Variational principles for conservative and dissipative diffusions

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A stochastic variational principle is formulated for the conservative diffusions of stochastic mechanics in terms of the classical action. A suitable class of variations is chosen taking into account the time-reversal invariance of the theory. The resulting equations of motion are the stochastic Euler-Lagrange equations. Moreover a derivation of the Navier-Stokes equation is presented.

I. INTRODUCTION

Stochastic mechanics, introduced by Nelson in 1966,[1,2] has been formulated in a variational way in recent years. Different variational formulations of stochastic mechanics have been proposed, each one suggested by the different forms of the variational principle in classical mechanics.

The Guerra-Morato variational approach[3] is a generalization to the stochastic case of the Hamilton-Jacobi scheme. The starting point is an action functional defined as follows: for an R^d-valued Markov diffusion process $q(t)$ consider the expectation

$$\int dt\, E\{\tfrac{1}{2}m\dot{q}^{\,i}(t)\dot{q}^{\,i}(t)-V(q(t))\} \tag{1.1}$$

for some scalar potential V. This formal expression becomes meaningful after the subtraction of an infinite constant.[2] The functional of the process so obtained is called the mean classical action. The stationarity condition of this action functional with respect to variations of the drift field leads to the stochastic Hamilton-Jacobi equation.

The formulation above relies on the variations of the velocity field. It is also natural to consider variations of the trajectories of the process. Such an approach is usually called pathwise. The first attempt in this direction is due to Yasue[4] (see also Ref. 5 for recent developments) who considers a different action functional: namely, an energy-type one in the sense that the value of the functional on the physical process $q(t)$ is the time integral of the quantum Hamiltonian $(\psi,H\psi)$ on the state ψ described by the process $q(t)$. On the contrary the mean classical action (1) evaluated on the physical velocity field is connected to the standard quantum Lagrangian.[3]

Recently Morato[6,7] has proposed a stochastic pathwise variational principle using the mean classical action. The equations of motions are different from the stochastic Newton law since in this new formulation nonirrotational drift fields are also present. If the gradient condition for the drift field is assumed then the equation reduces to the stochastic Newton equation. Therefore the Hamilton-Jacobi variational formulation and the Lagrangian pathwise one give different stochastic dynamics.

The main result of this paper is that the two variational approaches are actually equivalent in stochastic mechanics, as they are in classical mechanics, if a suitable notion of variation is introduced. It is shown that two different kinds of variations have to be considered in order to recover this equivalence. The time-reversal symmetry properties of such variations play an important role for achieving the time reversibility of the stochastic mechanics. In fact a characteristic feature of the variational principle in Ref. 6 is that the solutions do not share the same temporal symmetry as the action functional. Because of that, this approach is very natural in order to describe dissipative diffusions.[8]

Finally a perspective of this work is that a pathwise picture of the stochastic mechanics, derived from a variational principle, could open the way to the construction of the canonical structure of the theory.

The paper is organized as follows. In Secs. II and III the time-reversal symmetry properties of the stochastic variational structure are analyzed. In Sec. IV the pathwise variational principle is formulated for the conservative diffusions of the stochastic mechanics. In Sec. V an example is given of a variational principle for the dissipative diffusions describing a Navier-Stokes incompressible fluid.

II. TIME-REVERSAL SYMMETRY BREAKING IN MORATO'S PATHWISE VARIATIONAL PRINCIPLE

In this section the structure of the stochastic calculus of variations given in Refs. 6 and 7 is investigated. Let $q(t)$ be an R^d-valued diffusion Markov process whose evolution is ruled by the stochastic integral equation

$$q(t)=q(t_0)+\int_{t_0}^{t} v_t(q(s),s)\,ds+\sqrt{\alpha}\int_{t_0}^{t} dw(s)\ . \tag{2.1}$$

Consider the conditional expectations

$$E\{\Delta^+q(t)\mid\mathcal{P}_t\},\quad E\{\Delta^-q(t)\mid\mathcal{F}_t\}\ ,$$

where

$$\Delta^+q(t)=q(t+\Delta t)-q(t)\ ,$$

$$\Delta^-q(t)=q(t)-q(t-\Delta t)$$

are the forward and the backward increments, $\mathcal{P}_t$ ($\mathcal{F}_t$) is an increasing (decreasing) family of σ algebras such that each $q(t)$ is $\mathcal{P}_t$ measurable ($\mathcal{F}_t$ measurable). Then

$$\lim_{\Delta t\to 0+}(\Delta t)^{-1}E\{\Delta^+q(t)\mid\mathcal{P}_t\}=v_+(q(t),t)\ , \tag{2.2}$$

$$\lim_{\Delta t\to 0+}(\Delta t)^{-1}E\{\Delta^-q(t)\mid\mathcal{F}_t\}=v_-(q(t),t)\ . \tag{2.3}$$

Moreover v_+ and v_- are related by

$$(v_+ - v_-)(x,t) = \alpha \nabla \ln\rho(x,t) , \tag{2.4}$$

where ρ is the density.

