to point out that there are difficult and currently unanswered questions which merit further consideration.

The starting point is the generalization of the QED vertex (2.3). Such a vertex takes the form

$$H_I(t)=\frac{g}{(2\pi)^{3/2}}\int d^3p\frac{d^3k}{(2\omega_k)^{1/2}}\frac{p^\mu}{\epsilon_p}\times\left[a_\mu^{i\dagger}(\mathbf{k})\exp\left[i\frac{kp}{\epsilon_p}(t-t_-)\right]+a_\mu^i(\mathbf{k})\exp\left[-i\frac{kp}{\epsilon_p}(t-t_-)\right]\right]\rho^i(p)\ , \tag{4.1a}$$

where

$$\rho^i(p)=\sigma_{ab}^i\sum_{s=1}^{2}[b_s^{\dagger a}(\mathbf{p})b_s^b(\mathbf{p})-d_s^{\dagger a}(\mathbf{p})d_s^b(\mathbf{p})]\ , \tag{4.1b}$$

where σ^i is the ith Pauli spin matrix. Such a vertex has already been considered by Nelson and co-workers.[14] They demonstrate that dressing the spinor states with a pseudounitary operator derived from this vertex renders the non-Abelian theory infrared finite at least to $O(g^2)$ in the coupling constant. Again, the form (4.1) is obtained from the full vertex of the standard theory by first assuming free interaction picture fields and then constraining the spinor momenta to be unchanged by the vertex. As a result, any external gauge field lines generated by this vertex will necessarily carry zero momentum.

In configuration space (4.1) takes the form

$$H_I(t)=g\int d^3x\, d^3y\, d^3z\, \Psi_\alpha^{a\dagger}(\mathbf{x},t)\times V_{\alpha\beta}^{ab}(\mathbf{x},\mathbf{y},\mathbf{z},t)\Psi_\beta^b(\mathbf{y},t)\ , \tag{4.2}$$

where $V_{\alpha\beta}^{ab}$ is given by

$$V_{\alpha\beta}^{ab}(\mathbf{x},\mathbf{y},\mathbf{z},t)=\int\frac{d^3p}{(2\pi)^3}\frac{mp^\mu}{\epsilon_p{}^2}A_\mu^i(\mathbf{z},t)\sigma_{ab}^i\times\delta^3\left[\mathbf{z}-\frac{\mathbf{p}}{\epsilon_p}(t-t_-)\right]\times[\Lambda_{\alpha\gamma}^+(p)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+\Lambda_{\alpha\gamma}^-(p)e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}]\gamma_{\gamma\beta}^0\ . \tag{4.3}$$

Clearly, forms (4.1) and (4.2) coincide only in the event that all fields are given a Fock decomposition in the Feynman gauge. Therefore, (4.1) can serve only as a motivation for the form (4.2). This must certainly be true if confinement is to be generated by (4.2), for in that case a Fock decomposition of the fields is a poor, if not completely incorrect, choice of basis states in the gauge sector. It will be assumed that (4.2) has validity as the infrared piece of the interaction above and beyond the Fock-space form (4.1). This was certainly true in the Abelian case of the previous section.

It is possible to motivate (4.2) in a somewhat more general manner. It is commonly believed that the spinor particles will remain pointlike, i.e., Fock, even when confined. Under such an assumption it is easy to see that (4.2) will allow only the zero-momentum part of the gauge field to contribute. It will be seen later in this section that the spinors will remain Fock-like up to a phase generated by the decoupling transformations.

