Traceless pole particles with intrinsic spin in spaces with torsion

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It is shown by the method of Papapetrou that a traceless pole particle with intrinsic spin in spaces with torsion satisfy $u^i u_i = 0$ and $Du^i/Dq = 0$, where u^i is the velocity of the center point, and D/Dq is the covariant derivative with Cartan torsion.

I. INTRODUCTION

In some previous papers[1–4] the method of moments that Papapetrou[5] introduced in general relativity was applied to particles with spin whose energy-momentum tensor had zero trace. The paths such traceless particles took were shown to be null geodesics. If u^i represents the velocity of the center of such a particle [$=dx^i/dq$, where q is a path parameter and $x^i(q)$ is the position of the center], then, with $\{{}_j{}^i{}_k\}$ being the Christoffel symbol,

$$u^i u_i = 0, \quad du^i/dq + \{{}^i_{jk}\} u^j u^k = 0 \ . \tag{1}$$

Heyl, von der Heyde, Kerlick, and Nestor[6] and Yasskin and Stoeger[7–10] studied particles in spaces with a Cartan torsion. In particular, Yasskin and Stoeger applied Papapetrou's method to the problem. The purpose of this paper is to calculate the trajectories of traceless particles with just *intrinsic* spin but for spaces with torsion. We simplify the calculation to refer to pole particles only, i.e., those for which integrals of the type $\int \delta x^i$ [ ] $dV = 0$, where δx^i is the distance of the volume integration point from the center X^i on a surface of constant time. The result obtained is that X^i satisfies the equations

$$u^i u_i = 0 \ , \tag{2a}$$

$$du^i/dq + \Gamma^i{}_{jk} u^j u^k = 0 \ , \tag{2b}$$

where $\Gamma^i{}_{jk}$ is the affine connection containing torsion. We shall call motion that satisfies Eq. (2b) Cartan transport.

In Sec. II, we introduce the basic notation and derive the basic equations, using the formalism of Ref. 6. Although Yasskin and Stoeger derive more general equations than ours, which are limited to pole particles and intrinsic spin, the derivation using the notation of Ref. 6 seems worthwhile to include here because of its simplicity in this case. In Sec. III, the traceless side condition is analyzed for spaces with torsion. For pole particles we get back the side conditions of Ref. 1. Since the spin equation found in Sec. II is also the same as in Ref. 1, the solution for the trajectory will also be the same, except that $\Gamma^i{}_{jk}$ must be used instead of $\{{}^i_{jk}\}$. A traceless pole particle with intrinsic spin performs null Cartan transport in spaces with torsion.

II. BASIC EQUATIONS

We follow the notation of Heyl, Heyde, Kerlick, and Nestor,[6] by and large, so that Σ^{ij} represents the nonsymmetric *canonical* energy-momentum tensor, $S^i{}_{jk}$ the Cartan torsion that appears in the connection $\Gamma^i{}_{jk}$, and τ^{ijk} the intrinsic spin density of the particle. However, to match up with Ref. 1, we use D_k or D/Dx^k (instead of $\overset{*}{\nabla}_k$) for the covariant derivative using the $\Gamma^i{}_{jk}$, and T^{ik} (instead of σ^{ik}) for the symmetrized energy-momentum tensor. Relations between Γ, { }, and between Σ and T are

$$\Sigma^{ij} = T^{ij} + (D_k + 2S_k{}^m{}_m)(\tau^{ijk} - \tau^{jki} + \tau^{kij}) \ , \tag{3}$$

$$\Gamma^k{}_{ij} = \{{}^k_{ij}\} + S_{ij}{}^k - S_j{}^k{}_i - S^k{}_{ij} \ , \tag{4}$$

as shown in Eqs. (3.8) and (2.11) of Ref. 6. $\Gamma^i{}_{jk}$ is not symmetric in j and k, but the Christoffel symbols $\{{}^i_{jk}\}$ are, while τ^{ijk} and S^{ijk} are antisymmetric in i and j.

