Symmetry behavior in curved spacetime: Finite-size effect and dimensional reduction

B. L. Hu
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742 and The Institute for Advanced Study, Princeton, New Jersey 08540

D. J. O'Connor
Center for Relativity, Department of Physics, University of Texas, Austin, Texas 78712
(Received 22 December 1986; revised manuscript received 14 May 1987)

We discuss the effect of certain aspects of curvature and topology on symmetry breaking in curved spacetime with the aim of understanding phase transitions in the early Universe. We show that for spacetimes with some compact spatial dimensions where the invariant operator of the scalar fluctuation field has a discrete spectrum, the most important contribution to the infrared behavior comes from its zero mode (or band). The decoupling of higher modes gives rise to dimensional reduction in the infrared domain. We introduce the notion of effective infrared dimension and explain how it can be useful for physically understanding the symmetry behavior in spacetimes of different topology. We also introduce an eigenvalue analysis to study dimensional reduction in both direct-product spaces and spaces which can approximate product spaces under extreme deformations. We illustrate this method by analyzing the symmetry behavior of the Taub universe in the small- and large-anisotropy limits. These geometric effects in curved space are of the same nature as finite-size effects in condensed matter and surface physics. We use the two-particle-irreducible formalism and the large-N approximation to derive the higher-loop corrections. We give the results of the effective mass and the effective potential for various systems and spacetimes with compact dimensions. The ideas, techniques, and results developed here are useful for the study of finite-size effects in field theory, cosmology, and condensed-matter physics.

I. INTRODUCTION

One major area of research in quantum field theory in curved spacetime in the 1980s has been on interacting quantum fields and their implications.[1] The emphasis has since gradually shifted from the analysis of local properties of fields and geometry such as ultraviolet-divergence and renormalization problems to more global aspects. Examples are gravitational instantons in Euclidean quantum gravity,[2] the Gauss-Bonnet terms in the higher-derivative corrections of gravitational Lagrangians,[3] the effect of the so-called kinetic terms and the quasilocal expansion in the effective action,[4] renormalization-group derivation of running coupling constants,[5] etc. Understanding the full character of physical processes such as particle production and back reaction in the early Universe[6] requires the full nonlocal form of the effective action. Symmetry breaking in curved space is another problem of this nature, wherein the global properties of spacetime are expected to play an important role.[7]

Work on symmetry breaking in curved space has progressed in several stages, from the somewhat formal and illustrative problems to the more physical and realistic ones. General relativists are naturally interested in the effect of geometry, topology, and boundary conditions (including twisted fields) on the symmetry behavior of a system described by quantum fields.[8] These preliminary investigations demonstrated the importance of the global effects of geometry, but the problems initially studied were perhaps mostly of academic interest. At the same time field theorists were interested in symmetry breaking in connection with unified theories at very high energies. Interest in the decay of the false vacuum has led Coleman and others[9] to consider the effect of gravity on tunneling as a form of symmetry breaking. This led to interesting results but the effects of curved spacetime considered were mainly classical and the approaches used were mainly flat-space techniques. The advent of inflationary cosmology[10] has spurred some serious interest in phase transitions in the early Universe.[11] Details even down to the precise form of the effective potential have become important. Formal inquiries and realistic problems merged to add the impetus for more in-depth investigation of symmetry breaking in curved spacetime. Systematic studies have since been carried out, some using full-fledged curved-space quantum-field-theoretical techniques.[12–18]

Our interest in this topic stems from attempts to understand how quantum gravitational effects related to global properties of spacetime such as topology and geometry enter in the quantum-field-theoretical description of systems undergoing symmetry breaking. The early Universe around the Planck time is a testing ground for these inquiries.[19] We are interested in the general properties as well as the particular characteristics describing specific spacetimes of realistic interest. For this purpose we have carried out detailed analysis of the symmetry behavior of systems (involving scalar and gauge fields) in the Einstein,[14] Taub,[15] and de Sitter[16] universes. This allows us to see how different geometric and field-theoretical effects including the effects of topology, curvature, deformation, and field coupling can become important under different conditions. In certain cases we were able to draw implications of these results for inflationary[17] and chaotic cosmologies.[18] By comparing their similarities we were able to see some general behavior emerging. As we soon come to realize, as much as topology and curvature effects are important in their own right, the physically more relevant factor in determining the symmetry behavior of systems in curved space with some compact (or finite) dimension is the so-called finite-size effect.[20] The finite-size effect in curved-space symmetry breaking is influenced both by the topology (or boundary condition) and the curvature of spacetime, but is itself neither of them. It is the finiteness of certain spatial (or temporal) dimensions which constrains the range of the fluctuation fields and modifies the critical behavior of these systems.

Despite the sizable amount of work on symmetry behavior in curved space carried out previously we find that the following two aspects remain somewhat deficient: the treatment of infrared behavior which is crucial to a complete and accurate description, and the physical meaning of the results. We attempted to address the first problem in a number of earlier papers, especially using the de Sitter universe as an example.[14–16] This paper is devoted to the second problem. We want to study the physical meaning of some of the results obtained by us and others previously, and discuss their significance in broader terms. We will discuss finite-size effects in field theory and cosmology and draw the parallels with critical phenomena in condensed-matter physics. How one can identify finite-size effects in more general systems using path-integral and spectral analysis methods will be discussed in a later paper.[21] This paper is organized as follows. In Sec. II we describe the effective action approach to the analysis of symmetry behavior with the example of an N-component $\lambda\phi^4$ field. We discuss the difference between classical and quantum effects of curvature in the nonminimal and minimal coupling cases and point out that for the massless minimally coupled scalar fields rather peculiar behavior near the symmetric state almost invariably occurs. As it is associated with the appearance of infrared divergence, this case is often unjustifiably ignored or mistreated.[12] But it is precisely the one closest in nature in curved space to the workings of the Coleman-Weinberg mechanism[22] responsible for new inflation. It is also where the finite-size effect is most significant in affecting the symmetry behavior. We show that in these cases the infrared behavior is dominated by the lowest-mode contribution of the scalar fluctuation operator. In Sec. III we discuss the decoupling of higher modes and infrared dimensional reduction. We introduce the notion of effective infrared dimension and present an eigenvalue analysis for the discussion of dimensional reduction. We treat as an example the symmetry behavior of the Taub universe in the large anisotropy limit. In Sec. IV we discuss the higher-loop corrections to the effective mass and the effective potential via the two-particle-irreducible formalism. In Sec. V we use these results to discuss the symmetry behavior of spacetimes with different number of compact dimensions: Examples were drawn from de Sitter, Einstein universes as well as finite-temperature and Kaluza-Klein theories. After this we give a general discussion of results in terms of finite-size effects in field theory and cosmology. Dynamical effects of symmetry breaking in curved space will be discussed in later communications.[23]

II. SYMMETRY BREAKING IN CURVED SPACETIME

A. Effective action of an N-component scalar field

In the discussion of symmetry behavior, it is important to know where and when minimum free energy states (local and global minimum) exist and how the system chooses between and evolves from these states. For these purposes the approach based on the effective action proves most powerful.[24] The effective action $\Gamma(\hat{\phi})$ gives the free energy density of the system as a functional of the order-parameter field $\hat{\phi}$. One can compute directly from it the field-theoretical and thermodynamic quantities of interest in the system with relative ease. Since contributions from the quantum and thermal fluctuations[25] are built in, one does not need to solve separately the equations of motion for the background field and the fluctuation field and worry about their self-consistency in the iteration, as in the effective-mass approach often used in stability analysis.[26] For dynamical spacetimes, as the background field (order parameter) $\hat{\phi}$ depends on time, phase-transition processes will possibly be accompanied by vacuum particle production. To concentrate on the effect of symmetry breaking, we can in this study restrict our attention to static spacetimes (or spacetimes which admit static coordinatization like the de Sitter universe) where the order parameter is a constant. For constant background metrics and background fields, one can work with the effective potential $V(\hat{\phi})=-(\mathrm{vol})^{-1}\Gamma(\hat{\phi})$, where the spacetime four-volume (vol) is factored out from the effective action.