The theory to be examined is described by the Lagrangian density

$$L=L_{\text{matter}}+L_{\text{gauge}}+L_{\text{ghost}}+L_I\ , \tag{4.4}$$

where

$$L_{\text{matter}}=i\bar{\Psi}^a(\gamma^\mu\partial_\mu+im)\Psi^a\ , \tag{4.4a}$$

$$L_{\text{gauge}}=\tfrac{1}{2}\text{Tr}(F_{\mu\nu}F^{\mu\nu})-\tfrac{1}{2}(\partial^\mu A_\mu^a)^2\ , \tag{4.4b}$$

$$F_{\mu\nu}=(\partial_\mu A_\nu^i-\partial_\nu A_\mu^i-g\epsilon^{ijk}A_\mu^jA_\nu^k)\sigma^i\ , \tag{4.4c}$$

$$L_{\text{ghost}}=\partial^\mu\bar{\eta}^i(\partial_\mu\eta^i+g\epsilon^{ijk}\eta^jA_\mu^k)\ , \tag{4.4d}$$

$$\int d^3x\, L_I(\mathbf{x},t)=-g\int d^3x\, d^3y\, d^3z\, \Psi_\alpha^{a\dagger}(\mathbf{x},t)\times V_{\alpha\beta}^{ab}(\mathbf{x},\mathbf{y},\mathbf{z},t)\Psi_\beta^b(\mathbf{y},t)\ , \tag{4.4e}$$

so that the standard vertex has been replaced by (4.2). Form (4.4d) is the Faddeev-Popov ghost Lagrangian in the Feynman gauge.

As before, the expectation value of the Hamiltonian in the two-spinor state is given by

$$N^{-1}E(t)=\int d^3x_1\cdots d^3x_4u_p^\dagger(\mathbf{x}_1,t)\cdots u_q(\mathbf{x}_4,t_-)\int[dA_\mu d\Psi^\dagger d\Psi]\Psi^a(\mathbf{x}_1,t)\Psi^b(\mathbf{x}_2,t)H(t)\times\Psi^{a\dagger}(\mathbf{x}_3,t_-)\Psi^{b\dagger}(\mathbf{x}_4,t_-)\exp\left[i\int_{t_-}^t dt'\int d^3x\, L(\mathbf{x},t')\right]\ . \tag{4.5}$$

In order to evaluate (4.5) the spinors are decoupled from the gauge fields as in the Abelian case of Sec. III. This is accomplished by an infinite series of gaugelike transformations on the spinors. In order to give an explicit representation for the transformations the gauge fields are given the basis representation

$$\sigma^i A^i_\mu = \begin{vmatrix} A^1_\mu & A^2_\mu - iA^3_\mu \\ A^2_\mu + iA^3_\mu & -A^1_\mu \end{vmatrix} . \tag{4.6}$$

The spinors are then subjected to the change of variable in the path integral given by the unitary transformation

$$\Psi^a(\mathbf{x},t) = \int d^3x' \frac{d^3p}{(2\pi)^3} \left[\frac{m}{\epsilon_p}\right] [\Lambda^+(p) e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} + \Lambda^-(p) e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}] \times \gamma^0 U_{ab}(\theta,\mathbf{p},t)\Phi^b(\mathbf{x}',t) , \tag{4.7}$$

where the $\Lambda^\pm$ are defined as in (2.6). The matrix $U(\theta,\mathbf{p},t)$ is given by

$$U(\theta,\mathbf{p},t) = \begin{vmatrix} e^{-i\theta_1}(\cos\theta_2\cos\theta_3 - i\sin\theta_2\sin\theta_3) & e^{-i\theta_1}(-\sin\theta_3\cos\theta_2 - i\cos\theta_3\sin\theta_2) \\ e^{i\theta_1}(\sin\theta_3\cos\theta_2 - i\cos\theta_3\sin\theta_2) & e^{i\theta_1}(\cos\theta_2\cos\theta_3 + i\sin\theta_2\sin\theta_3) \end{vmatrix} . \tag{4.8}$$