The current velocity v and the osmotic velocity u are defined by

$$v_+ + v_- = 2v, \quad v_+ - v_- = 2u . \tag{2.5}$$

The Morato variational principle assumes as action functional the mean classical action

$$A_{[t_0,t_1]}(q) = \int_{t_0}^{t_1} dt\, E\{L(q(t),t)\} , \tag{2.6}$$

where

$$L(x,t) = \tfrac{1}{2} m (v_+ \cdot v_-)(x,t) - V(x) . \tag{2.7}$$

The class of variations δq is defined in such a way that $(q+\delta q)(t) \equiv q'(t)$ is a diffusion Markov process with the same diffusion coefficient and the same initial density $\rho_0 \equiv \rho(\ ,t_0)$ as $q(t)$. Denoting by $\{t_i\}_{i=1,\dots,N}$ a partition of $[t_0,t_1]$ in N intervals of mesh ϵ and by p_{t_1} an arbitrary random variable, Morato's result[6,7] can be stated as

$$\lim_{N\to\infty} \delta \sum_i \epsilon E\{\tfrac{1}{2} m \epsilon^{-2} \Delta^+(q(t_i)) \cdot \Delta^-(q(t_i)) - V(q(t_i))\} - E\{p_{t_1} \cdot q(t_1)\} = 0 \tag{2.8}$$

to first order in δq for any δq such that $\delta q(t_0)=0$ if and only if the current and the osmotic velocities satisfy the equation

$$\partial_t v + (v\cdot\nabla)v - [(\alpha/2)\nabla^2 u + (u\cdot\nabla)u] + u \wedge (\nabla \wedge v) + (\alpha/2)\nabla \wedge (\nabla \wedge v) = -(1/m)\nabla V \tag{2.9}$$

with the final condition $v(\ ,t_1) = p_{t_1}$.

For the sake of simplicity we consider the case $d=3$. Taking into account the continuity equation

$$D\rho = -\rho(\nabla \cdot v) \quad \text{and} \quad u = (\alpha/2)\nabla \ln\rho$$

the evolution equations for u and v can be written as

$$\begin{aligned} &\partial_t u + (\alpha/2)\nabla^2 v + (u\cdot\nabla)v + (v\cdot\nabla)u + (\alpha/2)\nabla\wedge(\nabla\wedge v) + u\wedge(\nabla\wedge v) = 0 , \\ &\partial_t v + (v\cdot\nabla)v - (\alpha/2)\nabla^2 u - (u\cdot\nabla)u + (\alpha/2)\nabla\wedge(\nabla\wedge v) + u\wedge(\nabla\wedge v) = -(1/m)\nabla V . \end{aligned} \tag{2.10}$$

The Morato equation (2.9) differs from the Newton stochastic law[1,2]

$$\partial_t v + (v\cdot\nabla)v - (\alpha/2)\nabla^2 u - (u\cdot\nabla) = -(1/m)\nabla V \tag{2.11}$$

by the presence of the terms

$$u\wedge(\nabla\wedge v) + (\alpha/2)\nabla\wedge(\nabla\wedge v) . \tag{2.12}$$

On the other hand, the Guerra-Morato variational principle, based on the same action functional and on the variation of the drift field, leads to Madelung's fluid equations which are equivalent to the stochastic Newton law. Therefore the two different ways to vary the same action functional, by varying the paths or the drifts of the process, are not equivalent. Further Eq. (2.9) is not time-reversal invariant. In fact let us denote by T the time-reversal transformation (see, e.g., Ref. 9)

$$\begin{aligned} T:\ & t \to t^* = -t , \\ & x(t) \to x^*(t^*) = x(t) , \\ & v(t) \to v^*(t^*) = -v(t), \quad u(t) \to u^*(t^*) = u(t) ; \end{aligned} \tag{2.13}$$

then under T (2.9) changes to

$$\partial_t v + (v\cdot\nabla)v - (\alpha/2)\nabla^2 u - (u\cdot\nabla)u - u\wedge(\nabla\wedge v) - (\alpha/2)\nabla\wedge(\nabla\wedge v) = -(1/m)\nabla V . \tag{2.14}$$

Note that the terms involving $\nabla \wedge v$ have changed sign.

On the contrary, the action functional (2.6) is time-reversal invariant as one can see from (2.5) and (2.13). Then a typical phenomenon of symmetry breaking is involved: the solutions of the variational principle do not share the symmetry property of the action. The diffusion processes whose drift evolves according to (2.9) may be dissipative processes. The possibility of describing dissipative systems in a variational framework is an interesting aspect of the Morato variational calculus. A development along this line is contained in Ref. 8, where Eq. (2.9) is interpreted as the evolution equation for a viscous quantum fluid near zero temperature in some kind of mean-field approximation.