Consider now an N-component self-interacting scalar field Φ^a $(a=1,\ldots,N)$ on a manifold of dimension D, coupled to the background spacetime with curvature R and coupling constant ξ (conformal coupling for $\xi=0$ and minimal coupling for $\xi=1$) described by the action

$$S[\Phi]=\int d^Dx\sqrt{g}\left[\frac{1}{2}\Phi^a\Delta\Phi^a+\frac{1}{2}\left[m^2+\frac{(1-\xi)R}{6}\right]\Phi^2+\frac{\lambda}{4!}\Phi^4\right], \tag{2.1}$$

where $\Delta=-\sqrt{g}\,\partial_\mu(\sqrt{g}\,g^{\mu\nu}\partial_\nu)$ is the Laplace-Beltrami operator on scalars. For convenience we choose to develop the formalism here in the Euclidean version. Perform a background-field decomposition of the field Φ^a into a background field $\hat{\phi}^a$ and a fluctuation field ϕ^a, i.e., $\Phi^a=\hat{\phi}^a+\phi^a$. The background field $\hat{\phi}^a$ is required to satisfy the classical equations of motion with an arbitrary external source. Such a shift eliminates the linear term in the fluctuation field (it is equivalent to performing a Legendre transform). The resultant action is

$$S[\hat{\phi},\phi]=S[\hat{\phi}]+\int d^Dx\sqrt{g}\left\{\frac{1}{2}\phi^a\left[\left[\Delta+m^2+(1-\xi)\frac{R}{6}+\frac{\lambda}{6}\hat{\phi}^2\right]\delta^{ab}+\frac{\lambda}{3}\hat{\phi}^a\hat{\phi}^b\right]\phi^b+\frac{\lambda}{6}\hat{\phi}^a\phi^a\phi^2+\frac{\lambda}{4!}\phi^4\right\} . \tag{2.2}$$

The effective action $\Gamma[\hat{\phi}]$ is obtained by functionally integrating over the fluctuation fields:

$$e^{-\Gamma[\hat{\phi}]}=\int [d\phi]e^{-S[\hat{\phi},\phi]} . \tag{2.3}$$

The wave operator A^{ab} for the fluctuating field is given by

$$A^{ab}=\left[\Delta+M^2+\frac{\lambda}{2}\hat{\phi}^2\right]\frac{\hat{\phi}^a\hat{\phi}^b}{\hat{\phi}^2}+\left[\Delta+M^2+\frac{\lambda}{6}\hat{\phi}^2\right]\left[\delta^{ab}-\frac{\hat{\phi}^a\hat{\phi}^b}{\hat{\phi}^2}\right] , \tag{2.4}$$

where $M^2=m^2+(1-\xi)R/6$. Here $\delta^{ab}-\hat{\phi}^a\hat{\phi}^b/\hat{\phi}^2$ and $\hat{\phi}^a\hat{\phi}^b/\hat{\phi}^2$ are orthogonal projectors, the former into an $(N-1)$-dimensional subspace orthogonal to the direction in the internal space picked out by $\hat{\phi}^a$, and the latter projects along the direction of $\hat{\phi}^a$. Note that the operator Δ does not commute with the projectors unless $\hat{\phi}^a$ is a constant.

When the direction in group space picked out by $\hat{\phi}^a$ does not vary from point to point around the manifold (this does not necessarily imply $\hat{\phi}^2$ is a constant[27]) the Green's function for A^{ab} is easily seen to be given by

$$G^{ab}=G_1\frac{\hat{\phi}^a\hat{\phi}^b}{\hat{\phi}^2}+G_2\left[\delta^{ab}-\frac{\hat{\phi}^a\hat{\phi}^b}{\hat{\phi}^2}\right] , \tag{2.5}$$

where the G_i $(i=1,2)$ are the Green's functions for the operators $\Delta+M_i^2$ and

$$M_1^2=M^2+\frac{\lambda}{2}\hat{\phi}^2,\quad M_2^2=M^2+\frac{\lambda}{6}\hat{\phi}^2 . \tag{2.6}$$

The one-loop correction is related to the sum of the logarithms of the determinants of the fluctuation operators. So in this case, by using the projection operators, the one-loop effective action is given by

$$\begin{aligned}\Gamma[\hat{\phi}]&=S[\hat{\phi}]-\frac{1}{2}\mathrm{Tr}\ln G_1-\frac{N-1}{2}\mathrm{Tr}\ln G_2\\&=S[\hat{\phi}]+\frac{1}{2}\sum_l d_l\ln\lambda_{1l}+\frac{N-1}{2}\sum_l d_l\ln\lambda_{2l} ,\end{aligned} \tag{2.7}$$

where $S[\hat{\phi}]$ is the classical action, λ_{il} and d_l are the eigenvalues and degeneracies of the operators $\Delta+M_i^2$. We will discuss the N-component theory further in Secs. III and IV, but for the purpose of this section it is sufficient to just use the example of the one-component theory in which case the effective action is given by (2.7) without the last term.

The determinant of A is formally divergent and needs to be regularized. There are a number of commonly used regularization methods.[1] If the space is Riemannian and has sufficient symmetry so that the spectrum of the invariant operator is known explicitly, then the ζ-function method is probably most convenient.[28] The generalized ζ function is defined by

$$\zeta(\nu)=\sum_n(\mu^{-2}\lambda_n)^{-\nu} , \tag{2.8}$$

where λ_n are the eigenvalues of the operator A on the Euclideanized metric obtained by a Wick rotation to imaginary time $\tau=it$. Here a constant mass scale μ is introduced to make the measure $d[\phi]$ of the functional integral dimensionless. Using the regularization method of Dowker and Critchley,[28] one can express the one-loop effective potential as[14]

$$V^{(1)}(\hat{\phi})=-\tfrac{1}{2}\hbar(\mathrm{vol})^{-1}[\zeta'(0)+\zeta(0)/\nu] . \tag{2.9}$$

The divergence in $V^{(1)}$ will be canceled by the addition of counterterms. A discussion of renormalization can be found in standard references.[1–5]

B. Effective action in cosmological spacetimes

Consider the class of spatially homogeneous cosmologies[29] with metric

$$ds^2=dt^2-\sum_{a,b=1}^{3}\gamma_{ab}(t)\sigma^a(x)\sigma^b(x) , \tag{2.10}$$

where $\gamma_{ab}(t)$ is the metric tensor and $\sigma^a(x)$ are the invariant basis one-forms on the homogeneous hypersurfaces, satisfying the structure condition $d\sigma^a=\frac{1}{2}C^a_{bc}\sigma^b\Lambda\sigma^c$, where C^a_{bc} are the structure constants of the underlying symmetry group. For a diagonal type-IX (mixmaster)[30] universe, $C^a_{bc}=\epsilon_{abc}$ the totally antisymmetric tensor, and $\gamma_{ab}=l_a{}^2\delta_{ab}$, where l_a are the three principal radii of curvature. The case when two of the three l_a's are equal gives the Taub universe,[31] that when all three being equal gives the Robertson-Walker (RW) universe. We will consider just the static geometries, in which case the RW spacetime is known as the Einstein universe. The spatial metric $d\Omega_3{}^2$ being maximally symmetric is that of a three-sphere with radius a:

$$ds^2=dt^2-a^2d\Omega_3{}^2 . \tag{2.11}$$

The de Sitter universe on the other hand is a maximally symmetric (four-dimensional) spacetime which can be viewed as a dynamical (three-dimensional) homogeneous space. Depending on its spatial section there are many ways to put coordinates on it.[32] In the $R^1\times S^3$ Robertson-Walker coordinates which cover the whole de Sitter space, the line element is

$$ds^2=(H\cos\eta)^{-2}(d\eta^2-d\Omega_3{}^2),\quad -\frac{\pi}{2}\le\eta\le\frac{\pi}{2} \tag{2.12}$$

$$=dt^2-H^{-2}\cosh^2(Ht)d\Omega_3{}^2,\quad -\infty<t<\infty , \tag{2.13}$$

where $d\Omega_3{}^2$ is the interval on the unit three-sphere. The time coordinates t and η are related by

$$(\cos\eta)^{-1}=\cosh Ht, \quad \tan\eta=\sinh Ht \; . \tag{2.14}$$

Define

$$a=H^{-1}\cosh(Ht) \; . \tag{2.15}$$

Notice that at late times the scale factor of S^3 goes as $a\sim H^{-1}e^{Ht}$, where $H=\dot{a}/a$ is the expansion rate (Hubble constant). The Euclidean metric on S^4 is obtained by replacing t by $\tau=it$:

$$ds_E{}^2=-[d\tau^2+(H^{-1}\cos H\tau)^2 d\Omega_3{}^2] \; . \tag{2.16}$$

One needs to know the spectrum of the invariant operators to compute the ζ function. For many-component fields [see (2.4)] in static spatially homogeneous spaces $\lambda_n=k_0{}^2+\kappa_N{}^2+M_2{}^2$ where $k_0=(-\infty,\infty)$, and κ_N are the eigenvalues of the Laplace operator Δ on the homogeneous three-space, and M_2 is defined in (2.6). Here N denotes the collective spatial quantum numbers. If a periodic condition is imposed on the imaginary time as in the finite-temperature theory, then $k_0=2\pi n_0/\beta$, $n_0=0,\pm1,\pm2,\ldots$. For the Einstein universe ($R^1\times S^3$), the characteristic function is the hyperspherical function with quantum numbers $N=(n,l,m)$ with ranges $n=0,1,2,\ldots; l=0,1,\ldots,n$; and $m=-l,-l+1,\ldots,l$. For the mixmaster or the Taub universe, one can use the SO(3)-invariant representation function D^J_{KM} with quantum numbers $N=(J,K,M)$ where $J=1,1,\ldots;K,M=-J$ to J (Ref. 33). For the Euclidean de Sitter space, n are the quantum numbers of S^4. The spectrum for these spaces are well known. The effective potentials for the Einstein, Taub, and de Sitter spaces have been calculated before.

