Instability of Killing-Cauchy horizons in plane-symmetric spacetimes

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(Received 18 February 1987)

It is well known that when plane-symmetric gravitational waves collide, they produce singularities. Presently known exact solutions representing such collisions fall into two classes: those in which the singularities are spacelike, and those in which timelike singularities appear preceded by a Killing-Cauchy horizon. This paper shows that Killing-Cauchy horizons in plane-symmetric spacetimes are unstable against plane-symmetric perturbations and thence argues that generic spacetimes representing colliding plane waves are likely to have spacelike singularities without Killing-Cauchy horizons. More specifically, this paper gives an explicit definition of Killing-Cauchy horizons in plane-symmetric spacetimes and classifies these horizons into two types: those which are smooth surfaces, called "type I," and those which are singular, called "type II." It is then shown that type-I horizons are unstable with respect to any generic, plane-symmetric perturbation data posed on a suitable initial null boundary and evolved with arbitrarily nonlinear field equations satisfying some very general requirements; linearized gravitational perturbations constitute a special case of this instability. Horizons of type II are shown to be unstable with respect to generic, plane-symmetric perturbations satisfying linear evolution equations; a special case again is linearized gravitational perturbations.

I. INTRODUCTION AND SUMMARY

It has been known since the early 1970s[1] that when two plane gravitational waves propagating in an otherwise flat background collide, they focus each other so strongly as to produce a spacetime singularity. Until recently all the known solutions to the Einstein field equations describing such collisions[1,2] entailed all-encompassing, spacelike singularities that could not be avoided by any observer on any timelike world line. However, recently Chandrasekhar and Xanthopoulos[3] have constructed exact solutions in which the collision produces a Killing-Cauchy horizon, which in turn (if one continues the metric through the horizon analytically) is followed by a timelike singularity that is readily avoided by almost all observers traveling on timelike world lines. On the other hand, there are theorems[4,5] which suggest that spacetime singularities are the general outcome of arbitrary plane-wave collisions. However, these theorems establish the presence of singularities only in nonflat, plane-symmetric spacetimes which do not possess Killing-Cauchy horizons;[5] a stronger singularity theorem applicable to colliding plane-wave solutions which contain such horizons is not yet available.

Hence, the question naturally arises as to which of the above outcomes of plane-wave collisions is generic (if, indeed, any of them really is). The present paper makes no attempt to formulate this question precisely (which in itself is a nontrivial task to accomplish). However, this paper shows that the Killing-Cauchy horizons present in the recent Chandrasekhar-Xanthopoulos solutions cannot be generic, because such horizons in any plane-symmetric spacetime are unstable against linear vacuum perturbations (as well as nonvacuum perturbations) that preserve the plane symmetry. It is natural to expect that the growth of these instabilities, in a generic plane-symmetric situation, will convert the horizon into an all-encompassing spacelike singularity, and that such singularities are therefore the generic outcome of plane-wave collisions. However, this paper does not make any attempt at proving this speculation rigorously. [Independently of, and simultaneous with our proof of this instability, Chandrasekhar and Xanthopoulos discovered that the presence of a perfect fluid with (energy density) = pressure, or a null dust, in their solution destroys the horizon in the full nonlinear theory.]

Before turning to a detailed formulation and proof of the instability results, we illustrate them by two simple examples of plane-symmetric spacetimes with Killing-Cauchy horizons. In Sec. II of this paper we shall classify such horizons into two classes which we call type I and type II. A simple example of a spacetime with a Killing-Cauchy horizon of type I is the plane-polarized, plane sandwich wave[6] with the metric

$$g=-du\,dv+F^2(u)dx^2+G^2(u)dy^2\ , \tag{1.1}$$

where F, G are constant (hence g is flat) for $u\le 0$ and

$$F(u)=(f_1-u)\ ,\quad G(u)=(f_2-u)\ , \tag{1.2}$$

for $u\ge 1$, where $f_2\ge f_1>1$. In the region $0\le u\le 1$, F and G are determined by the spacetime curvature associated with the gravitational wave. The wave is sandwiched inside the region $0\le u\le 1$ since this spacetime is flat not only for $u\le 0$ but also for $u\ge 1$, as becomes evident after transforming to the global coordinate system (U,V,X,Y) given by (for $u\ge 1$)