The θ_i are functions of $\mathbf{p}$, A^i_μ, and t, and have the explicit representations in the basis (4.6):

$$\theta_1(\mathbf{p},t) = \int d^3z \int_{t_-}^{t} dt' g \left[\frac{p^\mu}{\epsilon_p}\right] A^1_\mu(\mathbf{z},t') S(t-t') \times \delta^3\left[\mathbf{z} - \frac{\mathbf{p}}{\epsilon_p}(t'-t_-)\right] , \tag{4.9a}$$

$$\theta_2(\mathbf{p},t) = \int d^3z \int_{t_-}^{t} dt' g \left[\frac{p^\mu}{\epsilon_p}\right] [\cos(2\theta_1) A^2_\mu(\mathbf{z},t') + \sin(2\theta_1) A^2_\mu(\mathbf{z},t')] \times S(t-t')\delta^3\left[\mathbf{z} - \frac{\mathbf{p}}{\epsilon_p}(t'-t_-)\right] , \tag{4.9b}$$

$$\theta_3(\mathbf{p},t) = \int d^3z \int_{t_-}^{t} dt' g \left[\frac{p^\mu}{\epsilon_p}\right] [\cos(2\theta_1) A^3_\mu(\mathbf{z},t') - \sin(2\theta_1) A^2_\mu(\mathbf{z},t')] \times S(t-t')\delta^3\left[\mathbf{z} - \frac{\mathbf{p}}{\epsilon_p}(t'-t_-)\right] . \tag{4.9c}$$

In expressions (4.9b) and (4.9c) the phase angle θ_1 is a function of $\mathbf{p}$ and t', but is otherwise identical to (4.9a). It is apparent that (4.8) has the form of a product of finite SU(2) transformations. The phases θ_2 and θ_3 are nonlinear in the gauge fields due to the appearance of the phase θ_1 in their definition. This nonlinearity is present for the following reason. Expressions (4.8) and (4.9) were determined by first decoupling the color-diagonal piece of the interaction, involving A^1_μ. However, the matrix in the gauge fields, appearing in the definition (4.3) of the vertex, does not commute with the matrix which decouples A^1_μ, and leaves a residue of the phase θ_1 in the off-diagonal elements of the gauge field matrix after the transformation. Of course, this aspect of decoupling the spinors was absent in the Abelian case of Sec. III. The matrix (4.8) is then the product of an SU(2) transformation which decouples the spinors from A^1_μ with another which would decouple the spinors from the forms of A^2_μ and A^3_μ. However, because of the SU(2) algebra, the fields A^2_μ and A^3_μ have been altered by the application of the first transformation, and the forms of (4.9b) and (4.9c) reflect this. Of course, this second transformation decouples the gauge field only to $O(\theta)$, or, equivalently, to $O(g)$. Under the action of the transformation (4.7), the vertex (4.2) becomes

$$H_I(t) = \int d^3x\, d^3y\, d^3z\, \Phi^{a\dagger}(\mathbf{x},t) V'_{ab}(\mathbf{x},\mathbf{y},\mathbf{z},t)\Phi^{b\dagger} , \tag{4.10}$$

where V' is a complicated function of the θ_i, but is $O(\theta^2)$.

Clearly, the process of decoupling the spinors from the gauge fields could be continued through an infinite series of transformations similar to (4.7). Such a process quickly generates a complicated set of nonlinear phases for the spinors. For the sake of simplicity the diagonalization procedure will be terminated with (4.7). To be consistent with this termination solutions for the A^i_μ will be sought which render θ_i small. Such a restriction will allow the remaining interaction (4.10), and all the transformations necessary to decouple it, to be ignored. To see this it need only be noted that the spinor variables in the path integral may be formally integrated keeping the action term (4.10). Doing so gives the fermionic determinant $\det(\not{D} + V')$ where $\not{D}$ is the free Dirac operator and V' is the potential appearing in (4.10). This determinant can be given a power-series representation in the θ_i whose first nonvanishing term in the θ_i can be shown to be $O(\theta^2)$. The phases induced by the change of variable (4.7) are clearly $O(\theta)$. Since a solution for small θ is being sought, the $O(\theta^2)$ terms, and V', will now be suppressed for consistency. To this order of approximation the spinors are therefore decoupled, and the new spinor variables Φ are governed by a free spinor action.