C. Symmetry behavior

For the study of symmetry behavior in curved spacetime one should use a high curvature approximation ($M_2{}^2a^2 \ll 1$) to the effective action instead of a local (e.g., Schwinger-DeWitt small-proper-time) expansion, as was erroneously assumed in some previous work. This would correspond to examining the infrared domain. One can distinguish between classical versus quantum effects and conformal versus minimal couplings,[7,14–16] but the case of special interest here is when $M_2{}^2=m^2+(1-\xi)R/6+\lambda\hat{\phi}^2/6=0$. This corresponds to massless minimal coupling or $R=0$ with arbitrary coupling at the symmetric state. In such cases, symmetry considerations are determined solely by quantum effects. We found that in the large-curvature limit the one-loop effective potential $V^{(1)}$ for these cases has a leading $|\hat{\phi}|$ behavior near the symmetric state $\hat{\phi}=0$ in the Einstein and Taub universes,[14,15] and a leading $\ln|\hat{\phi}|$ behavior in the de Sitter universe.[16] This hitherto undetected behavior has interesting implications in cosmological phase transitions.[17,18]

As remarked in our earlier papers[15–17] this rather peculiar behavior near the symmetric state is due to the zero mode of the fluctuation operator and varies with the topology of the underlying spaces. We learned that in spacetimes with some compact dimensions the lowest mode of the fluctuation operator has the strongest effect on the symmetry behavior of the system. In the next section we will show that this mode in fact dominates the infrared behavior: the system can be described by a field theory in a lower dimension. We will give a formal derivation of infrared dimensional reduction by examining the result of the decoupling of the higher modes (or bands) in the functional integral for the effective action. We will then give a physical explanation in terms of correlation lengths and the notion of effective IR dimension (EIRD). It then becomes clear that the infrared behavior of S^4 de Sitter universe is equivalent to a zero-dimensional system and the $S^3\times R^1$ Einstein universe has a one-dimensional infrared behavior. An alternative way of seeing this problem of dimensional reduction is by spectral analysis. This is applied to direct product spaces with some compact dimensions and to spaces which can be reduced to product spaces. We use the Taub universe as example. We also mention analogous behavior in quantum-mechanical perturbation problems.

III. INFRARED BEHAVIOR AND DIMENSIONAL REDUCTION

A. Decoupling of the higher modes (or bands)

The discussion in this section applies to cases where the eigenvalues of the fluctuation operator takes on a band structure. By band structure we mean that the eigenvalues occur in continua with each continuum having a higher lowest eigenvalue than the previous one. This is true for fields on spacetimes with compact sections or for operators with discrete spectrum (e.g., the harmonic oscillator). The procedure is to expand the fields in terms of the band eigenfunctions and convert the functional integral over the fields to an integral over the amplitudes of the individual modes. When the lowest mode is massless it will give the dominant contribution to the effective action. The low-energy behavior corresponds to a lower-dimensional system.

Consider quantum fields on a manifold with topology $R^d\times B^b$ where B is compact. Consider the situation where the fluctuation operator A has the general form of a direct sum of operators D and B:

$$A^{ab}(x,y)=D^{ab}(x)+B^{ab}(y) \tag{3.1}$$

with coordinates x on R^d and y on B^b. Assume that the eigenvalues ω_n of the operator B are discrete:

$$B^{ab}\psi_n(y)=\omega_n^{ab}\psi_n(y) \; . \tag{3.2}$$

Decomposing the field $\phi^a(x,y)$ in terms of the eigenfunctions $\psi_n(y)$ of B^{ab}

$$\phi^a(x,y)=\sum_n \phi_n^a(x)\psi_n(y) \tag{3.3}$$

one obtains for the quadratic part of the action

$$\tfrac{1}{2}\int dx\,dy\,\phi^a A^{ab}\phi^b=\tfrac{1}{2}\int dx(\phi_n^a f_{nm}D^{ab}\phi_m^b + \omega_n^{ab}f_{nm}\phi_n^a\phi_m^b) \; , \tag{3.4}$$

where $f_{nm}=\int dy\,\psi_n(y)\psi_m(y)$. When ϕ_n are properly normalized $f_{nm}=\delta_{nm}$ (we will make such a choice here) the resulting theory in terms of the new fields ϕ_n^a will involve massive fields with masses determined by the matrix λ_n^{ab}, even if the fields in terms of the old variables appeared massless. We will take the smallest eigenvalue to be given by $n=0$ and assume that its only degeneracy is labeled by the indices a and b. Assume also that the operator D^{ab} is simply minus the Laplacian on R^d times δ^{ab}, the $n=0$ mode is then governed by the action whose quadratic term is

$$\frac{1}{2}\int d^dx(\phi_0^a\Delta\phi_0^a+\omega_0^{ab}f_{00}\phi_0^a\phi_0^b) \ . \tag{3.5}$$

The apparent mass matrix of this field is ω_0^{ab}. For the case of an N component $\lambda\phi^4$ theory the action after this decomposition takes the form

$$S[\hat{\phi}+\phi]=\int d^dx\left[\frac{1}{2}\phi_n^a(\Delta^2\delta^{ab}+\omega^{ab})\phi_n^b + \frac{\lambda}{6}g_{nlm}^a\phi_l^b\phi_m^b+\frac{\lambda}{4!}f_{knlm}\phi_k^a\phi_n^a\phi_l^b\phi_m^b\right] , \tag{3.6}$$

where $g_{nlm}^a=\int dy\,\hat{\phi}^a\psi_n\psi_l\psi_m$ and $f_{knlm}=\int dy\,\psi_k\psi_n\psi_l\psi_m$. The effective action is now given by the functional integral

$$e^{-\Gamma[\hat{\phi}]}=\int[d\phi_n^a]e^{-S[\hat{\phi}+\phi]} \ . \tag{3.7}$$

The interesting case occurs when the lowest eigenvalue approaches zero. At low energy the Appelquist-Carazzone decoupling theorem[34] assures us that with higher modes decoupled from the dynamics, the infrared behavior is governed by the lowest band. We are then left with a purely lower-dimensional theory.[35] The higher modes do play a role in the ultraviolet divergences present in the theory (e.g., renormalization problem in Kaluza-Klein theories[36]) and therefore determine the high-energy running of coupling constants, but the infrared region of the theory is governed by the lower d-dimensional theory.

B. Correlation length and effective infrared dimension

The above result of dimensional reduction from a formal derivation of mode decoupling can be understood in a more physical way by using the concept of effective infrared dimensions (EIRD). By EIRD we mean the dimension of space or spacetime wherein the system at low energy effectively behaves. One well-known example is the Kaluza-Klein theory of unification and cosmology.[37] After spontaneous compactification an 11-dimensional spacetime with full diffeomorphism symmetry reduces at energy below the Planck scale to the physical four-dimensional space with GL(4,R) covariance and a seven-dimensional internal space with symmetry group containing the standard SU(3)×SU(2)×U(1) subgroups of strong and electroweak interactions. For observers today of very low energy the effective IR dimension of spacetime is four, even though the complete theory is eleven dimensional. By the same token, Einstein's theory of general relativity can presumably be regarded as the EIR limit of an otherwise more complete theory of higher-derivative gravity,[38] induced gravity,[39] superstrings,[40] or possibly some other as yet undetermined theory. For curved-space symmetry-breaking considerations, the EIRD which the system "feels" is governed by a parameter η which is the ratio of the correlation length Ξ and the scale length L of the background space $\eta\equiv\Xi/L$. For compact spaces like S^4, L is simply 2π times the "radius" of S^4, the only geometric scale parameter. For product spaces $R^d\times B^b$ with B^b some compact space, there are two scale lengths, L_b is finite in the compact dimensions and $L_d=\infty$ in the noncompact dimensions. Examples are Kaluza-Klein theories $d=4$, $b=1$, 6, 7, or others, finite-temperature field theory (imaginary-time formalism $L=\beta$, inverse temperature) $d=3$, $b=1$, and Einstein (or RW) universe $d=1$, $b=3$.

The symmetry behavior of the system (described here by a $\lambda\phi^4$ scalar field as example) is determined by the *correlation length* Ξ defined as the inverse of the effective mass M_{eff} related to the effective potential V_{eff} by (we use subscript eff to denote quantities containing higher loop corrections)

$$\Xi^{-2}=\left.\frac{\partial^2V_{\text{eff}}}{\partial\hat{\phi}^2}\right|_{\hat{\phi}_{\min}}\equiv M_{\text{eff}}{}^2(=\text{curvature-induced mass } M_{1,2}{}^2+\text{radiative corrections}) \ . \tag{3.8}$$

It measures the curvature of the effective potential at a minimum-energy state $\hat{\phi}_{\min}$ ($\hat{\phi}=0$ for the symmetric state or the false vacuum, $\hat{\phi}=\sigma$ for the broken-symmetry state or the true vacuum.) The effective mass is defined to include radiative corrections to the same order corresponding to the effective potential. (This quantity is called the generalized susceptibility function in condensed-matter physics.) The critical point of a system is reached when $\Xi\to\infty$ or $M_{\text{eff}}\to 0$. In flat or open spaces or for bulk systems, the critical point can be reached without restriction from the geometry (note that in dynamical situations, exponential expansion can effectively introduce a finite-size effect equivalent to event horizons, see Refs. 7 and 23). However, in spaces with compact dimensions, the correlation length of fluctuations can only extend to infinity in the remaining noncompact dimensions, and thus the critical behavior becomes effectively equivalent to a lower d-dimensional system. One can also think of Ξ as the Compton wavelength $\Lambda=2\pi/M_{\text{eff}}$ of a system of quasiparticles with effective mass M_{eff}. Any fine structure of the background spacetime with scale L is relevant only if $\Lambda\lesssim L$. Thus when Λ is small or $\eta\ll 1$ (far away from critical point, at higher energy, higher modes contribute) it sees the details of a spacetime of full dimensionality. At this wavelength, the apparent size of the universe is large in both compact and noncompact dimensions. When $\Lambda \to \infty$ or $\eta \gg 1$ (near critical point, IR limit, lowest mode dominant) structures of finite-size or compact dimensions will not be important. The apparent size of the universe will be determined by the scales associated with the noncompact space and the EIRD is measured by the number of noncompact dimensions. η very large is an indicator of when dimensional reduction can take place.