$$x=\frac{X}{f_1-U}\ ,$$
$$y=\frac{Y}{f_2-U}\ ,$$
$$u=U\ , \tag{1.3}$$
$$v=V+\frac{X^2}{f_1-U}+\frac{Y^2}{f_2-U}\ ,$$

in which the metric is

$$g=-dU\,dV+dX^2+dY^2\ ,\quad U\geq 1. \tag{1.4}$$

As is clear from the form of the metric in Eq.(1.1), the plane-wave spacetime admits the two spacelike Killing vectors $\xi_1=\partial/\partial x$ and $\xi_2=\partial/\partial y$ as plane-symmetry generators. In the global Minkowskian chart (U,V,X,Y) that covers the whole spacetime including the surface $\{u=U=f_1\}$ in a nonsingular fashion, these Killing vectors are given by the expressions

$$\xi_1=\frac{\partial}{\partial x}=(f_1-U)\frac{\partial}{\partial X}-2X\frac{\partial}{\partial V}\ ,$$
$$\xi_2=\frac{\partial}{\partial y}=(f_2-U)\frac{\partial}{\partial Y}-2Y\frac{\partial}{\partial V}\ . \tag{1.5}$$

The Killing vector ξ_1 or both ξ_1 and ξ_2 become null on the Killing-Cauchy horizon $\mathcal{S}=\{U=f_1\}$ according to whether $f_2>f_1$ or $f_2=f_1$. (See Fig. 1 for the case $f_1=f_2$.) In either case they are both spacelike before the horizon ($U<f_1$) and become tangent to the horizon as U approaches f_1, one (or both) of them pointing along the null generators of the Killing-Cauchy horizon when $U=f_1$. In the case $f_1=f_2$, both ξ_1 and ξ_2 vanish on the null line $\mathcal{C}=\{U=f_1,X=Y=0\}$ in $\mathcal{S}$, whereas on any neighborhood of $\mathcal{C}$ in $\mathcal{S}$ at least one of ξ_i $(i=1,2)$ is nonzero. In the case $f_2>f_1$, ξ_1 vanishes on the null two-plane $\mathcal{P}=\{U=f_1,X=0\}$ in $\mathcal{S}$, whereas it is nonzero on any neighborhood of $\mathcal{P}$ in $\mathcal{S}$. On the other hand, ξ_2 remains spacelike and nonzero on $\mathcal{S}$ in this case.[7] Figure 1 depicts the Killing vector field $\xi_1=\partial/\partial x$, surfaces of constant u and v, and the Killing-Cauchy horizon $\mathcal{S}=\{u=U=f_1\}$ for this example, in the case $f_1=f_2$ and in Minkowskian coordinates with the Y direction suppressed. As one sees from this figure or from Eqs. (1.3) and (1.4), the horizon $U=f_1$ is a smooth hypersurface in spacetime generated by endless null geodesics. This turns out to be the feature that distinguishes type-I horizons from type II.

In Sec. III we study the propagation of a wide class of classical fields on a plane-symmetric spacetime having a Killing-Cauchy horizon of type I as in the above example. The class of fields we work with is constrained only by the type of wave equation they satisfy and these constraints are very weak; for example, they admit linear scalar waves satisfying $\Box\phi=0$, linearized gravitational perturbations, and fields satisfying arbitrarily nonlinear evolution equations that respect the causal structure of the unperturbed background spacetime (e.g., the $\lambda\phi^4$ field theory); but not (in general) the fully nonlinear gravitational perturbations. Section III shows that when generic,