In conclusion the presence of dissipative diffusions means that a direction of time has been chosen at some step of the construction.

Let us look more closely at the class of variations. Given a process $q(t)$ with drift v_+ and initial density ρ_0 denoted by $x(t)$ the path starting in x_0 at time t_0 constructed from a specific path $w(t)$ of a standard Wiener process by means of

$$x(t) = x_0 + \int_{t_0}^{t} dx\, v_+(x(s),s) + \sqrt{\alpha} \int_{t_0}^{t} dw(s) . \tag{2.15}$$

Now associate to the same realization $w(t)$ a different path $x'(t)$ in such a way that

$$x'(t) = x_0 + \int_{t_0}^{t} ds\, v'_+(x'(s),s) + \sqrt{\alpha}\int_{t_0}^{t} dw(s) \tag{2.16}$$

for some smooth drift v'_+.

By this procedure we obtain a mapping from the sample path space of the Wiener process to a sample path space of a smooth Markovian diffusion $q'(t)$, with drift v'_+ and initial density ρ_0. So the variation process $\delta^+ q(t) \equiv q'(t) - q(t)$ satisfies the equation

$$\delta^+ q(t) = \int_{t_0}^{t} ds[v'_+(q'(s),s) - v_+(q(s),s)] . \tag{2.17}$$

Therefore the process $\delta^+ q(t)$ has differentiable sample paths and it is $\mathcal{P}_t$ measurable since it depends on $q(s)$ for $t_0 \le s \le t$, but it is not independent of $\mathcal{F}_t$. These peculiar properties of the variation processes are responsible for the presence of non-time-reversal-invariant solutions.[6,7]

Let us now exploit another possible construction of variation processes. A diffusion process can be viewed as a solution of the backward differential stochastic equation

$$q(t)=x_1-\int_t^{t_1} ds\, v_-(q(s),s)-\sqrt{\alpha}\int_t^{t_1} d\tilde{w}(s) \tag{2.18}$$

where $\tilde{w}(s)$ is the Wiener backward process. In this way $q(t)$ can be reconstructed backward in time from a final condition $q(t_1)=x_1$. Starting from the representation (2.18) we can construct a varied process with the same rule as before. For any realization $w(t)$ such that

$$x(t)=x_1-\int_t^{t_1} ds\, v_-(x(s),s)-\sqrt{\alpha}\int_t^{t_1} d\tilde{w}(s) \tag{2.19}$$

we construct a trajectory $x''(t)$ from

$$x''(t)=x_1-\int_t^{t_1} dx\, v''_-(x''(s),s)-\sqrt{\alpha}\int_t^{t_1} d\tilde{w}(s) \tag{2.20}$$

for some smooth function $v''_-(x,t)$. In this way we obtain a smooth Markov process $q''(t)$ with the same diffusion constant as $q(t)$ and the same final density. The variation process

$$(\delta^- q)(t)\equiv q''(t)-q(t) \tag{2.21}$$

satisfies the equation

$$(\delta^- q)(t)=-\int_t^{t_1} ds\,[v''_-(q''(s),s)-v_-(q(s),s)]\ . \tag{2.22}$$

Then the variations, $\delta^- q$ have differentiable sample paths, but have different measurability properties than $\delta^+ q$; in fact $\delta^- q(t)$ is $\mathcal{I}_t$ measurable, since it depends on $q(s)$ for $t\le s\le t_1$. So we can obtain the set of the varied processes with the same diffusion constant by means of two kinds of variations $\delta^+ q$ and $\delta^- q$ that we call forward and backward variations, respectively.

Up to now the emphasis has been put on the process, but in this kind of pathwise variational approach the paths play the main role. Therefore let us look at the variations of the paths rather than the variations of the processes. As an example let us consider two smooth Markovian diffusions q and q', with drifts v_+ and v'_+, respectively, for a fixed diffusion constant α, having the same initial density ρ_0 and also the same final density ρ_1. Let $x(t)$ be a sample path of the process $q(t)$ starting in x_0 at time t_0 and such that $q(t_1)=x_1$. Suppose that the process $q'(t)$ has been chosen in such a way that, if we regard q' as $q+\delta^+ q$, with $\delta^+ q(t_0)=0$, there exists a sample path of q', $x^+(t)$, such that

$$x^+(t)-x(t)=\int_t^{t_1} ds[v'_+(x^+(s),s)-v_+(x(s),s)] \tag{2.23}$$

and $x^+(t_1)=x_1$.