Notice that in flat-space critical phenomena the effective potential V (free energy density) depends on the coupling constants of fields which run with energy and temperature. In curved-space coupling parameters run also with curvature or the scale length of the space. This makes the concept of EIRD even more interesting, as there is now an interplay between Ξ and L; and η can either decrease or increase with curvature. For example, for $\lambda\phi^4$ fields in the Einstein universe[15] near the symmetric state $\hat{\phi}=0$, the EIRD is equal to 1, but near the global minimum of the broken-symmetry state, $\hat{\phi}=\sigma$ (called $\hat{\phi}_{\rm min}$ in Ref. 15), it is equal to 4. Near the symmetric state, $\eta \gg 1$ signifies reduction of EIRD to one. This is consistent with the theorem of Hohenberg, Mermin, and Wagner[41] (for statistical mechanics on a lattice) and Coleman (for continuum field theory)[42] which states that in dimensions less than or equal to two, the infrared divergence of the scalar field is so severe that there could be no possibility of spontaneous symmetry breaking: the only vacuum expectation value for $\hat{\phi}$ allowed is zero. Away from the region $\hat{\phi}\simeq 0$ the one-dimensional behavior no longer prevails. Indeed a global minimum of the effective potential exists at $\hat{\phi}_{\rm min}$ (see discussion in Sec. IV of Ref. 15). Near $\hat{\phi}_{\rm min}$, $\eta \ll 1$ and decreases with curvature. Thus the apparent size of the Universe near the global minimum actually increases with increasing curvature. There is therefore no dimensional reduction and the system has a full four-dimensional IR behavior. A transition to the asymmetric ground state is not precluded as symmetry breaking via tunneling is in principle possible. The complete picture extending from $\hat{\phi}=0$ to $\hat{\phi}=\hat{\phi}_{\rm min}$ is a combination of one-dimensional and four-dimensional infrared behavior. Similar arguments can be applied to other spacetimes or field theories. Using this notion one can understand, for example, why it is often said that at high-temperatures (small radius limit of S^1) the finite-temperature theory becomes an effective three-dimensional theory.

C. Dimensional reduction: An eigenvalue analysis

In the above we have introduced the notion of EIRD and suggested the ratio of the correlation length Ξ to the geometric scale of spacetime L as a measure of the conditions for the system to behave effectively in the infrared regime as in a reduced dimension. We suggested that for product spaces $R^d \times B^b$ with some noncompact dimension d, the EIRD is usually just d. In this section we will verify this assertion by analyzing the spectrum of the fluctuation operators in these spacetimes directly. We will first discuss direct-product spaces and then discuss spacetimes which can be reduced to product spaces in some limit of continuous deformation.

1. Direct-product spaces

Some of the examples we discussed above are direct-product spaces: relativistic cosmology has topology $M^4=R^1\,(\text{time})\times S^3$ (or R^3, H^3, T^3), Kaluza-Klein cosmology has $M^4\times S^7$ (or other internal space), finite-temperature theory has $R^3\times S^1$. Let us analyze a simple example $S^2\times S^1$ for illustration. (This could be the spatial geometry of a "handled" Gowdy universe.[43]) Similar reasoning can be extended to a wide range of product spaces.

The wave operator A governing the fluctuation fields in the large-N limit is $A\equiv\Delta+M_2{}^2$, where Δ is the Laplace-Beltrami operator. For $S^2\times S^1$ with radii a_2 and a_1, respectively, Δ is a sum of the total angular momentum operator L on S^2, and L_z on S^1:

$$\Delta=\frac{L^2}{a_2{}^2}+\frac{L_z{}^2}{a_1{}^2}\ . \tag{3.9}$$

The eigenfunction is a product of $Y_{lm}(\theta,\phi)e^{in\chi}$ belonging to the eigenvalues

$$\kappa_N{}^2=\frac{l(l+1)}{a_2{}^2}+\frac{n^2}{a_1{}^2}\ , \tag{3.10}$$

where $N=(l,m,n)$, $l=0,\ldots,\infty$, $m=-l$ to l, $n=0,\pm1,\ldots$. The eigenvalues of A are then $\lambda_N=\kappa_N{}^2+M_2{}^2$. In the infrared region we are interested in the contribution of the lowest eigenvalue (zero mode) to the effective potential. We will consider the two limiting cases of (a) $S^2\times R^1$ and (b) $R^2\times S^1$ obtained when a_1 and $a_2\to\infty$, respectively, and show that the EIRD is equal to 1 and 2, respectively.

The effective potential V can be constructed from the ζ function (2.8), which in this setting reads

$$\zeta(\nu)=\mu^{2\nu}\sum_{l,n}(2l+1)\left[\frac{l(l+1)}{a_2{}^2}+\frac{n^2}{a_1{}^2}+M_2{}^2\right]^{-\nu}\ , \tag{3.11}$$

where $2l+1$ is the degeneracy of m. In the limiting cases the summation over discrete quantum numbers will be replaced by integrals of the form $\int k^{D-1}dk$. We will derive the dimensionality of the reduced system by finding D. In the following, constants independent of $\hat{\phi}$ will be ignored since our main focus is on the dependence of Green's functions and the effective potential on $\hat{\phi}$. Thus in case (a) $a_1\to\infty$, the lowest eigenvalues belong to the band $l=0$. As $a_1\to\infty$, define $k=n/a_1$, which assumes continuous value, $\zeta(\nu)$ becomes

$$\zeta(\nu)\sim a_1\int_{-\infty}^{\infty}dk\,(k^2+M_2{}^2)^{-\nu}\ ,\qquad a_1\to\infty,\quad M_2a_2\ll 1\ . \tag{3.12}$$

This is a one-dimensional integral. In case (b) $a_2\to\infty$, the lowest eigenvalues come from the lowest band $n=0$ given by the first term in (3.10), where l assumes continuous values as $a_2\to\infty$. Now call $k=l/a_2$, $\zeta(\nu)$ becomes $(n=0)$

$$\zeta(\nu) \sim 2a_2 \int_0^\infty dk\, k(k^2+M_2{}^2)^{-\nu} ,$$
$$a_2 \to \infty, \quad M_2\, a_1 \ll 1 . \quad (3.13)$$

The extra factor of k comes from the degeneracy of m for nonzero l. The integral is two dimensional, as expected.

The direct-product space in this example can also be obtained as a limit of $S^3 \to S^2 \times S^1$. The eigenfunctions of S^3 which are the hyperspherical harmonics $Y_{nlm}(\chi,\theta,\phi)$ can be expressed as a product of the spherical harmonics $Y_{lm}(\theta,\phi)$ and the Gegenbauer polynomials $G_{nl}(\chi)$. The eigenfunction $e^{in\chi}$ on S^1 we saw above are obtained as limits of $G_{nl}(\chi)$ as the angular momentum of S^2 is decoupled from the "radial" equations governing G_{nl}. This we will see in subsection 2 below.

Physically, the two limiting cases may represent two different symmetry states of the system: one with cylindrical symmetry where the quantum states with good quantum number m are eigenstates of the L_z operator (on S^2), the other with spherical symmetry with good quantum numbers l associated with the L^2 operator. As a result of symmetry breaking the original system can end up in one of these states with a different symmetry. External perturbations with a particular symmetry and sufficient magnitude will influence the selection of the end state of the system. A simple example in elementary quantum mechanics is the Zeeman versus the Paschen-Bach effects associated with many electron atoms subjected to an external magnetic field. The total magnetic quantum number of the system will become a good quantum number at very strong fields. More sophisticated examples can be found in the consideration of classes of internal spaces admitting larger symmetry groups with the strong and electroweak gauge groups as subgroups in the Kaluza-Klein theory.[44] By the same analysis, it is not difficult to see that the EIRD of any homogeneous cosmology with compact spatial section is one; the spatial metrics of the Einstein-Rosen[45] waves $S^2 \times R^1$, the Kantowski-Sachs[46] universe with spatial metric $dl^2 = dz^2 + a^2 d\Omega_2{}^2$, black-hole spacetimes,[47] and the whole class of stationary axisymmetric metrics[48] in relativistic astrophysics have EIRD$=2$. Similarly, the (imaginary-time) finite-temperature theory[49] $R^3 \times S^1$ has EIRD$=3$ and the Kaluza-Klein theory on $M^4 \times B^b$ with compact internal space B has EIRD$=4$.