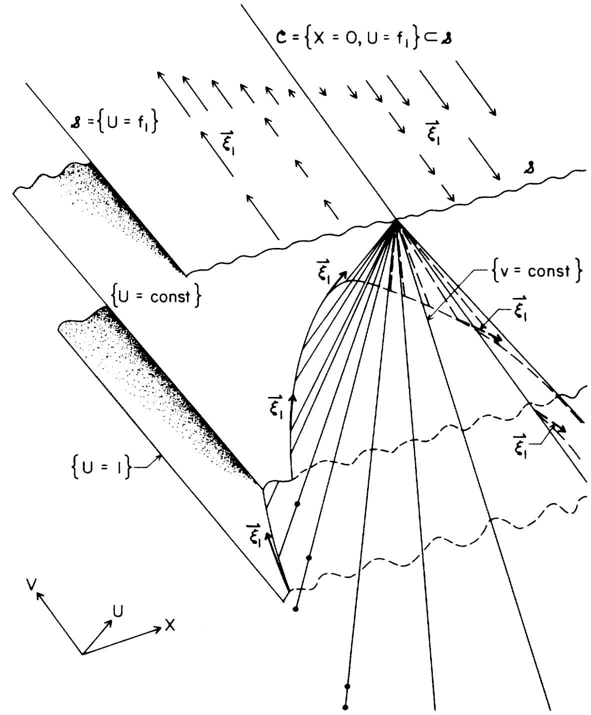

FIG. 1. The type-I Killing-Cauchy horizon $\mathcal{S}$ in the Minkowskian region of the plane-sandwich-wave spacetime described by Eqs. (1.1)– (1.5) with $f_1=f_2$. The Minkowski region is given by $U\geq 1$ and the horizon $\mathcal{S}$ is located at $U=f_1=f_2$. The Y dimension is suppressed. The Minkowskian null cone centered on the line $\mathcal{C}=\{X=0,\ U=f_1\}$ in $\mathcal{S}$ is (the closure of) a $\{v=\text{const}\}$ surface and has one generator in common with $\mathcal{S}$ along the line $\mathcal{C}$. The remaining generators of this cone are lines of constant v, x, and y on which $u=U$ ranges from 1 to f_1. The Killing vector field $\xi_1=\partial/\partial x$ is tangent to the intersections of $\{u=U=\text{const}\}$ surfaces with the $\{v=\text{const}\}$ cones which are obtained by rigidly translating the illustrated null cone along the line $\mathcal{C}$. On the Killing-Cauchy horizon $\mathcal{S}$, ξ_1 degenerates to a null vector tangent to the null generators of $\mathcal{S}$ and vanishes on $\mathcal{C}$.

plane-symmetric initial data for such fields are propagated with the corresponding field equations on a plane-symmetric spacetime with a Killing-Cauchy horizon of type I, the fields become singular as they approach the horizon.

This instability of Killing-Cauchy horizons of type I is well illustrated by the example of a linear scalar field satisfying the wave equation $\Box\phi=0$ in the above plane-sandwich-wave spacetime given by Eqs. (1.1) and (1.2). The scalar wave equation

$$\Box\phi=\frac{1}{\sqrt{-|g|}}\frac{\partial}{\partial x^\alpha}\left[\sqrt{-|g|}\,g^{\alpha\beta}\frac{\partial\phi}{\partial x^\beta}\right]=0$$

in this case takes the form [Eq. (1.1)]

$$-4\phi_{,uv}-2\left[\frac{F_{,u}}{F}+\frac{G_{,u}}{G}\right]\phi_{,v}+\frac{1}{F^2}\phi_{,xx}+\frac{1}{G^2}\phi_{,yy}=0\ . \tag{1.6}$$

For a plane-symmetric field $\phi(u,v)$ and for $u \geq 1$, this equation becomes [cf. Eq. (1.2)]

$$-2\phi_{,uv} + \left[\frac{1}{f_1 - u} + \frac{1}{f_2 - u} \right] \phi_{,v} = 0 \tag{1.7}$$

and has the general solution

$$\phi = \frac{a(v)}{[(f_1 - u)(f_2 - u)]^{1/2}} + b(u) , \tag{1.8}$$

where a and b are functions that are uniquely determined by initial data for ϕ on the null boundary consisting of the null surfaces $u = 1$ and $v = 0$. Clearly, for generic initial data, a and b will be nonzero and ϕ will diverge as $u = U \to f_1$. If initial data on the surface $\{v = 0\}$ and for $v \geq v_1 > 0$ on the surface $\{u = 1\}$ are zero, then this initial-value problem describes the collision of a scalar plane sandwich wave with the background gravitational plane wave. In that case the solution simply is

$$\phi = a(v) \left[\frac{(f_1 - 1)(f_2 - 1)}{(f_1 - u)(f_2 - u)} \right]^{1/2}$$
$$= a \left[V + \frac{X^2}{f_1 - U} + \frac{Y^2}{f_2 - U} \right] \left[\frac{(f_1 - 1)(f_2 - 1)}{(f_1 - U)(f_2 - U)} \right]^{1/2} , \tag{1.9}$$

where $a(v)$ is equal to ϕ on the initial surface $\{u = 1\}$ and vanishes for $v \geq v_1$ and for $v \leq 0$.