If we look at the process $q(t)$ backward in time then

$$x(t)=x_1-\int_1^{t_1} ds[v_-(x(s),s)]-\sqrt{\alpha}\int_t^{t_1} d\tilde{w}(s)\ .$$

On the other hand, $q'(t)=q+\delta^- q$, with $\delta^- q(t_1)=0$, implies that there exists a path $x^-(t)$ associated to the same $\tilde{w}(t)$ such that

$$x^-(t)=x_1-\int_t^{t_1} ds\, v'_-(x^-(s),s)-\sqrt{\alpha}\int_t^{t_1} d\tilde{w}(s) \tag{2.24}$$

so that

$$x^-(t)-x(t)=-\int_t^{t_1} ds[v'_-(x^-(s),s)-v_-(x(s),s)]\ ,\quad t\in[t_0,t_1] \tag{2.25}$$

is a sample path of $\delta^- q$.

It is evident that $x^+(t)$ and $x^-(t)$ are in general different. We can think of them as the trajectory varied forward and backward in time. Therefore two processes can be compared looking at the past or future in terms of trajectories. The Morato variational calculus chooses the forward way to confront the processes.

In conclusion the set of varied processes is the same in the pathwise and in the control variational principle. What is different is the topological structure, that is the notion of vicinity of processes differs in the two cases. In the Guerra-Morato approach two processes are considered "near" if $\delta v=v'-v$ is "small," while in the pathwise approach[6] the vicinity of the processes is induced by a notion of vicinity of the forward trajectories. As a consequence, different definitions of differentiability are involved. Therefore in order to state our variational principle we need a definition of the differential of a functional on the processes.

Let us denote by $\Omega(\alpha)$ the set of smooth Markovian diffusions corresponding to the diffusion constant α. For any $q,q'\in\Omega(\alpha)$ with the same density at some time T let us define

$$\|\delta^+ q\|=\sup_{w,t\ge T}|x^+(t)-x(t)|\ ,\qquad \|\delta^- q\|=\sup_{w,t\le T}|x^-(t)-x(t)|\ . \tag{2.26}$$

We call forward variation $\delta^+ F$ and backward variation $\delta^- F$ of a functional F on $\Omega(\alpha)$:

$$(\delta^\pm F)(q)=\frac{d}{d\lambda}F(q+\lambda\delta^\pm q)\bigg|_{\lambda=0}\ . \tag{2.27}$$

Then a functional F on $\Omega(\alpha)$ is said to be forward (backward) differentiable on $q(t)$ if

$$F(q+\delta^+ q)-F(q)=\delta^+ F+o(\|\delta^+ q\|)\ , \tag{2.28}$$

$$F(q+\delta^- q)-F(q)=\delta^- F+o(\|\delta^- q\|)\ . \tag{2.29}$$

This differential structure will allow us to formulate a symmetric variational principle.

III. TIME REVERSAL OF VARIATIONS

Classical mechanics is a time-reversible theory. The variation of the action functional or the variation of the time-inverted one yields the same equations of motion which are in fact time-reversal invariant. On the contrary, in stochastic mechanics, at least in the stochastic calculus we are investigating, this is no longer true. For a process $q\in\Omega(\alpha)$, corresponding to a drift v_+ and to a given density $\rho=\rho(\ ,T)$ for some instant T, let us consider the time-inverted process $q^*(t^*)$ satisfying the time-inverted equation

$$q^*(t'^*)=q^*(t^*)+\int_{t^*}^{t'^*} ds^* v_+^*(q^*(s^*),s^*)+\sqrt{\alpha}\int_{t^*}^{t'^*} dw^*(s^*),\quad t^*\le t'^*\ . \tag{3.1}$$

Then we can construct the set of varied processes with respect to q^*, with the same density ρ, by means of the forward variations in t^*. As explained in the preceding section, we can consider $(q^*+\delta q^*)(t^*)$ with $(\delta^+q^*)(T^*)=0$ and the corresponding variation δ^+q^* satisfying, for some $v_+^{*\prime}$,

$$(\delta^+q^*)(t^*)=\int_{T^*}^{t^*} ds^*[v_+^{*\prime}(q^{*\prime}(s^*),s^*)-v_+^*(q(s^*),s^*)] , \quad t^*\geq T^* . \tag{3.2}$$

The $\mathcal{P}_{t^*}$ measurability of $(\delta^+q^*)(t^*)$ implies the $\mathcal{F}_t$ measurability when it is regarded as a function of t. Taking into account that $v_+^*(q^*(t^*),t^*)=-v_-(q(t),t)$ and $v_+^{*\prime}(q^{*\prime}(t^*),t^*)=-v'_-(q(t),t)$, we have that

$$(\delta^+q^*)(t^*)=-(\delta^-q)(t) . \tag{3.3}$$

Note that the variations transform under time reversal in the same way as the forward and backward differentials dq and $d*q$. This is an essential requirement on the variations if one is interested in introducing a canonical structure.