2. *Reduced product spaces*

We now consider spaces which are not direct-product spaces but can approach product spaces in the limit of extreme deformations or large perturbations. One convenient example is the (static) Taub universe.[31,30] In Ref. 15 we have analyzed its symmetry behavior for (i) small anisotropy near the Einstein space. Here we want to study the large anisotropy limits, i.e., under the conditions (ii) $l_1 = l_2 \gg l_3$ ("oblate" configuration) and (iii) $l_1 = l_2 \ll l_3$ ("prolate" configuration). Intuitively one may be tempted to view the space to have geometry S^3 in case (i), $R^2 \times S^1$ in case (ii), and $R^1 \times S^2$ in case (iii). From discussions above one may also expect the system to have an EIRD$=0$ in case (i), 2 in case (ii), and 1 in case (iii). As we will see, using the eigenvalue analysis introduced above these intuitions are indeed correct—except for the "prolate" case. This is because our intuitive picture of the Taub universe (as described by a "spheroid") is formed with scale lengths substituting the curvature radii, which are what l_i really are. The extreme "prolate" limit $l_1 = l_2 \ll l_3$ simply does not correspond to a line in the ordinary sense, from which one may form the erroneous impression of reduction to a one-dimensional system. Let us consider the system case by case. The eigenvalues of the Laplace-Beltrami operator are given by[33]

$$\kappa_N{}^2 = \frac{J(J+1)}{l_1{}^2} + \left[\frac{1}{l_3{}^2} - \frac{1}{l_1{}^2}\right] K^2 , \quad (3.14)$$

where $N=(J,K,M)$, $J=0,1,\ldots,\infty$, $K,M=-J,-J+1,\ldots,J$.

In case (i) $l_1 = l_2 = l_3$ it is just the Einstein universe S^3. In this limit the ζ function is given by

$$\zeta(\nu) \sim \sum_{J=0}^{\infty} \sum_{K=-J}^{J} \sum_{M=-J}^{J} \left[\frac{J(J+1)}{l^2} + M_2{}^2\right]^{-\nu} . \quad (3.15)$$

Now the quantum number J corresponding to the Casimir operator in the SO(3)-symmetric eigenfunctions $D^J_{KM}(\theta,\phi,\psi)$ are related to the principal quantum number n corresponding to the Casimir operator in the SO(4)-symmetric eigenfunctions (hyperspherical harmonics) $Y_{nlm}(\chi,\theta,\phi)$ by $J = n/2$. Similarly $l = a/2$ where a is the radius of S^3. Rewriting ζ in terms of n and a gives

$$\zeta(\nu) \sim \sum_{q=1}^{\infty} q^2 \left[\frac{q^2-1}{a^2} + M_2{}^2\right]^{-\nu} \equiv Z(\nu,x) , \quad (3.16)$$

where $q = n+1$, $x = M_2{}^2 a^2 - 1$, and the Z function was defined in Ref. 14. Now when a is held finite and $M_2{}^2 a^2 \ll 1$ the lowest mode corresponding to $n=0$ or $q=1$ gives the most dominant contribution to ζ. This single-mode contribution comes from a zero-dimensional integral (just a c number, $\zeta \sim [M_2{}^2]^{-\nu}$). This shows explicitly why compact spaces should have zero EIRD. If however $a \to \infty$ we recover a three-dimensional flat space.

In case (ii) we increase the deformation in the oblate direction, i.e., we are interested in the limit $l_1 = l_2 \gg l_3$. To analyze this case it is useful to note the identity

$$\sum_{J=0}^{\infty} \sum_{K=-J}^{J} f(J,K) = \sum_{K=-\infty}^{\infty} \sum_{J=|K|}^{\infty} f(J,K) . \quad (3.17)$$

With this (3.15) can be reexpressed as

$$\zeta(\nu) = \sum_{J=0}^{\infty} (2J+1) \left[\frac{J(J+1)}{l_1{}^2} + M_2{}^2\right]^{-\nu} + 2 \sum_{K=1}^{\infty} \sum_{J=K}^{\infty} (2J+1) \left[\frac{J(J+1)}{l_1{}^2} + \left[\frac{1}{l_3{}^2} - \frac{1}{l_1{}^2}\right] K^2 + M_2{}^2\right]^{-\nu} . \quad (3.18)$$

Replacing J/l_1 by k and letting $l_1 \to \infty$ we get

$$\zeta(\nu)=2l_1^2\int_0^\infty dk\,k(k^2+M_2^2)^{-\nu}+4l_1^2\sum_{K=1}^{\infty}\int_0^\infty dk\,k\left[k^2+\frac{K^2}{l_3^2}+M_2^2\right]^{-\nu}. \quad (3.19)$$

So as long as l_3 is small relative to M_2^{-1}, this is dominated by the first term arising from $K=0$. This shows that in the IR limit

$$\zeta(\nu)\sim 4l_1^2\int_0^\infty dk\,k(k^2+M_2^2)^{-\nu}; \quad (3.20)$$

i.e., the system behaves like a two-dimensional system.

Finally for case (iii), the limit $l_1=l_2 \ll l_3$. In this limit the curvature of the Taub space

$$R=(4l_1^2-l_3^2)/2l_1^4 \quad (3.21)$$

becomes $R\simeq -l_3^2/(2l_1^2)$. In the limit $l_3\to\infty$ the curvature becomes infinitely negative. We are left with two possibilities. Either hold l_1 and l_3 fixed but finite, in which case the spectrum remains discrete and in the IR limit is dominated by the lowest mode $J=K=0$, giving a zero EIRD as in the Einstein universe; or hold the ratio of l_1 and l_3 fixed and send them both to infinity simultaneously, in which case the curvature becomes zero, and we recover the flat three-dimensional space with an EIRD of 3. So contrary to intuition (which uses the erroneous picture of "stretching" the scale factors), the deformed three-sphere does not approach a one-dimensional "line" (which is indeed the case for Bianchi type-I universe).[50] This can be seen clearly from a plot of the eigenvalues of the Laplace-Beltrami operator as a function of deformation. In Fig. 4 of Ref. 33 the eigenvalues of the mixmaster universe from $J=0$ to $J=8$ are plotted. The shape and deformation parameters (α,β) used there are related to l_i by[30]

$$l_1=l_0\exp\left[\beta\cos\left[\alpha-\frac{\pi}{3}\right]\right],$$
$$l_2=l_0\exp\left[\beta\cos\left[\alpha+\frac{\pi}{3}\right]\right], \quad (3.22)$$
$$l_3=l_0\exp(-\beta\cos\alpha).$$

The Taub universe corresponds to $\alpha=0°$ ("spheroid"). At $\beta=0$ [case (i), Einstein universe], all $(2J+1)$ levels of K are degenerate, the lowest eigenvalue is $J=0$. At large $\beta>0$ [case (ii) oblate deformation] the lowest K level of each multiplet decreases and they all converge to zero as the deformation increases. The "stacking up" of these sublevels as the "pancake" flattens into an almost open two-dimensional space gives rise to the degeneracy in the lowest mode. This is what makes the infrared behavior different from the "bulk" case of the Einstein universe. At large $\beta<0$ [case (iii), prolate deformation], the lower spectrum eigenvalue distribution is qualitatively not much different from case (i). The lowest mode is distinct and contains no degeneracy. Notice from Eq. (3.20) it is indeed the degeneracy factor in the lowest mode which gives a different infrared behavior in case (ii). The regrouping of eigenvalues according to different J and K values as the space is deformed is a manifestation of symmetry breaking in the system. Similar to our discussion in the last section, here geometry is playing the role of the magnetic field as in Zeeman effect in breaking and reconstituting the symmetry of the system.

Unlike product spaces discussed in the previous section which are unrelated and disjoint, the above cases can be related through continuous deformation. In the minisuperspace picture,[51] they correspond to points in the world history (trajectory) of the Universe. In the dynamical context, they represent different characteristic solutions of the mixmaster universe.[30,52] Specifically cases (i), (ii), and (iii) correspond to the quasi-isotropic, corner run (or small oscillation) and bounce (off the wall) solutions. The changing symmetry behavior of the system as a consequence of geometrodynamics is an interesting phenomenon worthy of further studies. The cosmological implications for inflation in the mixmaster universe has been discussed in Ref. 18. The spectral analysis we have discussed here for the type-IX universe can be performed for other Bianchi types. It would be interesting to see how the symmetry behavior of the system changes as spaces with higher symmetry groups reduce to lower ones as the Universe evolves[53] (e.g., type-$\mathrm{VII}_h\to V\to I$, etc.). The discussion in this section would then provide a spectral depiction of geometrodynamics, which contains more direct information about the low-energy infrared behavior.

IV. HIGHER-ORDER CORRECTED EFFECTIVE MASS AND POTENTIAL

In Secs. II and III we have used the effective mass M_{eff} and effective potential $V_{\text{eff}}(\hat{\phi})$ for the discussion of infrared behavior in curved spacetimes. Already to one-loop one sees that the effective potential in curved spacetimes (with some compact dimension) behaves quite differently from flat space results near the symmetric state, the difference due mainly to geometric effects. To get a more accurate picture one should include higher loop contributions. We will discuss in this section one way of doing this and present the result for general spacetimes. We find that when the dominant higher-order terms are included, the effective potential behaves only quantitatively different from the one-loop result. For example, the $\ln|\hat{\phi}|$ dependence in de Sitter space is replaced by a power law near $\hat{\phi}=0$. The qualitative features of geometric effects on symmetry breaking described in Sec. III are largely insensitive to these modifications. However, for a quantitative description of phase transitions in more realistic cosmological settings, one needs a more exact form of the effective potential including these corrections. A convenient treatment of higher-order corrections in the large-N (number of fields) limit is via the two-particle-irreducible effective action of Cornwall, Jackiw, and Tomboulis and others.[54,55] We will construct this scheme for quantum fields in curved space and use it to calculate the large-N limit of the N-component $\lambda\phi^4$ field.