Geometrically, the reason for this singular behavior is simple: The symmetry of the spacetime, as embodied in the Killing vector fields $\xi_1 = \partial/\partial x$ and $\xi_2 = \partial/\partial y$, forces the plane-symmetric field to focus onto the line $\mathcal{C}$ (Fig. 1); a line to which all curves of constant v,x,y converge as $u \to f_1$; and this focusing of the waves produces a divergence in their amplitude. The proof of instability in Sec. III shows that this behavior is quite general for plane-symmetric spacetimes with type-I Killing-Cauchy horizons.

Turn now to the second example, a spacetime with metric

$$g = -dt^2 + dz^2 + t^2 dx^2 + dy^2 , \quad t \leq 0 , \tag{1.10a}$$

or, by putting $u = t - z, v = t + z$,

$$g = -du\, dv + \tfrac{1}{4}(u+v)^2 dx^2 + dy^2 , \quad u + v \leq 0 , \tag{1.10b}$$

in which the plane-symmetry generating Killing vectors are again $\xi_1 = \partial/\partial x$ and $\xi_2 = \partial/\partial y$. In this case ξ_2 is everywhere spacelike, while ξ_1 becomes null on the Killing-Cauchy horizon $t = 0$ but is spacelike prior to the horizon ($t < 0$). This spacetime is actually flat as one sees from the coordinate transformation

$$T = t \cosh x , \quad X = t \sinh x , \tag{1.11}$$

$$Y = y , \quad Z = z ,$$
$$g = -dT^2 + dX^2 + dY^2 + dZ^2 . \tag{1.12}$$

Figure 2 depicts the Killing vector fields $\xi_1 = \partial/\partial x$, $\xi_2 = \partial/\partial y$, surfaces of constant t, and the horizon $\{t = 0\}$

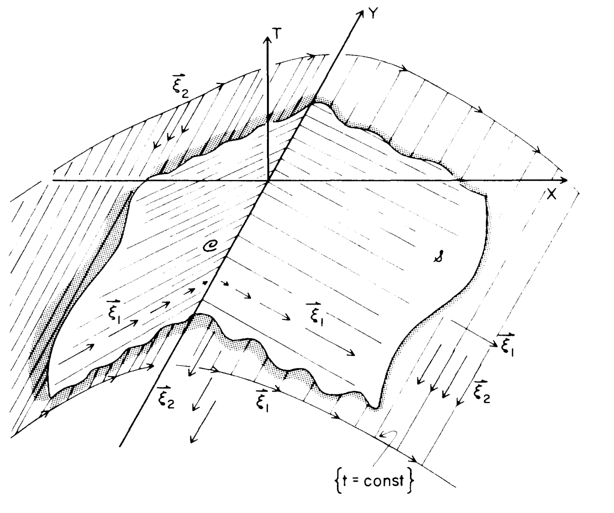

FIG. 2. The type-II Killing-Cauchy horizon $\mathcal{S}$ in Minkowski space described by Eqs. (1.10)–(1.12). The Z dimension is suppressed. The horizon $\mathcal{S}$ is located at $\{t = 0\}$, i.e., at $\{T = -|X|\}$ in Minkowskian coordinates. A $\{t = \text{const} < 0\}$ surface lying under $\mathcal{S}$ is shown along with the orbits of the plane-symmetry generating Killing vectors ξ_1 and ξ_2 on it. Even though it is spacelike below the horizon $\mathcal{S} = \{t = 0\}$, the Killing vector ξ_1 becomes null on the horizon $\mathcal{S}$ and points along its null generators, whereas the other plane-symmetry generator ξ_2 is everywhere spacelike. ξ_1 vanishes on the line (two-plane) $\mathcal{C}$ in $\mathcal{S}$ given by $\{T = X = 0\}$. The horizon $\mathcal{S}$ has a "crease" singularity on this line $\mathcal{C}$, at which the null generators of $\mathcal{S}$ have their future end points and onto which all lines of constant x, y, and z converge as $t \to 0$.

($= \{T = -|X|\}$) in the Minkowskian coordinates with the Z (z) dimension suppressed. As one sees from this figure, the horizon $t = 0$ is not everywhere smooth; it has a crease on the curve denoted by $\mathcal{C}$ in the figure; i.e., at $T = X = 0$. This kind of nonsmooth behavior characterizes type-II horizons; it shows up, for example, in the Killing-Cauchy horizons of the exact, colliding plane-wave solutions studied by Chandrasekhar and Xanthopoulos in Ref. 3 [their Eq. (124)].