Now we want to state the relation between $\delta^{\pm}F$ and δ^-F for a functional F on $\Omega(\alpha)$ such that $F^*(q^*)=F(q)$. From the definition (2.27) it follows that

$$\begin{aligned}(\delta^+F^*)(q^*)&=\frac{d}{d\lambda}F^*(q^*+\lambda\delta^+q^*)\bigg|_{\lambda=0}\\&=-\frac{d}{d\lambda}F(q+\lambda\delta^-q)\bigg|_{\lambda=0}\\&=-(\delta^-F)(q) .\end{aligned} \tag{3.4}$$

Therefore the operations δ^+ and δ^- play the same role as the forward and backward derivatives[2] D_+ and D_- when nontemporal increments are considered. The role of the time is played in this case by λ. Hence this differential structure seems to put the time and the other variables on the same footing. We plan to return to this subject in a forthcoming paper.[10]

We shall be interested in the following in an action functional defined by

$$A_{[0,T]}(q,S_T)=E\int_0^T L(q(s),s)ds-E\{S_T(q(T),T)\} , \tag{3.5}$$

where $S_T(\ ,T)$ is an arbitrary function which will play the role of a Lagrangian multiplier controlling the variation of the density at time T in the variational principle. Now we introduce the time-reversed version of this action functional by

$$A^*_{[T^*,0]}(q^*,S_0^*)=E\int_{T^*}^0 L(q^*(s^*),s^*)ds^* -E\{S_0^*(q^*(0),0)\} \quad (T^*\leq 0) . \tag{3.6}$$

Since it will turn out from the variational principle that the function S_0 is connected to the velocity field by the relation $mv(\ ,0)=\nabla S_0(\ ,0)$ we assume the following transformation property under the time reversal T for S_0:

$$S_0^*(q^*,0)=-S_0(q,0) . \tag{3.7}$$

Then the functional A^* is related to an action functional A' by

$$A^*_{[T^*,0]}(q^*,S_0^*)=A'_{[0,T]}(q,S_0) , \tag{3.8}$$

where

$$A'_{[0,T]}(q,S_0)=E\int_0^T L(q(s),s)ds+E\{S_0(q(0),0)\} . \tag{3.9}$$

A' differs from A in the fact that the Lagrangian multiplier S_0 is now a function of the variables at the initial time. Therefore A' has to be used in the variational principle when one wants to keep the final density fixed.

Finally for the variations we have

$$\delta^+A^*(q^*,S_0^*)=-\delta^-A'(q,S_0) . \tag{3.10}$$

We shall show in Sec. IV that the backward variation of A' and the forward variation of A give rise to different equations of motions, in which one turns out to be the time reversal of the other. Therefore the variational principle based on the forward variation gives different dynamics for the process and its time-inverted counterpart, even if the Lagrangian is time-reversal invariant. We could say that the variation does not commute with the time-reversal transformation; hence, the variational principle has to be formulated in a symmetric way.

IV. VARIATIONAL PRINCIPLE FOR CONSERVATIVE DIFFUSIONS: STOCHASTIC NEWTON EQUATION

Stochastic mechanics, as a stochastic formulation of quantum mechanics, is a time-reversible theory and the diffusions associated to the quantum states are conservative. The control variational principle, which preserves this time-reversal property, can be formulated starting from the functional A, keeping the initial density fixed, as well as for the functional A', with the final density fixed. In the first case the controlling velocity field is v_+, while in the second one it is v_-. In the pathwise variational principle instead, since the pathwise variations are intrinsically not symmetric in time, we have to consider the forward variation for A and the backward one for A', namely, keeping fixed the initial or the final density, respectively. The following proposition shows that in the pathwise case the forward and the backward variational formulations give different equations.

Proposition. A process $q(t)\in\Omega(\alpha)$ is extremal for the functional $A'_{[0,T]}(q,S_0)$ under the variations $\delta^-q(s)$, $s\in[0,T]$, such that $\delta^-q(T)=0$ if and only if its drift satisfies the equation

$$\partial tv+(v\cdot\nabla)v-[(\alpha/2)\nabla 2u+(u\cdot\nabla)u]-u\wedge(\nabla\wedge v) -(\alpha/2)\nabla\wedge(\nabla\wedge v)=-(1/m)\nabla V \tag{4.1}$$

and the boundary condition

$$mv(\ ,0)=\nabla S_0(\ ,0) .$$

Proof. Denote by $A'^N(q)$ the discretized action; that is,

$$A'^N(q)=\sum_i \epsilon E\{\tfrac{1}{2}m\epsilon^{-2}\Delta^+ q(t_i)\Delta^- q(t_i)-V(q(t_i))\}+E\{S_0(q(0),0)\} \quad (4.2)$$

so that

$$A'_{[0,T]}(q,S_0)=\lim_{N\to\infty} A'^N(q)\ . \quad (4.3)$$

Then the variation of A'^N is given by

$$A'^N(q+\delta q)-A'^N(q)=\sum_i \epsilon E\{\tfrac{1}{2}m\epsilon^{-2}[\Delta^+ q(t_i)\Delta^-(\delta^- q(t_i))+\Delta^+(\delta^- q(t_i))\Delta^- q(t_i)]-\nabla V(q(t_i))\delta^- q(t_i)\}$$
$$-E\{\nabla S_0(q(0),0)\cdot\delta^- q(0)\}+o(\|\delta^- q\|)\ . \quad (4.4)$$