A second important point is that the gauge fields and coupling constant appearing in the definitions of the θ_i and in the action are understood to be bare quantities. Normally wave function and coupling constant renormalizations are associated with ultraviolet divergences. However, here it is assumed that the renormalization constants for the infrared-finite theory factorize into two pieces: one to remove the infrared divergences of the vertex (4.2), the other to remove the ultraviolet divergences of the full vertex which are familiar from perturbation theory. Such a procedure has been examined previously by Stapp[15] for the spinor propagator in QED. It is clear that the photon field in QED requires no wave-function renormalization for the infrared theory of Sec. III. However, the non-Abelian fields of this section require it, and it must be done in a manner consistent with the Slavnov-Taylor identities. Denoting bare quantities with an overbar and the renormalized quantities without, it follows that

$$\overline{A}{}^{i}_{\mu}=Z_3{}^{1/2}A^{i}_{\mu}\ , \tag{4.11a}$$

$$g\overline{A}{}^{i}_{\mu}=Z_3 g A^{i}_{\mu}\ . \tag{4.11b}$$

It is now possible to calculate, in the small-θ approximation, the energy of a color singlet, where one particle is held fixed and the other adiabatically removed. In terms of the original spinor variables the Heisenberg state is given by

$$|\,\mathbf{p},\mathbf{q}=0\,\rangle=\frac{1}{\sqrt{2}}\int d^3x_1 d^3x_2 u_p(\mathbf{x}_1,t_-)u_0(\mathbf{x}_2,t_-)\times\epsilon^{ab}\Psi^{a\dagger}(\mathbf{x}_1,t_-)\Psi^{b\dagger}(\mathbf{x}_2,t_-)\,|\,0\,\rangle, \tag{4.12}$$

where ϵ^{ab} is the Levi-Civita tensor which guarantees antisymmetry on the two color indices available for SU(2). The spin arguments will be suppressed since they have no dynamical content. The evaluation of the energy is greatly simplified by using the following facts. First, the incoming state (or, alternatively, the field at the time t_-) is left unchanged by the transformation (4.7) since all the θ_i vanish at $t=t_-$. The state is thus given by

$$|\,\mathbf{p},\mathbf{q}=0\,\rangle=\frac{1}{\sqrt{2}}\int d^3x_1 d^3x_2 u_p(\mathbf{x}_1,t_-)u_0(\mathbf{x}_2,t_-)\times\epsilon^{ab}\Phi^{a\dagger}(\mathbf{x}_1,t_-)\Phi^{b\dagger}(\mathbf{x}_2,t_-)\,|\,0\,\rangle \tag{4.13}$$

after the transformation. Second, since the Φ's are being treated as free spinor fields after the transformation, when integrated against the external free spinor functions the fermionic variables' phases develop the same momentum dependence as the external momenta, i.e., $\mathbf{p}$ and 0. Thus, when the transformed spinor variables are integrated out of the path integral, apart from a trivial spinor determinant, the only residue will be the phases on the respective fields at the time t, and only the subset which resulted from contractions against the in state (4.13). This is, of course, identical to the Abelian case of Sec. III, except that the phases are nonlinear, and the results remain an approximation, whereas the Abelian case gave an exact result. In this approximation the normalization of the two-particle state is $N^{-1}=[\delta^3(0)]^2$. It is shown in the Appendix that the expectation value of the energy of the gauge field for the color singlet of (4.12) reduces to

$$E_{\text{gauge}}(t)\int[dA_\mu d\overline{\eta}\,d\eta][\cos(\theta_3-\theta_3')\cos(\theta_2-\theta_2')\cos(\theta_1-\theta_1')+\sin(\theta_3-\theta_3')\sin(\theta_2+\theta_2')\sin(\theta_1-\theta_1')]\times[H_{\text{gauge}}(t)+H_{\text{ghost}}(t)]\exp\left[i\int d^3x\int_{t_-}^{t}dt'(L_{\text{gauge}}+L_{\text{ghost}})\right]\ , \tag{4.14}$$

where θ_i is the ith phase evaluated at $\mathbf{p}$, and θ_i' is the ith phase evaluated at 0.