Section IV of this paper studies the propagation of fields satisfying linear wave equations (e.g., scalar fields or linearized gravitational perturbations) in a plane-symmetric spacetime with a type-II Killing-Cauchy horizon. When these fields are constrained to be plane symmetric and are evolved from generic initial data, they diverge as they approach the horizon. As an example, consider a scalar field satisfying $\Box\phi = 0$ in the spacetime with metric (1.10). The general plane-symmetric (i.e., x,y independent) solution to

$$\Box\phi = -\frac{\partial^2\phi}{\partial t^2} - \frac{1}{t}\frac{\partial\phi}{\partial t} + \frac{\partial^2\phi}{\partial z^2} + \frac{1}{t^2}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{1.13}$$

is

$$\phi = \int_{-\infty}^{+\infty} [A(\omega) J_0(\omega t) + B(\omega) Y_0(\omega t)] e^{i\omega z} d\omega , \tag{1.14}$$

where J_0, Y_0 are the Bessel functions of the first and second kind and the functions $A(\omega),B(\omega)$ are uniquely determined by initial data for ϕ on some initial $t=\text{const}$ surface prior to the horizon $t=0$. As we approach the horizon $t=0$, $J_0(\omega t)$ remains well behaved but $Y_0(\omega t)$ diverges logarithmically

$$Y_0(\omega t)\sim\frac{2}{\pi}\ln|\omega t|+\text{const}\ ; \tag{1.15}$$

and correspondingly, unless $B(\omega)$ vanishes for all ω (a nongeneric case),

$$\phi\sim E(z)\ln|t|=\tfrac{1}{2}E(Z)\ln|T^2-X^2| \tag{1.16}$$

for some (generically nonzero) function $E(z)$.

As in the type-I case, the reason for this instability is geometrical: The ξ_1 symmetry of the field and of the spacetime forces the waves to focus onto the line $\mathcal{C}$ ($T=X=0$), to which all curves of constant x,y,z converge as $t\to 0$ (Fig. 2); and this focusing of the field produces a divergence in its amplitude. The proof of instability in Sec. IV shows that this behavior is quite general for linear fields in plane-symmetric spacetimes with type-II horizons.

In the concluding section (Sec. V) we briefly recapitulate the implications of these results for the general structure of singularities in plane-symmetric spacetimes.

Throughout this paper our notation and conventions are the same as those in Ref. 8, in particular the metric has signature $(-,+,+,+)$ and the Newman-Penrose equations are used in the "rationalized" form appropriate to that signature.[9,4]

II. CLASSIFICATION

By a plane-symmetric spacetime we shall mean a maximal spacetime $(\mathcal{M},g)$ with a C^2 metric g on which there exist (i) a pair of commuting Killing vectors $\xi_i\equiv\xi_1,\xi_2$ and (ii) a dense open subset at each point of which the ξ_i generate a spacelike two-dimensional plane in the tangent space. If the dense open subset is equal to $\mathcal{M}$, we call $(\mathcal{M},g)$ strictly plane symmetric as no breakdowns of plane symmetry occur on $\mathcal{M}$.

By a Killing-Cauchy horizon in a plane-symmetric spacetime $(\mathcal{M},g)$ we shall mean a null, achronal, edgeless[8] three-dimensional connected (C^{1-}) surface $\mathcal{S}$ in $\mathcal{M}$ on which at least one of the Killing vectors ξ_i degenerates to a null Killing vector (which is not identically zero on $\mathcal{S}$); and whose null geodesic generators have no past end points in $\mathcal{M}$ and are past complete. It follows from the definition of plane symmetry that both ξ_i must be tangent to $\mathcal{S}$, and hence the Killing vector(s) which degenerates to a null vector on $\mathcal{S}$ is tangent to the null generators of $\mathcal{S}$ on $\mathcal{S}$. As the spacetime is maximal and the generators of $\mathcal{S}$ are tangent to Killing directions, we assume (without loss of generality) that the null geodesics generating $\mathcal{S}$ are also future complete in $\mathcal{M}$ [or at least in a neighborhood of $\mathcal{S}$ in $\mathcal{M}$ (Ref. 8)].