The measurability properties of $\delta^- q$ imply

$$E\{\Delta^+(\delta^- q(t))\Delta^- q(t)\}=E\{\Delta^+(\delta^- q(t))E\{\Delta^- q(t)\mid \mathcal{J}_t\}\}=\epsilon E\{\Delta^+(\delta^- q(t))\cdot v_-(q(t),t)\}+o(\epsilon) \quad (4.5)$$

and

$$E\{\Delta^+ q(t)\Delta^-(\delta^- q(t))\}=E\{\epsilon v_+(q(t),t)\cdot\Delta^-(\delta^- q(t))+\sqrt{\alpha}\Delta^+ w\cdot\Delta^-(\delta^- q(t))\}+o(\epsilon)\ , \quad (4.6)$$

where we have exploited (2.1).

By means of summation by parts one has

$$E\{v_-(q(t),t)\cdot\Delta^+(\delta^- q(t))+v_+(q(t),t)\cdot\Delta^-(\delta^- q(t))\}$$
$$=E\{\Delta^+[v_-(q(t),t)\cdot\delta^- q(t)]+\Delta^-[v_+(q(t),t)\cdot\delta^- q(t)]-\Delta^+ v_-(q(t),t)\cdot\delta^- q(t)-\Delta^- v_+(q(t),t)\cdot\delta^- q(t)\}+o(\epsilon)\ . \quad (4.7)$$

Finally

$$E\{\Delta^+ v_-(q(t),t)\cdot\delta^- q(t)\}=E\{\epsilon D_+ v_-(q(t),t)\cdot\delta^- q(t)+\sqrt{\alpha}\nabla_i v_-(q(t),t)\Delta^+ w^i(t)\cdot\delta^- q(t)\}+o(\epsilon)\ , \quad (4.8)$$

$$E\{\Delta^- v_+(q(t),t)\cdot\delta^- q(t)\}=E\{\epsilon D_- v_+(q(t),t)\cdot\delta^- q(t)\}+o(\epsilon)\ , \quad (4.9)$$

because $\delta^- q$ is $\mathcal{J}_t$ measurable.

Collecting all the terms we have

$$A'^N(q+\delta q)-A'^N(q)=\sum_i \epsilon E\{\tfrac{1}{2}m[\epsilon^{-1}\Delta^+ v_-(q(t_i),t_i)\cdot\delta^- q(t_i)+\epsilon^{-1}\Delta^- v_+(q(t_i),t_i)\cdot\delta^- q(t_i)$$
$$-(D_+ v_- + D_- v_+)(q(t_i),t_i)\cdot\delta^- q(t_i)]-\nabla V(q(t_i))\cdot\delta^- q(t_i)\}$$
$$-E\{\nabla S_0(q(0),0)\cdot\delta^- q(0)\}$$
$$+\sqrt{\alpha}\sum_i \epsilon E\{\tfrac{1}{2}m\Delta^+ w^k(t)[\epsilon^{-2}\Delta^-(\delta^- q^k(t_i))-\epsilon^{-1}\nabla_k v_-^j(q(t_i),t_i)\delta^- q^j(t_i)]\}$$
$$+o(\epsilon)+o(\|\delta^- q\|)\ . \quad (4.10)$$

Let us examine the last term in (4.10). Taking into account the relations

$$\Delta^-(\delta^- q^k(t))=\epsilon\nabla_j v_-^k(q(t),t)\delta^- q^j(t)+\epsilon(v'_- - v_-)(q(t),t)+o(\|\delta^- q\|)\ , \quad (4.11)$$

$$\sqrt{\alpha}\Delta^+ w(t)=\sqrt{\alpha}\Delta^-\tilde{w}(t+\epsilon)-2u(q(t),t)\epsilon+o(\epsilon)\ , \quad (4.12)$$

the result is

$$\sqrt{\alpha}E\{\tfrac{1}{2}m\Delta^+ w^k(t)[\epsilon^{-2}\Delta^-(\delta^- q^k(t))-\epsilon^{-1}\nabla_k v_-^j(q(t),t)\delta^- q^j(t)]$$
$$=\sqrt{\alpha}E\{\tfrac{1}{2}m\epsilon^{-1}\Delta^+ w^k(\nabla_j v_-^k - \nabla_k v_-^j)\delta^- q^j(t)$$
$$=\sqrt{\alpha}E\{\tfrac{1}{2}m\epsilon^{-1}\Delta^-\tilde{w}^k(t+\epsilon)F_{kj}(q(t),t)\delta^- q^j(t)\}-mE\{(u^k F_{kj})(q(t),t)\delta^- q^j(t)\}+o(\epsilon)+o(\|\delta^- q\|)\ , \quad (4.13)$$

where we have set $F_{kj}=\nabla_j v_-^k - \nabla_k v_-^j$.