Now the sine terms can be dropped since they are odd in the gauge fields, while the action in the adiabatic limit (equivalent to keeping only the quadratic terms) is even. In addition, the ghost part of the Hamiltonian can be ignored since it contributes only a zero-point energy even when A_μ develops an expectation value. Under these circumstances the cosine terms may be replaced with

$$\cos(\theta_1-\theta_1')\cos(\theta_2-\theta_2')\cos(\theta_3-\theta_3')=\exp[i(\theta_1-\theta_1'+\theta_2-\theta_2'+\theta_3-\theta_3')]\ , \tag{4.15}$$

which allows the phases to be absorbed into the action. If the adiabatic limit is taken it is again obvious that only the A_0 component of the phases survives. In the limit that $\mathbf{A}^i\to 0$, only the quadratic part of the gauge field action will remain. Thus, in the adiabatic limit, the gauge field action reduces to

$$L_{\text{gauge}}(\mathbf{x},t)=-\tfrac{1}{2}Z_3\partial_\mu A^i_0\partial^\mu A^i_0-Z_3A^1_0[j_p-j_0]-Z_3A^2_0[(\cos2\theta_1+\sin2\theta_1)j_p-(\cos2\theta_1'+\sin2\theta_1')j_0]+Z_3A^3_0[(\cos2\theta_1-\sin2\theta_1)j_p-(\cos2\theta_1'-\sin2\theta_1')j_0]\ , \tag{4.16}$$

where j_p is given by

$$j_p(\mathbf{x},t)=g\delta^3\left[\mathbf{x}-\frac{\mathbf{p}}{\epsilon_p}(t-t_-)\right]\ . \tag{4.17}$$

Expression (4.16) gives rise to the following equations of motion for the A^i_0:

$$\partial^2A^1_0=j_p-j_0\ , \tag{4.17a}$$

$$\partial^2A^2_0=(\cos2\theta_1+\sin2\theta_1)j_p-(\cos2\theta_1'+\sin2\theta_1')j_0\ , \tag{4.17b}$$

$$\partial^2A^3_0=(\cos2\theta_1-\sin2\theta_1)j_p-(\cos2\theta_1'-\sin2\theta_1')j_0\ . \tag{4.17c}$$

The time t appearing in the equations must coincide with time t of the Hamiltonian in order for the equations to have the form given. This is because the variation of the phase θ_1 with respect to A_0^1 vanishes only if the time of the phase coincides with the out-state. Of course, choosing this time has the added advantage that it coincides with the time of the Hamiltonian, which is a functional of the field configurations at the time t. From the form of (4.17a) it follows that $\dot{A}_0^1 \approx 0$ when $\mathbf{p} \approx 0$. It is not possible to make a similar assumption for the solutions for A_0^2 and A_0^3 because of the appearance of the θ_1 phase in Eqs. (4.17b) and (4.17c). However, this in no way affects the validity of the adiabatic approximation or the form of the equations derived so far, as the diligent reader may verify.

Equation (4.17a) is identical to the Abelian case, and thus it appears possible to obtain a Coulombic solution over all space. However, because of the appearance of θ_1 in the equations for A_0^2 and A_0^3, such a solution is not possible. It is straightforward to show that if A_0^1 is given the Coulombic solution to (4.17a) then the phase θ_1 is divergent, in direct contradiction to the basic assumption that θ_i is small. The solution to this dilemma is to seek a non-Coulombic form for the A_0^i, and this is accomplished by invoking different boundary conditions on the gauge fields. This is done by using a flux tube Green's function to solve the equations of (4.17). Such a Green's function has been discussed previously by Baker, Ball, and Zachariasen.[15] The Green's function is constructed to be symmetric about the axis defined by the particle motion, which will be chosen to be the z axis for convenience. The flux tube has a radius a which will be given the limit $a \to 0$. This limit is employed to simplify calculations tremendously. The Green's function must satisfy Neumann boundary conditions in the radial direction, for otherwise, in the small-a limit it would become trivially zero everywhere. The static Green's function then takes the form