The equations we need are derived from those for Σ^{ij} and τ^{ikj} given by Eqs. (3.12) and (3.13) of Ref. 6. However, we need to convert these to densities. This is done by utilizing

$$e = (-\det g_{ik})^{1/2}, \quad \partial_j e = \{{}^k_{kj}\} e \ , \qquad \Gamma^k{}_{kj} = \{{}^k_{kj}\} - 2S_{jk}{}^k \ . \tag{5}$$

Equations (3.12) and (3.13) of Ref. 6 can then be written as

$$\partial_j(e\Sigma^{ij}) + (\Gamma^i{}_{jm} - 2S^i{}_{jm}) e\Sigma^{mj} = R^{iljk} e\tau_{jkl} \ , \tag{6}$$

$$\partial_j(e\tau^{ikj}) + \Gamma^i{}_{jm} e\tau^{mkj} + \Gamma^k{}_{jm} e\tau^{imj} = e\Sigma^{[ik]} \ , \tag{7}$$

where $A^{[ik]} = \frac{1}{2}(A^{ik} - A^{ki})$.

The integration of Eqs. (6) and (7) is made over a surface at constant laboratory time t, it being assumed that Σ and τ are localized. They eventually are confined to a point. Space divergences integrate to zero, so that we get

$$(d/dc\,t)\int \Sigma^{i0}d\omega+\int(\Gamma^{i}{}_{jm}-2S^{i}{}_{jm})\Sigma^{mj}d\omega =\int R^{iljk}\tau_{jkl}d\omega\ , \quad (8)$$

$$(d/dc\,t)\int \tau^{ik0}d\omega+\int(\Gamma^{i}{}_{jm}\tau^{mkj}+\Gamma^{k}{}_{jm}\tau^{imj})d\omega =\int \Sigma^{[ik]}d\omega\ , \quad (9)$$

where $d\omega=e\,dV$, where dV is the volume differential in laboratory coordinates on the surface of constant t.

Following the method of Papapetrou, we need to obtain from Eqs. (6) and (7) equations for $x\Sigma$ and $x\tau$ where x^i is a position in space-time. We have

$$\partial_j(x^m e\Sigma^{ij})=e\Sigma^{im}+x^m\partial_j(e\Sigma^{ij})\ , \quad (10)$$

$$\partial_j(x^m e\tau^{ikj})=e\tau^{ikm}+x^m\partial_j(e\tau^{ikj})\ . \quad (11)$$

The divergences in the last terms of Eqs. (10) and (11) are to be taken from Eqs. (6) and (7), but we omit writing the whole expression out here.

Integrating Eqs. (10) and (11) over space at constant time t gives

$$(d/dc\,t)\int x^m\Sigma^{i0}d\omega=\int \Sigma^{im}d\omega+\int x^m\partial_j(e\Sigma^{ij})dV\ , \quad (12)$$

$$(d/dc\,t)\int x^m\tau^{ik0}d\omega=\int \tau^{ikm}d\omega+\int x^m\partial_j(e\tau^{ikj})dV\ . \quad (13)$$

We now make the expansion

$$x^m=X^m(t)+\delta x^m \quad (m=1,2,3)\ ,$$
$$x^0=X^0=ct \quad (m=0)\ , \quad (14)$$

on the surface of constant t, about the center point $X^m(t)$. Also expanded this way are $\Gamma^{i}{}_{jk}$, $S^{i}{}_{jm}$, and R^{iljk}. Then making the pole approximation that integrands containing the first or higher powers of δx^i integrate to zero, we find that Eqs. (8) and (9) reduce to

$$(d/dc\,t)\int \Sigma^{i0}d\omega+(\Gamma^{i}{}_{jm}-2S^{i}{}_{jm})\int \Sigma^{mj}d\omega =R^{iljk}\int \tau_{jkl}d\omega\ , \quad (15)$$