If $(\mathcal{M},g)$ is a spacetime with a Killing-Cauchy horizon $\mathcal{S}$ for which the above definitions are satisfied only on $I^-(\mathcal{S})\cup\mathcal{S}$, we will still regard $(\mathcal{M},g)$ as plane symmetric for it will become clear later that this is all we need to prove our results. (See the remarks following Theorems 1 and 2.)

On any plane-symmetric spacetime there are local coordinate systems (u,v,x,y) [covering at least $I^-(\mathcal{S})$] such that $\xi_i=\partial/\partial x^i$ ($x^1\equiv x, x^2\equiv y$). By plane symmetry, in any such coordinate system a Killing-Cauchy horizon $\mathcal{S}$ in $\mathcal{M}$ will be given by an expression of the form $\{f(u,v)=\text{const}\}$ since ξ_i are tangent to $\mathcal{S}$. Then there are two possible cases.

If there exists a local coordinate system (u,v,x,y) in which $\xi_i=\partial/\partial x^i$ and $\mathcal{S}$ is given by $\mathcal{S}=\{f(u,v)=\text{const}\}$ where ∇f is a smooth, everywhere nonvanishing vector field on $\mathcal{S}$, then we will say that $\mathcal{S}$ is a Killing-Cauchy horizon of type I.

If in every local coordinate system of the above kind and for every $f(u,v)$ such that $\mathcal{S}=\{f(u,v)=\text{const}\}$, ∇f either vanishes or blows up at some points on $\mathcal{S}$, then we call $\mathcal{S}$ a Killing-Cauchy horizon of type II.

Clearly, the first example of a Killing-Cauchy horizon which we described in the last section [Sec. I, Eqs. (1.1)–(1.5)] is of type I since it was given by $\mathcal{S}=\{u=U=f_1\}$ and $\nabla u=-2\partial/\partial v=-2\partial/\partial V$ is a smooth everywhere nonzero vector field on $\mathcal{S}$. On the other hand, our second example [Eqs. (1.10)–(1.12)] was of type II as it was given by $\mathcal{S}=\{t=\frac{1}{2}(u+v)=0\}$ where

$$\nabla t=-\frac{\partial}{\partial t}=-\frac{T}{(T^2-X^2)^{1/2}}\frac{\partial}{\partial T}-\frac{X}{(T^2-X^2)^{1/2}}\frac{\partial}{\partial X}$$

which blows up on $\mathcal{S}$. An alternative choice for f, $f(u,v)=t^2=\frac{1}{4}(u+v)^2$ leads to $\nabla f=2t\nabla t$, which vanishes on the crease line $\mathcal{C}=\{T=X=0\}\subset\mathcal{S}$. It is not possible to describe $\mathcal{S}$ globally by any $f(u,v)=0$ where ∇f is smooth and everywhere nonzero on $\mathcal{S}$.

III. INSTABILITY OF HORIZONS OF TYPE I

Before stating our instability theorem for horizons of type I, we formulate some of our assumptions.

Assumption (A1). $(\mathcal{M},g)$ is a plane-symmetric vacuum spacetime.

Assumption (A2). There is an open subset in $\mathcal{M}$ on which g is flat.

Assumption (A2) is not true of all plane-symmetric spacetimes, but it is true of spacetimes containing nothing but plane-symmetric gravitational waves (possibly coupled with matter or electromagnetic radiation), since such spacetimes are flat before any of the waves arrive.

By (A1) we can define a canonical null tetrad on $(\mathcal{M},g)$: $\boldsymbol{l},\mathbf{n}$ are the null geodesic congruences everywhere orthogonal to the ξ_i and Lie parallel along ξ_i; $\mathbf{m},\mathbf{m}^*$ are linearly independent linear combinations of the ξ_i, normalized such that $-g(\boldsymbol{l},\mathbf{n})=g(\mathbf{m},\mathbf{m}^*)=1$, $g(\mathbf{m},\mathbf{m})=0$. Then as is shown by Szekeres,[10] it follows from the presence of only two nontrivial dimensions that we can find a local chart (u,v,x,y) with $\xi_i=\partial/\partial x^i$ such that

$$\begin{aligned}\boldsymbol{l}&=\frac{\partial}{\partial u}+P^i(u,v)\frac{\partial}{\partial x^i}\ ,\\ \mathbf{n}&=R(u,v)\frac{\partial}{\partial v}+Q^i(u,v)\frac{\partial}{\partial x^i}\ ,\\ \mathbf{m}&=\frac{1}{F(u,v)}\frac{\partial}{\partial x}+\frac{1}{G(u,v)}\frac{\partial}{\partial y}\ ,\end{aligned} \tag{3.1}$$