The main idea behind the construction is that, just as one can solve by a variational calculation for the full background field in a given situation, one can equally well solve for the full propagator via a variational calculation. The relevant object is the two-particle-irreducible (2PI) effective action $\Gamma[\hat{\phi},G]$, which is the Legendre transform of a generating functional containing an additional external current. Functional derivatives with respect to this external current generate vacuum expectation values of an even number of products of ϕ^i. The construction proceeds as follows. The generating functional $W[J,K]$ is given by the functional integral

$$e^{-W[J,K]}=\int [d\Phi]\mu[\Phi]e^{-S[\Phi]-\Phi^i J_i-\Phi^i K_{ij}\Phi^j}\ , \tag{4.1}$$

where we have used DeWitt's condensed index notation. Defining

$$\begin{aligned}\frac{\delta W}{\delta J_i}&=\langle\Phi^i\rangle_{J,K}\equiv\hat{\phi}^i\ ,\\ \frac{\delta W}{\delta K_{ij}}&=\langle\Phi^i\Phi^j\rangle_{J,K}\equiv G^{ij}+\hat{\phi}^i\hat{\phi}^j\ ,\end{aligned} \tag{4.2}$$

we can perform the Legendre transform to get

$$\Gamma[\hat{\phi},G]=W[J,K]-\hat{\phi}^i J_i-(\hat{\phi}^i\hat{\phi}^j+G^{ij})K_{ij}\ . \tag{4.3}$$

K_{ij} is coupled to $\hat{\phi}^i\hat{\phi}^j+G^{ij}$ so that G^{ij} agrees with $\langle\Phi^i\Phi^j\rangle-\langle\Phi^i\rangle\langle\Phi^j\rangle$ when the currents are turned off. From (4.2) $\Gamma[\hat{\phi},G]$ satisfies

$$\frac{\delta\Gamma}{\delta\hat{\phi}^i}-\hat{\phi}^j\frac{\delta\Gamma}{\delta G^{ij}}=-J_i,\quad \frac{\delta\Gamma}{\delta G^{ij}}=-K_{ij}\ . \tag{4.4}$$

Solving (4.4) with J_i and K_{ij} set to zero then gives the true ground state and propagator for the field.

The generating functional is then given by

$$e^{-\Gamma[\hat{\phi},G]}=\int [d\Phi]\mu[\Phi]\exp\left[-S[\Phi]+(\Phi^i-\hat{\phi}^i)\left[\frac{\delta\Gamma}{\delta\hat{\phi}^i}-\hat{\phi}^j\frac{\delta\Gamma}{\delta G^{ij}}\right]+(\Phi^i\Phi^j-\hat{\phi}^i\hat{\phi}^j-G^{ij})\frac{\delta\Gamma}{\delta G^{ij}}\right] \tag{4.5}$$

which is to be solved iteratively. A change of variable to the background and fluctuation fields is appropriate at this point. Thus, defining $\Phi^i=\hat{\phi}^i+\phi^i$ and changing the variable of integration to the fluctuation field ϕ yields

$$e^{-\Gamma[\hat{\phi},G]}=\int [d\phi]\mu[\hat{\phi}+\phi]\exp\left[-S[\hat{\phi}+\phi]+\phi^i\frac{\delta\Gamma}{\delta\hat{\phi}^i}+\phi^i\phi^j\frac{\delta\Gamma}{\delta G^{ij}}-G^{ij}\frac{\delta\Gamma}{\delta G^{ij}}\right]\ . \tag{4.6}$$

The expression for this effective action can be written in the following form which makes its structure more apparent:

$$\exp\left[\mathrm{Tr}\left[G\frac{\delta\Gamma}{\delta G}\right]-\Gamma\right]=\int [d\phi]\mu[\hat{\phi}+\phi]\exp\left[-S[\hat{\phi}+\phi]+\phi^i\frac{\delta\Gamma}{\delta\hat{\phi}^i}+\phi^i\phi^j\frac{\delta\Gamma}{\delta G^{ij}}\right]\ . \tag{4.7}$$

From the right-hand side of (4.7) it is apparent that G does for the two-point function what $\hat{\phi}$ does for the one-point function. The background propagator is then to be taken in all internal lines as the propagator G, in analogy with the background field always being kept $\hat{\phi}$ in the ordinary effective action. A little work allows one to write down the resulting expression for Γ in the loop expansion to be

$$\Gamma[\hat{\phi},G]=S[\hat{\phi}]+\tfrac{1}{2}\mathrm{Tr}\ln[G^{-1}]+\tfrac{1}{2}A_{ij}(\hat{\phi})G^{ij}-\tfrac{1}{2}\mathrm{Tr}(1)+\tilde{\Gamma}[\hat{\phi},G]\ , \tag{4.8}$$

where $\tilde{\Gamma}[\hat{\phi},G]$ is the sum of all two-particle-irreducible diagrams occurring in the ordinary effective action but evaluated with the propagator G. We will now apply the above scheme to the $\lambda\phi^4$ field.

In the large-N limit the dominant contribution to the one-loop effective action comes from fluctuations of the modes transverse to the direction of $\hat{\phi}^a$, i.e., from the G_2 internal lines [see Eq. (2.7)]. The two-loop contribution can be written as

$$\begin{aligned}\Gamma^{(2)}[\hat{\phi}]=&\frac{\lambda}{4!}\int d^Dx\sqrt{g}\,\{[\mathrm{tr}G(x,x)]^2+2\mathrm{tr}[G(x,x)^2]\}\\ &-\frac{\lambda^2}{36}\int d^Dx\sqrt{g}\int d^Dy\sqrt{g}\ \hat{\phi}^a(x)\{G^{ab}(x,y)\mathrm{tr}[G(x,y)^2]+2G^{ac}(x,y)G^{cd}(y,x)G^{db}(x,y)\}\hat{\phi}^b(y)\ .\end{aligned} \tag{4.9}$$

Using the projection operators defined earlier this expression takes the simplified form

$$\begin{aligned}\Gamma^{(2)}[\hat{\phi}]=&\frac{\lambda}{4!}\int d^Dx\sqrt{g}\,[3G_1(x,x)+2(N-1)G_1(x,x)G_2(x,x)^2+(N^2-1)G_2(x,x)^2]\\ &-\frac{\lambda^2}{36}\int d^Dx\sqrt{g}\int d^Dy\sqrt{g}\,\hat{\phi}(x)[3G_1(x,y)^3+(N-1)G_2(x,y)^3]\ .\end{aligned} \tag{4.10}$$

Again it is evident that the dominant contribution in the large-N limit comes from the G_2 internal lines and the four-point vertex.

It is possible to obtain an expression for the sum of all diagrams of this double bubble character (usually called daisy or cactus diagrams) in the large-N limit. Taking into account the dominant two-loop graph in the large-N limit the effective action (4.8) to this order becomes

$$\Gamma[\phi,G] \simeq S[\hat{\phi}] + \frac{N}{2}\mathrm{Tr}\ln G^{-1} + \frac{N}{2}\int d^Dx\sqrt{g}\; A_2 G(x,x) - \frac{N}{2}\mathrm{Tr}(1) + \frac{\lambda}{4!}\int d^Dx\sqrt{g}\; G^2(x,x)\ . \tag{4.11}$$

Varying with respect to G we obtain

$$G^{-1}(x,y) = A_2(x,y) + \frac{\lambda N}{6} G(x,x)\delta(x,y)\ . \tag{4.12}$$

Since

$$A_2(x,y) = (\Delta + M_2{}^2)\delta(x,y)\ , \tag{4.13}$$

if we write

$$G^{-1}(x,y) = (\Delta + \chi)\delta(x,y) \tag{4.14}$$

then we have the consistency equation for χ:

$$\chi(x) = M_2{}^2 + \frac{\lambda N}{6} G(x,x)\ . \tag{4.15}$$

We can now use (4.12) to eliminate A_2 from (4.11) to obtain

$$\Gamma[\phi,G] = S[\phi] + \frac{N}{2}\mathrm{Tr}\ln G^{-1} - \frac{\lambda N^2}{4!}\int d^Dx\sqrt{g}\; G^2(x,x)\ . \tag{4.16}$$

Using (4.14) and (4.15) with χ a constant this becomes

$$\Gamma[\hat{\phi},\chi] = S[\hat{\phi}] + \frac{N}{2}\mathrm{Tr}\ln[(\Delta+\chi)\mu^{-2}] - \frac{3}{2\lambda}\int d^Dx\sqrt{g}\,(\chi - M_2{}^2)^2\ . \tag{4.17}$$

Finally, the iterative equation becomes

$$\chi = M_2{}^2 + \frac{\lambda N}{6\Omega}\mathrm{Tr}(\Delta+\chi)^{-1}\ . \tag{4.18}$$

This can in principle be solved for $\chi = \chi(\hat{\phi})$ and substituted back into (4.17) to recover the one-particle-irreducible effective action with the sum of all daisy-type diagrams included.

When $\hat{\phi}$ and χ are constants, (4.17) defines a two-particle-irreducible effective potential

$$V_{\mathrm{eff}}(\hat{\phi},\chi) = V(\hat{\phi}) - \frac{3}{2\lambda}(\chi - M_2{}^2)^2 + \frac{N}{2\Omega}\mathrm{Tr}\ln[(\Delta+\chi)\mu^{-2}]\ . \tag{4.19}$$

We will use this in the next section to obtain the minimum effective mass allowed for different EIRD's.