$$G(\mathbf{x}-\mathbf{x}') = -\int \frac{dk}{4\pi^2} I_0(k\rho_<) \left[K_0(k\rho_>) + \frac{K_1(ka)}{I_1(ka)} I_0(k\rho_>) \right] \cos k(z-z') , \tag{4.18}$$

if $\rho_>$ and $\rho_<$ are less than a, and is set zero if $\rho_>$ or $\rho_<$ is greater than a. In (4.18) the I's and K's are the standard modified Bessel functions. In the limit of an infinitely thin flux-tube expression (4.18), with suitable choice of contour for k, reduces to

$$G(\mathbf{x}-\mathbf{x}') = \frac{1}{\pi a^2}(z-z')S(z-z') \quad \text{for } \rho_<, \rho_> < a , \tag{4.19}$$

and vanishes everywhere outside the flux tube. It is now straightforward to show that the solution to (4.17a) is given by

$$A_0^1(\mathbf{x}) = \frac{g}{\pi a^2}\{[z-R(t)]S(z-R(t)) - zS(z)\} , \tag{4.20}$$

if $\rho < a$ and is zero if $\rho > a$. $R(t)$ is given by

$$R(t) = \left| \frac{\mathbf{p}}{\epsilon_p} \right| (t-t_-) , \tag{4.21}$$

and is clearly the distance of separation as a function of time. Obviously, (4.20) describes a flux tube along the axis of motion joining the current positions of the two spinors. The phase θ_1 can now be evaluated to obtain

$$\theta_1(\mathbf{p},t) = -Z_3 \frac{g^2}{2\pi a^2} R(t)(t-t_-) . \tag{4.22}$$

The small-a limit and the assumption of θ_i small are made consistent in (4.22) with the choice of renormalization constant

$$Z_3 = \frac{E_0}{3R_0} \frac{2\pi a^2}{g^2} , \tag{4.23}$$

where it will be seen shortly that E_0 is the energy of the color singlet at the distance of separation R_0. For the moment these constants may be viewed as nothing more than a choice of length scale for the theory. If an SU(2) theory were a realistic model of nature, the values of E_0 and R_0 would be chosen for phenomenological fit. For this choice of Z_3 the phase θ_1 becomes

$$\theta_1 = -E(t)(t-t_-) , \tag{4.24}$$

where

$$E(t) = \frac{E_0}{3R_0} R(t) . \tag{4.25}$$

The solutions to the other two equations of (4.17) can now be found by employing the same static Green's function after inserting the value of θ_1. It follows that

$$A_0^2(\mathbf{x}) = \frac{g}{\pi a^2} \left\{ [\cos E(t)(t-t_-) - \sin E(t)(t-t_-)] \times \frac{\sin 2E(t)[z-R(t)]}{2E(t)} \times S(z-R(t)) - zS(z) \right\} \tag{4.26a}$$

and

$$A_0^3(\mathbf{x}) = \frac{g}{\pi a^2} \left\{ [\cos E(t)(t-t_-) + \sin E(t)(t-t_-)] \times \frac{\sin 2E(t)[z-R(t)]}{2E(t)} \times S(z-R(t)) - zS(z) \right\} . \tag{4.26b}$$