$$(d/dc\,t)\int \tau^{ik0}d\omega+\Gamma^{i}{}_{jm}\int \tau^{mkj}d\omega+\Gamma^{k}{}_{jm}\int \tau^{imj}d\omega =\int \Sigma^{[ik]}d\omega\ , \quad (16)$$

where the Γ,S, and R components are evaluated at the center point X^m. We find that Eqs. (12) and (13) reduce to

$$(dX^m/dc\,t)\int \Sigma^{i0}d\omega=\int \Sigma^{im}d\omega\ , \quad (17)$$

$$(dX^m/dc\,t)\int \tau^{ik0}d\omega=\int \tau^{ikm}d\omega\ . \quad (18)$$

The final equations are obtained by defining the momentum and spin tensors

$$p^i=\int \Sigma^{i0}d\omega\ , \quad (19)$$

$$S^{ik}=2\int \tau^{ik0}d\omega\ . \quad (20)$$

The spin has a factor of 2 in the definition so as to conform with the notation in Ref. 1. With these definitions, and noting that

$$u^m=dX^m/dq=(dX^m/dc\,t)dc\,t/dq=u^0dX^m/dc\,t\ , \quad (21)$$

where q is the path parameter, we find from Eqs. (17) and (18) that

$$u^m p^i=u^0\int \Sigma^{im}d\omega\ , \quad (22)$$

$$u^m S^{ik}=2u^0\int \tau^{ikm}d\omega\ . \quad (23)$$

Multiplying Eqs. (15) and (16) by u^0, and using Eqs. (22) and (23) we get

$$dp^i/dq+(\Gamma^{i}{}_{jm}-2S^{i}{}_{jm})u^jp^m=R^{iljk}u_lS_{jk}/2\ , \quad (24)$$

$$dS^{ik}/dq+\Gamma^{i}{}_{jm}u^jS^{mk}+\Gamma^{k}{}_{jm}u^jS^{im}=p^iu^k-u^ip^k\ . \quad (25)$$

With the notation D/Dq representing the absolute derivative involving the connections Γ, these become

$$Dp^i/Dq-2S^{i}{}_{jm}u^jp^m=\tfrac{1}{2}R^{iljk}u_lS_{jk}\ , \quad (26)$$

$$DS^{ik}/Dq=p^iu^k-u^ip^k\ . \quad (27)$$

Except for the term in $S^{i}{}_{jm}$, these equations have the Papapetrou *form* although the covariant derivative is now constructed with the Γ, as can be seen by comparison with Eqs. (2.9) and (2.10) of Ref. 1. Our Eqs. (22), (23), (26), and (27) are the pole-particle intrinsic-spin limits of Eqs. (111)–(114) of Ref. 7. Since R^{iljk}, u^i, and $S^{i}{}_{jm}$ all have tensor character, we may infer from Eqs. (26) and (27) that p^i and S^{ik} also have tensor character, something that is not assured at the outset from Eqs. (19) and (20).

III. THE SIDE CONDITIONS

We shall define a traceless particle as one whose *symmetrized* energy-momentum tensor satisfies

$$T^{i}{}_{i}=0\ . \quad (28)$$

This is the condition in the nontorsion problem for a photon and neutrino, and it seems reasonable to continue with this definition. With Eq. (3), converted to densities again, this leads to

$$e\Sigma^{i}{}_{i}=\partial_k[e(\tau_i{}^{ik}-\tau_i{}^{ki}+\tau^{ki}{}_{i})]$$
$$=2\partial_k(e\tau^{ki}{}_{i})\ , \quad (29)$$

since τ^{ikm} is antisymmetric in the first two indices.

The integral of this equation over a constant-time surface, using Eq. (23) yields

$$\int \Sigma^{i}{}_{i}d\omega=2(d/dt)\int \tau^{0i}{}_{i}d\omega=(d/dt)(u^iS^{0}{}_{i}/u^0)\ . \quad (30)$$

There is no problem forming a trace under the integral sign in the pole approximation, since g_{ik} may be brought in and out of the integral. Multiply by u^0 and use Eq. (22) to obtain

$$u^ip_i=-da/dq\ , \quad (31)$$

where

$$a=(u^i/u^0)S^{i}{}_{0}\ . \quad (32)$$

Since Eq. (31) is valid in any reference frame, a must be a scalar. Equation (31) is the first side condition.