V. FINITE-SIZE EFFECT IN FIELD THEORY AND COSMOLOGY

The modification of the infrared properties of systems with finite dimensions or constraints (from geometry or from the boundedness of the fluctuation operator) and the corresponding change in the symmetry behavior is generally known in condensed-matter physics as the finite-size effect.[20] It was first introduced in the context of trying to understand the universality of critical exponents of thermodynamic functions in specimens of finite extent as distinct from infinite or bulk systems, where the Wilson-Fisher theory[56] of critical phenomena originated. Numerical studies like Monte Carlo simulations which assume finite samples also show deviations from the theory of infinite systems. As the finite system becomes infinite, while approaching the infinite-system critical point, the finite-size corrections become simpler and obey a scaling law first introduced by Fisher.[57] Later Nightingale[58] and others devised the so-called phenomenological renormalization (real-space renormalization) approach for finite systems. As expected this subject is of primary importance for surface physics, where the systems under study are usually of lower dimensions and constrained. The use of field-theoretical methods was developed by Brezin[59] and others. Two canonical examples often studied are the so-called (i) slab and (ii) cylinder geometries as finite-size corrections to the bulk system. The boundary conditions used for the fields include the (i) periodic, (ii) antiperiodic, and (iii) open conditions. In the following we are interested in a fixed finite size rather than the approach to the bulk system (the scaling regime). We will focus on the occurrence of infrared divergences and a universality associated with dimensional reduction in the IR regime.

A. Finite-size effect in curved spacetime

As mentioned in the Introduction earlier work has dealt with the effect of curvature and topology on symmetry breaking in curved spacetime. These global properties are expected to influence the critical behavior. The effect we wish to discuss is neither of these exactly but rather the generalization of the finite-size concept to curved spacetimes. Viewing the problem in this way may enhance our intuitive understanding of phase transitions in curved spacetime (e.g., it allows us to analyze the admissibility of second-order phase transitions). Since it is the finite effective range of space or spacetime which gives rise to dimensional reduction and other interesting phenomena, this effect should more appropriately be identified as the finite-size effect. The finite-size effect is not a topological effect. Topological effects[8] on phase transition are relevant for multiply connected manifolds and for twisted fields. [In surface physics problems case (i) mentioned above corresponds to untwisted, case (ii) corresponds to twisted boundary conditions, and case (iii) to finite systems.] Finite-size effects nevertheless exist for spaces of trivial topology. In general it is neither purely topological nor geometrical. The generic form of the effective action for a finite-size system depends on both the geometry and the boundary conditions. However, the form of the finite-size corrections simplifies when the small fluctuation operator has a near-zero eigenvalue. For the corresponding infinite system, this would be the region in the neighborhood of a second-order critical point. In this region many of the details of the physical problem become irrelevant and the physics is governed by the bulk behavior of the system in its EIRD.

The finite-size effect is usually used in a restricted sense referring to the modification of the correlation functions of a finite system. However, in the general sense as we have used above it refers to the effect of constraints on quantum fluctuations. It is in this sense that the Casimir effect should be viewed as a manifestation of the finite-size effect. The properties of the Casimir effect in general depend on the details of the system since in the calculation of the vacuum energy leading to the Casimir effect, the full spectrum is taken into account. Whereas in our study of the infrared behavior of quantum systems (in spaces with some compact dimensions), only the contribution of the low-lying modes are shown to be important. The dominant contribution to the Casimir force comes from the $\hat{\phi}$-independent part of the effective potential, whereas the contribution to the correlation functions comes purely from the $\hat{\phi}$-dependent part, which is dominated by the low-lying modes in the near-critical region. This explains why the "finite-size effect" for the three-sphere S^3 and the three-torus T^3 are similar. The detailed nature of the spectral contribution accounts for the difference in the manifestation of their Casimir force (for S^3 is repulsive while that for T^3 is attractive). Useful information about the Casimir effect can be extracted from an analysis of the infrared divergences by the methods we have presented, especially with respect to the massless minimal case in curved spacetime. But we will not pursue these questions here.

In the last section we have derived two formal equations for the effective mass $M_{\text{eff}}=\chi$ (4.18) and the effective potential V_{eff} (4.19) using the two-particle-irreducible effective action formalism of Cornwall, Jackiw, and Tomboulis for composite operators. We will now discuss the solutions for different geometries. This will be followed by some remarks on the nature of finite-size effect in curved space and its relation with its counterpart in condensed-matter physics.

B. Symmetry behavior of product spaces

For spacetimes with topology $R^d \times B^b$ (where B is a b-dimensional compact space) χ is simply

$$\chi = M_2{}^2 + \frac{\lambda N}{6\Omega(B)} \int \frac{d^d k}{(2\pi)^d} \sum_n \frac{1}{k^2+\kappa_n+\chi} , \tag{5.1}$$

where $\Omega(B)$ is the volume of the subspace B and κ_n are the eigenvalues of Δ restricted to B. Considering only the dominant lowest mode contribution ($\kappa_0=0$), we obtain from integrating this lowest band the general expression for the effective mass:

$$\chi = M_2{}^2 + \frac{\lambda N}{6\Omega\chi} \left[\frac{\chi}{4\pi} \right]^{d/2} \Gamma(1-d/2) . \tag{5.2}$$

We express this explicitly for each individual dimensions:

$$\chi = M_2{}^2 + \frac{\lambda N}{6\Omega(B)\chi} \text{ for } d=0 , \tag{5.3}$$

$$\chi = M_2{}^2 + \frac{\lambda N}{12\Omega(B)} \chi^{-1/2} \text{ for } d=1 , \tag{5.4}$$

$$\chi = M_2{}^2 - \frac{\lambda N}{24\pi\Omega(B)} \left[\ln \left[\frac{\chi\mu^{-2}}{4\pi} \right] + \gamma_E \right] \text{ for } d=2 , \tag{5.5}$$

$$\chi = M_2{}^2 - \frac{\lambda N}{24\pi\Omega(B)} \chi^{1/2} \text{ for } d=3 . \tag{5.6}$$

From these formulas we see that the solution $\chi=0$ for $M_2{}^2=0$ is not permissible for $d \le 2$. This can be interpreted to mean that there cannot be a second-order phase transition for an O(N) model in two or less dimensions, or, as Coleman[42] put it: "there are no Goldstone bosons in two or less dimensions." This is what we have referred to above as the Mermin-Wagner-Coleman theorem.[41,42]

Let us consider each case individually.

(a) For $d=0$ we are dealing with *spaces with finite volume* (e.g., Euclideanized de Sitter S^4). The effective mass or inverse correlation length is given by solving (5.3) when $M_2{}^2=0$:

$$M_{\text{eff}} = \chi^{1/2} = (\lambda N/6\Omega)^{1/2} \tag{5.7}$$

for N-component scalar fields to leading order in large N. It depends on the size of the system and vanishes as $\Omega \to \infty$. (For S^4, $\Omega = 8\pi^2 a^4/3$.) A critical point which should exist in the infinite-volume system will disappear in a finite-volume theory—the finite-size effect thus precludes a second-order phase transition from occurring in finite systems. This can be seen also from the form of the effective potential. For massless, minimally coupled scalar fields in S^4 near $\hat{\phi}=0$ (Refs. 16 and 17), the effective potential obtained by inserting the solution of (5.3) for χ with $M_2{}^2=(\lambda/6)\hat{\phi}^2$ into (4.19) is given by

$$V(\hat{\phi}) = \frac{8\pi^2}{3\Omega}\alpha(\lambda)_{\text{IR}} + \tfrac{1}{2}(\hbar\lambda N/6\Omega)^{1/2}\hat{\phi}^2 + \frac{\lambda}{48}\hat{\phi}^4 + \cdots , \tag{5.8}$$

where

$$\alpha_{\text{IR}}(\lambda) = -3\hbar N[1+\ln(6\Omega a^4/\hbar\lambda N)]/32\pi^2 .$$

Note that the $\ln|\hat{\phi}|$ dependence on $\hat{\phi}$ near $\hat{\phi}=0$ from one-loop calculation is now modified to a power law. As the volume becomes large in comparison with the volume sustained by the correlation length, the effect of the finite size of the system decreases and the higher mode contributions to the effective potential can no longer be neglected. To extract this large volume behavior a different approximation which treats the system as a small deviation from the infinite-volume limit is necessary. In general the approach to the infinite-volume limit depends on the higher modes. For example, in the case of S^4, in the large-a limit, the deviations due to the finite size of S^4 drop off as inverse powers of a; by contrast, for T^4 the finite-size effect drops off exponentially. The effective potential for T^4 in the large volume limit is given by[60]

$$V_{\rm eff}(\hat{\phi})=V_{\rm eff}^{\infty}(\hat{\phi})-\frac{N\chi^2}{2(2\pi^3)^{1/2}}[(\chi L_1{}^2)^{-5/4}e^{-(\chi L_1)^{1/2}}+\cdots+(\chi L_4{}^2)^{-5/4}e^{-(\chi L_4)^{1/2}}]+\cdots . \quad (5.9)$$

We see an exponential fall-off of the finite-size effect in this case. It is worth emphasizing that the exponential fall-off seems to be characteristic of periodic boundary conditions (i.e., of the torus) and not a generic property of the approach to the bulk system; it depends on the boundary conditions in the finite-volume setting (cf. Ref. 59). This rapid fall-off makes periodic boundary conditions very attractive for Monte Carlo simulations where finite systems are used to mimic the behavior of infinite-volume systems.