It is not difficult to see that if $E(t) \approx 0$, which occurs if either $E_0/R_0 \approx 0$ or $(t-t_-) \approx 0$, the forms (4.26) are equivalent to (4.20). Of course, this restriction is also necessary to make θ_1 small. It is straightforward to evaluate the gauge field Hamiltonian in this same limit. The gauge field Hamiltonian develops the expectation value

$$E_{\text{gauge}}(t)=Z_3 \int d^3x \tfrac{1}{2}(\partial_j A_0^i \partial_j A_0^i + \dot{A}_0^i \dot{A}_0^i)=Z_3 \frac{3g^2}{2\pi a^2}\int_{-\infty}^{\infty} dz[S(z)-S(z-R(t))]$$
$$+Z_3\frac{4g^2}{\pi a^2}\int_{R(t)}^{\infty} dz(1-\cos\{2E(t)[z-R(t)]\}\cos[E(t)(t-t_-)]) , \quad (4.27)$$

where terms proportional to $\dot{E}(t)$ and $\dot{R}(t)$ have been suppressed for consistency with the adiabatic limit. The second integral is formally divergent. However, expanding the integral in a power series in $E(t)$ shows that the lowest-order term is proportional to $E^3(t)$, and so it is argued that this divergence is an artifact of the truncation process in θ. In the small-θ limit the second integral will be ignored, so that, in the limit that $E(t) \gg E^3(t)$, expression (4.27) reduces to

$$E_{\text{gauge}}(t)=\frac{E_0}{R_0}R(t) . \quad (4.28)$$

Result (4.28) shows that the energy of the gauge field, which is the energy of interaction of the spinors, is consistent with a confining potential. It is to be noted that the forms of the solutions for the A_0^i which led to this confining potential were determined by demanding consistency with the infrared form of the interaction. The path-integral formulation of the theory then allows a ready incorporation of these classical configurations of the gauge fields into the quantized version of the theory.

It is straightforward to examine a nonsinglet state, e.g., a single spinor state, using this technique. It becomes readily apparent that the phases θ' and/or θ enter into the action of the path integral with the same sign. Solutions to the equations equivalent to (4.17) lead immediately to an infinite-energy configuration, describing a flux tube from the particle(s) to infinity.

One of the most esthetic aspects of this result is that states which are "gauge invariant" under the transformation (4.7) are effectively free. Such states are comoving spinor true gauge-singlet pairs. However, it is stressed that these results have been obtained only in the approximations made in this paper. The uniqueness of these solutions, as well as whether they persist, at least qualitatively, to higher orders of θ is unknown.

V. CONCLUSIONS

A method for extracting the long-distance forces present in an abridged infrared vertices has been developed and applied to several models. In order for this particular approach to be applicable the spinor particle must be massive. Further, the transformation which decouples the massive particle from the gauge field has been found only for the spinor field so far. It is thus impossible to apply this method to a theory where both spinors and gauge fields begin as massless. Thus, this approach cannot explain chiral-symmetry breaking in the spinor sector.

However, this approach has the distinct advantage that it can readily be incorporated into the quantized theory through the path integral. It also exhibits the relation of the quantized theory to the underlying classical field configurations, and relates these to the energy of interaction in a very obvious manner. In the case of non-Abelian fields it allows an escape from perturbative calculations of particle spectra and interaction energies, and demonstrates the fundamental differences between non-Abelian and Abelian field theories.

APPENDIX

In order to derive expression (4.14) in a reasonable amount of space a compact notation will be used. A phase angle without a prime will denote the phase angle evaluated at the momentum **p**, while primed phase angles are evaluated at zero momentum. The expressions c_i and s_i will refer to $\sin(\theta_i)$ and $\cos(\theta_i)$, respectively, while c_i' and s_i' are $\cos(\theta_i')$ and $\sin(\theta_i')$. It is straightforward to show that

$$\int d^3x\, u_p^\dagger(\mathbf{x},t)\Psi^a(\mathbf{x},t)=\int d^3x\, u_p^\dagger(\mathbf{x},t) \times U_{ab}(\theta,\mathbf{p},t)\Phi^b(\mathbf{x},t) \quad (\text{A1})$$

after the transformation. It has already been pointed out in (4.13) that the incoming color-singlet configuration is unchanged by the transformation. The only contractions (or alternatively functional integrations of spinor fields) which will give a nonzero contribution to the energy are those between the fields at time t and those at time t and those at time t_-. This means that only the field products in (4.5) of the form $\Phi^1\Phi^2$ need be kept, since all products of the form $\Phi^i\Phi^i$ will vanish when contracted against the in state (4.13). It follows that the only field products which need be kept take the form