Using the Ito differential rule

$$F_{kj}(q(t),t)=F_{kj}(q(t+\epsilon),t+\epsilon)-\nabla_s F_{kj}(q(t+\epsilon),t+\epsilon)\Delta^-\tilde{w}^s(t+\epsilon)+o(\epsilon)\ , \quad (4.14)$$

the last term in (4.10) becomes

$$\sqrt{\alpha}\tfrac{1}{2}m \sum_i E\{\Delta^- w^k(t_i+\epsilon)\nabla_s F_{kj}(q(t_i+\epsilon),t_i+\epsilon)\Delta^-\tilde{w}^s(t_i+\epsilon)\delta^- q^j(t_i+\epsilon)\}$$
$$-m\sum_i \epsilon E\{(u^k F_{kj})(q(t_i),t_i)\delta^- q^j(t_i)\}+o(\epsilon)+o(\|\delta^- q\|) \; . \tag{4.15}$$

Finally using again the measurability property of $\delta^- q$ (4.15) transforms in

$$\sum_i E\{\epsilon\sqrt{\alpha}\tfrac{1}{2}m[(\nabla_k F_{kj})(q(t_i+\epsilon),t_i+\epsilon)\delta^- q^j(t_i+\epsilon)]-m(u^k F_{kj})(q(t_i),t_i)\delta^- q^j(t_i)+o(\epsilon)+o(\|\delta^- q\|) \tag{4.16}$$

In conclusion, the backward variation is given by

$$\delta^- A'_{[0,T]}(q,S_0)=-\lim_{N\to\infty}\sum_i \epsilon m E\{\delta^+ q(t_i)[\tfrac{1}{2}(D_+v_- + D_-v_+)-\nabla V+\tfrac{1}{2}\alpha\nabla\wedge(\nabla\wedge v)+u\wedge(\nabla\wedge v)](q(t_i),t_i)\}$$
$$+E\{\delta^- q(0)\cdot(mv-\nabla S_0)(q(0),0)\} \; , \tag{4.17}$$

where we have used $\delta^- q(T)=0$.

Then $\delta^- A'_{[0,T]}(q,S_0)=0$ if the velocity field v satisfies Eq. (4.1) and there exists a function S_0 such that $mv(x,0)=\nabla S_0(x,0)$.

The proof of the necessarity of these conditions can be modeled on the proof given in Ref. 6 and it will not be reported here.

Now we have to restore explicitly the temporal symmetry. Since the variation δ^+ transforms under time reversal into $-\delta^-$, we look for the diffusions which are extremal for $A_{[0,T]}(q,S_T)$ under the variations $\delta^+ q$: $\delta^+ q(0)=0$ *and* also extremal for $A'_{[0,T]}(q,S_0)$ under the variations $\delta^- q$: $\delta^- q(T)=0$. Then we claim $\delta^- A'_{[0,T]}(q,S_0)=0$ and $\delta^+ A'_{[0,T]}(q,S_T)=0$ if and only if the process $q(t)$ satisfies the Newton stochastic law

$$\tfrac{1}{2}(D_+v_- + D_-v_+)=-(1/m)\nabla V \tag{4.18}$$

and there exist some function $S(x,t)$ such that $mv(x,t)=\nabla S(x,t)$ for $t\in[0,T]$.

In fact the vanishing of the forward and backward differentials is equivalent to the following conditions:

$$\partial_t v+(v\cdot\nabla)v-[(\alpha/2)\nabla^2 u+(u\cdot\nabla)u]=-(1/m)\nabla V \; , \tag{4.19a}$$

$$(\alpha/2)\nabla\wedge(\nabla\wedge v)+u\wedge(\nabla\wedge v)=0 \; , \tag{4.19b}$$

$$\exists S_0 \text{ and } S_T\text{: } mv(x,0)=\nabla S_0(x,0) \; ,$$
$$mv(x,T)=\nabla S_T(x,T) \; . \tag{4.19c}$$

The gradient condition at time 0 and the Newton stochastic law (4.19a) imply a gradient condition for any time. In fact, (4.19a) implies the Euler equation for the vorticity $\omega=\nabla\wedge v$:

$$D\omega\equiv\partial_t\omega+(v\cdot\nabla)\omega=(\omega\cdot\nabla)v-\omega(\nabla\cdot v) \; . \tag{4.20}$$

Then as in the Euler case if ω is zero at some instant it is zero for any time. Therefore (4.19b) is identically satisfied as a consequence of (4.19a) and (4.19c).