(b) $d=1$, one open dimension. The next class of geometries to consider are those with one open dimension. In this case we are dealing with $R\times B^b$. We have seen examples from spacetimes such as the Einstein universe with compact $B=S^3$. The present discussion is also applicable to noncompact B but with fluctuation operators on B possessing discrete eigenvalues. In this case the infrared contribution to the effective potential corresponds to that of a scalar field in one dimension. When the lowest eigenvalue is zero in the bulk system, by solving Eq. (5.4) with $M_2{}^2=(\lambda/6)\hat{\phi}^2$ for small $\hat{\phi}$, we obtain

$$\chi=\left[\frac{\lambda N}{12\Omega}\right]^{2/3}+\frac{1}{9}\lambda\hat{\phi}^2+\left[\frac{12\Omega}{\lambda N}\right]^{2/3}\left[\frac{\lambda\hat{\phi}^2}{18}\right]^2 . \quad (5.10)$$

The effective mass around $\hat{\phi}=0$ is therefore given by

$$M_{\rm eff}{}^2=\left[\frac{\lambda N}{12\Omega(B)}\right]^{2/3} . \quad (5.11)$$

Note that this is similar to the $d=0$ case except for the difference in the exponent. Again there is no second-order phase transition in this system. Substituting this back into (4.19) yields the effective potential near $\hat{\phi}=0$ (higher-loop corrected, large N)

$$V_{\rm eff}(\hat{\phi})=\frac{9}{2}\left[\frac{\lambda N}{12\Omega}\right]^{4/3}+\frac{1}{2}\left[\frac{\lambda N}{12\Omega}\right]^{2/3}\hat{\phi}^2+\frac{2}{3}\frac{\lambda}{4!}\hat{\phi}^4+\cdots . \quad (5.12)$$

We see that again the one-loop $V^{(1)}\sim|\hat{\phi}|$ behavior is modified to a power law. The nonzero curvature of the effective potential near the symmetric state leads to cosmological consequences similar to that of the $d=0$ case.

(c) $d=2$, two open dimensions. This includes examples ranging from a column of binary fluid (open with transverse x,y dimension) under the influence of Earth's gravity (variation in the z direction)[60] to cylindrically symmetric gravitational wave or axisymmetric stationary solutions[45–48] (with t,z as open dimension) in general relativity. The two-dimensional case is rather unusual since it is the transition dimension for an allowed to a forbidden second-order transition in an O(N) model. The cutoff correlation length is in fact not determinable from considering solely the two-dimensional infrared problem. Rather, in the case of the two-sphere or for periodic boundary conditions, the higher bands determine the scale. When the contribution of these bands is taken into account one finds that (5.5) becomes the form

$$\chi=M_2{}^2+\left[\frac{\lambda N\beta}{6\Omega}\right]-\frac{\lambda N}{24\pi\Omega}\ln(\chi a^2) , \quad (5.13)$$

where a is a characteristic length scale and β is a number that depends on the details of the problem.

In the case when the infrared dimension is two, as in a binary fluid problem, there are two different directions to consider since variation in the z direction breaks the local symmetry of the problem. When we consider the horizontal subspace with Ω now a quasivolume with effective size given by the transverse correlation length the problem is that of a two-dimensional scalar field. Again there are large infrared fluctuations that randomize any ordering tendencies. Solving (5.13) with $M_2{}^2=0$, one finds a minimum mass for the system given by

$$M_{\rm eff}=\frac{\lambda N}{48\pi\Omega}\left[1+4\beta-\ln\left[\frac{\lambda N a^2}{6\pi\Omega}\right]\right] . \quad (5.14)$$

(d) $d=3$, with one compact dimension. A familiar example is the imaginary-time finite-temperature theory.[49] This case also has infrared divergences. The 2PI effective potential[54] is given by

$$V_{\rm eff}(\hat{\phi},\chi)=\frac{\lambda}{4!}\hat{\phi}^4-\frac{3}{2\lambda}\left[\chi-\frac{\lambda}{6}\hat{\phi}^2\right]-\frac{N}{12\pi\Omega}\chi^{3/2} . \quad (5.15)$$

From Eq. (5.6) we see that $\chi=0$ is a permitted solution; therefore, there is an allowed second-order phase transition. The higher-order modes give a contribution beginning at order Ω^{-2} which to leading order can be neglected. Solving (5.6) for χ near $\hat{\phi}=0$ we obtain

$$\chi^{1/2}=\frac{4\pi\Omega}{N}\hat{\phi}^2-\frac{1}{8}\left[\frac{N\lambda}{48\pi\Omega}\right]^{-3}\left[\frac{\lambda}{6}\hat{\phi}^2\right]^2 . \quad (5.16)$$

Substituting this back into the effective potential (5.15) yields

$$V_{\rm eff}(\hat{\phi})=-\frac{2\pi^2\Omega^2}{3N^2}\hat{\phi}^6+\cdots . \quad (5.17)$$

In this case the effect of the lowest band when taken to all orders becomes negligible. One would need to include all higher modes. For larger values of $\hat{\phi}$, solving (5.6) and substituting into (5.15) for χ yields

$$V_{\text{eff}}(\hat{\phi})=-\frac{\lambda N}{72\pi\Omega}\left[\frac{\lambda}{6}\right]^{1/2}|\hat{\phi}|^3+\frac{\lambda}{4!}\hat{\phi}^4+\cdots . \tag{5.18}$$

For finite-temperature theory the volume Ω is given by the inverse temperature β. For Kaluza-Klein theory it is the size L of the fifth dimension.

In the large-temperature or small-radius limit in these two cases the effective potential becomes[60]

$$V_{\text{eff}}(\hat{\phi},\chi)=\frac{\lambda}{4!}\hat{\phi}^4-\frac{3}{2\lambda}\left[\chi-\frac{\lambda}{6}\hat{\phi}^2\right]^2-\frac{\chi^{1/2}}{(2\pi^3\beta^5)^{1/2}}e^{-\beta\sqrt{\chi}}[1+\tfrac{15}{8}(\beta^2\chi)^{-1/2}+\tfrac{105}{128}(\beta^2\chi)^{-1}+\cdots] . \tag{5.19}$$

The potential rapidly approaches the zero temperature form as the temperature is lowered.

C. Deviations from renormalization-group behavior

The renormalization-group (RG) technique is often used in the description of critical behavior.[56] Such attempts on finite or constrained systems can lead to erroneous results. For instance, using RG arguments one would infer incorrectly the existence of second-order transitions in some of the above examples. This is because continuing the renormalization-group transformations means taking an ever larger cutoff. For a constrained system, after a finite number of iterations, the finite size of some of the directions renders further iteration meaningless. This is more apparent in a lattice system where the renormalization-group transformations are implemented by decimation.[56,58] Repeating the decimation process one eventually runs out of lattice in the finite directions. When the lattice is not infinite in all directions what happens is that the smallest direction drops out first, followed successively by the remaining directions that are constrained. The final result is that the renormalization-group transformations can proceed *ad infinitum* only in the unconstrained directions and the resultant transformation is that appropriate to the lower-dimensional system. If this lower-dimensional transformation has a fixed point then a second-order transition occurs, otherwise it does not.

In condensed-matter physics much attention in this subject has been focused on finite-size scaling.[57] The rounding effect seen in two or less dimensions corresponds to the appearance of a lower limit on the correlation length, and the shift in the critical point to a pseudocritical point (the point of maximum correlation length in the absence of phase transition) arises from a nonzero induced mass in the effective potential. From the way we see it, the existence of a relationship among the rounding exponent θ, the correlation exponent ν, and the shift exponent $\lambda(\theta=\nu^{-1}=\lambda)$ is derivable from the existence of a minimum induced effective mass or, more generally, a shift in the effective mass. We can also understand why the phenomenon occurs in systems with less than or equal to two dimensions.

VI. CONCLUSION

In this work we began with an inquiry into the symmetry behavior of curved space and found that the geometry and topology of spacetime could strongly influence the infrared behavior of the system. In particular for spacetimes with some compact dimensions, the symmetry behavior of the system could effectively be that of a lower-dimensional one, that of the noncompact dimensions. (More precisely, it is determined by the ratio of the effective mass, which runs with energy or curvature, to the scale of the space in question.) This phenomenon of dimensional reduction can be understood by seeing that the system has a discrete spectrum and that the lowest band influences most strongly the infrared behavior. We used the two-particle-irreducible representation of the effective action to derive an expression for the effective mass (or inverse correlation length) for systems in direct product spaces. We then discuss how the geometric effects in curved space under study are the manifestation of a much more general phenomenon known as the finite-size effect in condensed-matter physics. How the finite-size effect can affect phase transitions in the early Universe has been discussed in our earlier work (Refs. 17 and 18). Our later work will deal with dynamical effects in symmetry breaking in curved space and how the finite-size effect can manifest in certain classes of dynamical spacetimes.

ACKNOWLEDGMENTS

We thank Dr. Chris Stephens for interesting discussions on possible extensions of the present viewpoint and methodology to other finite-size systems. This work was supported in part by the National Science Foundation under Grants Nos. PHY84-18199 and PHY83-18635.

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