$$\frac{1}{\sqrt{2}}\int d^3x\, d^3y\, u_p^\dagger(x)u_0^\dagger(y)[\Psi^1(x)\Psi^2(y)-\Psi^2(x)\Psi^1(y)]$$
$$=\frac{1}{\sqrt{2}}\int d^3x\, d^3y\, u_p^\dagger(x)u_0^\dagger(y)$$
$$\times\{[e^{-i(\theta_1-\theta_1')}(c_2c_3-is_2s_3)(c_2'c_3'+is_2's_3')-e^{i(\theta_1-\theta_1')}(c_2s_3-is_2c_3)(-c_2's_3'-is_2'c_3')]\Phi^1(x)\Phi^2(y)$$
$$+[e^{-i(\theta_1-\theta_1')}(-c_2s_3-is_2c_3)(c_2's_3'-is_2'c_3')-e^{i(\theta_1-\theta_1')}(c_2c_3+is_2s_3)(c_2'c_3'-is_2's_3')]\Phi^2(x)\Phi^1(y)\} . \quad (\text{A2})$$

In contracting against the incoming fields it is useful to note that since the spinors are now effectively free fields,

$$\int d^3x\, d^3y\, d^3x'\, d^3y'\, u_p^\dagger(\mathbf{x},t)u_0^\dagger(\mathbf{y},t)\langle \Phi_1(\mathbf{x},t)\Phi_2(\mathbf{y},t)\Phi_2^\dagger(\mathbf{x}',t_-)\Phi_1^\dagger(\mathbf{y}',t_-)\rangle u_p(\mathbf{x}',t_-)u_0(\mathbf{y}',t_-)=0 \quad \text{(A3)}$$

and

$$\int d^3x\, d^3y\, d^3x'\, d^3y'\, u_p^\dagger(\mathbf{x},t)u_0^\dagger(\mathbf{y},t)\langle \Phi_1(\mathbf{x},t)\Phi_2(\mathbf{y},t)\Phi_1^\dagger(\mathbf{x}',t_-)\Phi_2^\dagger(\mathbf{y}',t_-)\rangle u_p(\mathbf{x}',t_-)u_0(\mathbf{y}',t_-)=[\delta^3(0)]^2 , \quad \text{(A4)}$$

where angular brackets stand for the functional expectation value over the spinor variables only, and it is assumed that **p** is nonzero. Using these results it follows that the spinor variables may be integrated out of the theory and the normalization factor N cancels the δ functions of (A4). The result is a functional integral over the gauge field with a set of phase factors given from (A2) with the form

$$\cos(\theta_1-\theta_1')(c_2c_2'c_3c_3'+s_2s_2's_3s_3'+s_2s_2'c_3c_3'+c_2c_2's_3s_3')-\sin(\theta_1-\theta_1')(c_2s_2'c_3s_3'-c_2s_2's_3c_3'+s_2c_2'c_3s_3'-c_2s_2's_3c_3')$$
$$=\cos(\theta_1-\theta_1')\cos(\theta_2-\theta_2')\cos(\theta_3-\theta_3')-\sin(\theta_1-\theta_1')\sin(\theta_2+\theta_2')\sin(\theta_3-\theta_3') \quad \text{(A5)}$$

which is the desired result.

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[1]Manifestly covariant formulations of quantum electrodynamics employing a careful application of the Gupta-Bleuler condition generate a Coulomb interaction in the Feynman rules. However, as numerous authors have pointed out, quantum electrodynamics remains unitary without this procedure because of charge conservation for the spinors. Hence, in practice, the gauge condition is usually dropped with no damage to unitarity, and that is the justification for ignoring it in this paper. See K. Haller, Nucl. Phys. **B57**, 589 (1973); Acta Phys. Austriaca **42**, 163 (1975); M. S. Swanson, Phys. Rev. D **25**, 2086 (1982).

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