It is worthwhile to remark that the equation for the vorticity derived from (2.9) is

$$D_-\omega=(\omega\cdot\nabla)v_- -\omega(\nabla\cdot v_-) \; . \tag{4.21}$$

In conclusion this symmetric variational principle implies the following equations for the velocity fields:

$$\partial_t u+(\alpha/2)\nabla^2 v+(u\cdot\nabla)v+(v\cdot\nabla)u=0 \; ,$$
$$\partial_t v+(v\cdot\nabla)v-(\alpha/2)\nabla^2 u-(u\cdot\nabla)u=-(1/m)\nabla V \; , \tag{4.22}$$

which are equivalent, taking into account the gradient condition for v, to the Schrödinger equation.

V. VARIATIONAL PRINCIPLE FOR DISSIPATIVE DIFFUSIONS: NAVIER-STOKES EQUATION

An example of dissipative diffusions derived from a variational principle is the one satisfying the Morato equation, that is, a process extremal for the classical mean action with respect to forward variations.

Another example of a dissipative system which can be described in this variational framework is the Navier-Stokes incompressible fluid. The Navier-Stokes equation was derived by means of a variational principle first by Inoue-Funaki[11] and then by Yasue[12] (see also Ref. 13) who extended the Arnold variational principle for the Euler flow to the stochastic case.

The stochastic calculus developed in Ref. 6 allows us to give another derivation of the Navier-Stokes equation. Let us emphasize that in this case the set of varied processes is formed by the smooth diffusions corresponding to the same viscosity coefficient.

If one imposes the incompressibility condition Eqs. (2.10) become similar to the Navier-Stokes equations, if one regards the potential V as the pressure. But the interpretation is different since the pressure for an incompressible Navier-Stokes fluid is an unknown quantity while V in (2.10) is *a priori* given. In order to avoid this unsatisfactory feature one has to introduce the pressure field as a Lagrangian multiplier associated to the constraint of incompressibility.

We consider as action functional the integral of the energy of the viscous fluid

$$A_{\rm NS}(q)=\int_0^T E\{\tfrac{1}{2}v^2(q(t),t)\}dt \; . \tag{5.1}$$

$A_{\rm NS}$ is defined on the space of the volume-preserving processes, that is on the space of smooth Markovian diffusions such that the density ρ is constant. Therefore $\mathrm{div} v=0$ and $v_+=v_-$. Hence we have

$$E\{\tfrac{1}{2}v^2(q(t),t)\}=E\{\tfrac{1}{2}(v_+\cdot v_-)(q(t),t)\}$$
$$=\tfrac{1}{2}\epsilon^{-2}E\{\Delta^+q(t)\cdot\Delta^-q(t)\}$$
$$+o(\epsilon)\ . \quad (5.2)$$

We construct $A_{\rm NS}$ as the limit

$$A_{\rm NS}(q)=\lim_{N\to\infty}\sum_i \epsilon E\{\Delta^+q(t_i)\Delta^-q(t_i)\}\ . \quad (5.3)$$

We want to point out that it is impossible to derive the Navier-Stokes equation from the action functional (5.2) by varying the velocity field. We must use the pathwise calculus. An analogous phenomenon is encountered in the Euler fluid case. If one wants to derive the Euler equation by varying v, only the potential Euler flows are found. In order to describe also nonirrotational velocity fields more conditions have to be taken into account.[14]

We look for the volume-preserving processes such that $\delta^+A_{\rm NS}=0$ where the variation is constructed by means of the forward variation constrained by

$$\mathrm{div}\delta^+q=0 \quad (5.4)$$

almost surely, in order that the varied process $q+\delta^+q$ also have the volume-preserving property. We consider the variations δ^+q such that $\delta^+q(0)=0$. Henceforth we have to modify the functional into

$$A'_{\rm NS}(q)=A_{\rm NS}+E\{v_T\cdot q(T)\}\ , \quad (5.5)$$

where v_T is an arbitrary random variable that will turn out to be the final value of the velocity field.

Performing the same calculations as in Sec. IV we obtain the expression of the forward variation of $A'_{\rm NS}$

$$\delta^+A'_{\rm NS}(q)=\lim_{N\to\infty}\sum_i \epsilon E\{(Dv-\tfrac{1}{2}\alpha\Delta v)(q(t_i),t_i)\cdot\delta^+q(t_i)\}$$
$$+E\{[v(q(T),T)-v_T]\delta^+q(T)\}\ . \quad (5.6)$$

Taking into account the constraint of incompressibility on the variations (5.4) it follows that $\delta^+A'_{\rm NS}(q)=0$ for any δ^+q such that $\mathrm{div}\delta^+q=0$ if and only if there is a scalar function p such that

$$Dv-\tfrac{1}{2}\alpha\Delta v=-\nabla p,\quad v(x(T),T)=v_T\ . \quad (5.7)$$

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