where P^i, Q^i, R are real and F, G are complex, with $F^*G - G^*F \neq 0$ throughout the region on which strict plane symmetry holds and the tetrad (3.1) and the coordinate chart (u,v,x,y) are well behaved. The commutation relations[4] for the tetrad (3.1) yield

$$RP^1{}_{,v} - Q^1{}_{,u} = \frac{4\alpha}{F} + \frac{4\alpha^*}{F^*} - \frac{R_{,u}}{R} Q^1 \ ,$$
$$RP^2{}_{,v} - Q^2{}_{,u} = \frac{4\alpha}{G} + \frac{4\alpha^*}{G^*} - \frac{R_{,u}}{R} Q^2 \ , \tag{3.2}$$

where α denotes the Newman-Penrose spin coefficient. We can eliminate the $P^i\partial_i$ and $Q^i\partial_i$ terms from (3.1) by a coordinate transformation of the form

$$u' = u \ , \quad v' = v \ , \quad x^{i'} = x^i + \zeta^i(u,v) \ , \tag{3.3}$$

if $P^i + \zeta^i{}_{,u} = Q^i + R\zeta^i{}_{,v} = 0$. But the integrability conditions for $P^i + \zeta^i{}_{,u} = Q^i + R\zeta^i{}_{,v} = 0$ are

$$RP^i{}_{,v} - Q^i{}_{,u} = -\frac{R_{,u}}{R} Q^i \ ,$$

which by (3.2) are equivalent to $\alpha \equiv 0$. However, it follows by standard arguments[10] using the Ricci identities[10,4] in the vacuum case that assumption (A2) guarantees $\alpha \equiv 0$ on $\mathcal{M}$ when (3.1) is suitably set in the flat region. Hence we can, by a coordinate change (3.3), put our tetrad into the form

$$\boldsymbol{l} = \frac{\partial}{\partial u} \ , \quad \mathbf{n} = R(u,v)\frac{\partial}{\partial v} \ ,$$
$$\mathbf{m} = \frac{1}{F(u,v)}\frac{\partial}{\partial x} + \frac{1}{G(u,v)}\frac{\partial}{\partial y} \ . \tag{3.4}$$

The Newman-Penrose commutation relations for the tetrad (3.4) give zero values for the following combinations of spin coefficients

$$\kappa = \nu = \alpha = \beta = \tau = \pi = \gamma + \gamma^* = \rho - \rho^* = \mu - \mu^* = 0 \ ;$$

and the field equations then imply that two of the components of the Weyl tensor vanish:

$$\Psi_1 = \Psi_3 = 0 \ .$$

The other spin coefficients can also be calculated using the commutation relations. Of them we will only need the complex expansion

$$\rho = \frac{1}{2(F^*G - G^*F)} \left[F^*G\left[\frac{F_{,u}}{F} + \frac{G^*{}_{,u}}{G^*}\right] - G^*F\left[\frac{F^*{}_{,u}}{F^*} + \frac{G_{,u}}{G}\right]\right] . \tag{3.5}$$

Assumption (A3). There is a Killing-Cauchy horizon $\mathcal{S}$ of type I in $(\mathcal{M},g)$.

Assumption (A4). The metric g is analytic in a neighborhood of $\mathcal{S}$ in $\mathcal{M}$; i.e., there are admissible coordinate systems in a neighborhood of $\mathcal{S}$ in which the metric coefficients are analytic functions.

Assumption (A4) guarantees [as $g(\xi_i,\xi_i)$ are analytic functions near $\mathcal{S}$] that strict plane symmetry holds on a neighborhood $\mathcal{W}$ of $\mathcal{S}$ in $\mathcal{M}$, with the exception of breaking down on $\mathcal{S}$ itself.