Now multiply Eq. (29) by δx^i and integrate over the space at constant t:

$$\int \delta x^m \Sigma^i{}_i d\omega = 2 \int \delta x^m \partial_k (e\tau^{ki}{}_i) dV \ . \tag{33}$$

In the pole approximation, the left-hand side is zero. The right-hand side can be integrated by parts. The integrated-out part vanishes since the particle is localized, and the other term can be evaluated by using

$$\partial_k \delta x^m = \delta_k{}^m - \delta_k{}^0 u^m / u^0 \ , \tag{34}$$

since $\delta x^m = x^m - X^m(t)$. Equation (33) gives then

$$0 = \int \tau^{mi}{}_i d\omega - (u^m/u^0) \int \tau^{0i}{}_i d\omega \ . \tag{35}$$

Using Eq. (23) and the definition of a in Eq. (32), we get from this

$$u_i S^{im} = a u^m \ , \tag{36}$$

which is our second side condition.

Equations (31) and (36) are the side conditions for a traceless pole particle with intrinsic spin in a space with torsion. We see that they have the same form as the side conditions (3.3) and (3.4) obtained in Ref. 1.

IV. CONCLUSIONS

Since Eqs. (27), (31), and (36) are the same equations as in Ref. 1, the solution is also the same.[11] From Ref. 2, it follows that

$$u^k u_k = 0, \quad \dot{u}_k u^k = 0, \quad \dot{u}_k \dot{u}^k = 0 \ , \tag{37}$$

where $\dot{u}^k = Du^k/Dq$. These imply that $\dot{u}^k \sim u^k$, which means that a path parameter q can be found for which $\dot{u}^k = 0$, where now the covariant derivative D involves torsion. This is, of course, just Eq. (2). The conclusion is that the particle whose trajectory follows null Cartan transport is a traceless pole particle with intrinsic spin.

A photon does not qualify as such a particle because of the problem with gauge invariance.[12] A neutrino could. However, Heyl *et al.*[13] have remarked that a neutrino should obey ordinary parallel transport (without a torsion effect). The derivation described above indicates nevertheless that a neutrino would proceed by Cartan transport, i.e., with the $\Gamma^i{}_{jk}$, not the $\{{}^i{}_{jk}\}$.

As for the spin equation (27), this is no different from the case without torsion except for the interpretation of the operator D. Thus for massive particles, we use $u^i u_i = 1$ and $p_i u^i = m$ so that multiplication of (27) by u_k yields

$$p^i = m u^i + u_k \dot{S}^{ik} \ . \tag{38}$$

If now the side condition $u_k S^{ik} = 0$ is inserted in this (i.e., if $\dot{u}_k S^{ik} + u_k \dot{S}^{ik} = 0$ is used), then substitution of Eq. (38) into (27) yields the equation for Fermi-Walker transport. This argument is applicable to Papapetrou's original paper, and has been found by many other authors (see for example Ref. 9).

However, in the traceless case, we do not have Eq. (38), but at best

$$p^i = (p^0/u^0) u^i + (1/u^0) \dot{S}^{i0} \tag{39}$$

and we cannot proceed in this way. In Ref. 2, however, it was shown that the helicity vector

$$H_m = \tfrac{1}{2} e_{kmij} S^{ij} u^k \tag{40}$$

satisfies $\dot{H}_m = 0$, provided that $\dot{u}_m = 0$. (e_{kmij} is the alternating tensor.) That is, the helicity vector for a traceless particle is parallel propagated. In spaces with torsion, the same arguments hold,[11] but with the operator D containing torsion. The spin gets Cartan parallel propagated.

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[12]See Ref. 6, p. 407.
[13]See Ref. 6, p. 408.