Also note that the tetrad component $R(u,v)$ [Eq. (3.4)] is bounded and nonzero on $\mathcal{S}$ since the vanishing or divergence of R at $\mathcal{S}$ will cause curvature singularities (in Ψ_2 and Ψ_4) to appear on $\mathcal{S}$. (See, e.g., Refs. 10 and 11.) On the other hand, by (A3) $\mathcal{S}$ is of the form $\{f(u,v) = \text{const}\}$ where ∇f is smooth and everywhere nonzero on $\mathcal{S}$. Therefore by the implicit function theorem,[12] $\mathcal{S} = \{f(u,v) = \text{const}\}$ is a smooth (at least C^1) null surface and hence is generated by null geodesics without end points. Then, since the generators of $\mathcal{S}$ are future and past complete in $\mathcal{M}$ by assumption, by exactly the same argument as we will give in the proof of Theorem 2 below, it follows that we can find a function $\hat{f}$ which is smooth, vanishes on $\mathcal{S}$, and has a smooth, null nonzero gradient everywhere in a neighborhood of $\mathcal{S}$. As $\hat{f}$ has these properties globally on all of $\mathcal{S}$, it can be chosen to be a function of only u and v. (Since $\xi_i = \partial/\partial x^i$ are Killing and hence have zero convergence and since they become tangent to the horizon $\mathcal{S} = \{\hat{f} = 0\}$, they cannot be threading through every family of surfaces $\{\hat{f} = \text{const} \neq 0\}$ each of which consists of parallel null surfaces generated by complete null geodesics without end points.) Redefine $f \equiv \hat{f}$ since $\hat{f} = 0$ on $\mathcal{S}$. Then $\mathcal{S} = \{f = 0\}$ and $0 = g(\nabla f, \nabla f) = -Rf_{,u}f_{,v}$ [by (3.4)] in a neighborhood of $\mathcal{S}$. As $R \neq 0$ on $\mathcal{S}$, this implies either $f_{,u} \equiv 0$ or $f_{,v} \equiv 0$ (but not both since $\nabla f \neq 0$) in a neighborhood of $\mathcal{S}$, which clearly tells us that $\mathcal{S}$ is a surface of the form $\{u = \text{const}\}$ or $\{v = \text{const}\}$. We shall assume, without loss of generality, that $\mathcal{S} = \{u = f\}$ where f is a constant.

Theorem 1. Let $(\mathcal{M},g)$ be a spacetime satisfying assumptions (A1)–(A4). Let $\{Q^a\}$ denote an arbitrary multi-index field (e.g., scalar, tensorial, or spinorial) defined on the spacetime, which satisfies field equations obeying the following conditions.

(a) $Q^a \equiv 0$ is a solution of the field equations.

(b) The characteristic surfaces for the field equations are null surfaces of $(\mathcal{M},g)$ and the evolution of $\{Q^a\}$ is globally causal: if initial data for $\{Q^a\}$ are zero outside a closed set $\mathcal{K}$ in an initial surface Σ, then there exists an open neighborhood $\mathcal{U}(\Sigma)$ of $\overline{D^+(\Sigma)}$ in $\mathcal{M}$ such that whenever there exists a smooth extension of the solution on $D^+(\Sigma)$ to $\mathcal{U}(\Sigma)$ it can be chosen so that $Q^a = 0$ on $\mathcal{U}(\Sigma) - (J^+(\mathcal{K}) \cup J^-(\mathcal{K}))$.

(c) There is a consistent characteristic initial-value formalism for the field equations for $\{Q^a\}$: if $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$ is an initial null boundary consisting of three-dimensional null surfaces $\mathcal{N}_1$, $\mathcal{N}_2$ intersecting in a two-dimensional spacelike surface Z, then one can freely pose initial data on $\mathcal{N}$ (satisfying some constraint equations on $\mathcal{N}$). Moreover, uniqueness and local existence of solutions in $D^+(\mathcal{N})$ hold for both the general characteristic initial-value problem and for the plane-symmetric initial-value problem for $\{Q^a\}$; the latter being obtained from the field equations by assuming $(\mathcal{L}_{\xi_i} Q)^a \equiv 0$.

If these conditions are satisfied, then there is a null boundary $\mathcal{N}$ in $I^-(\mathcal{S})$ such that the evolution of any generic member of a class of plane-symmetric initial data for $\{Q^a\}$ on $\mathcal{N}$ that we will describe develops singularities on the Killing-Cauchy horizon $\mathcal{S}$.

Remarks

(i) First note that conditions (a), (b), (c) are universal properties of all physical fields that do not, by their stress energy, act back on the geometry of the background spacetime; hence in particular of linearized gravitational perturbations. Although we are primarily interested in fields satisfying linear evolution equations, it is clear that inclusion of higher-order terms in the equations will not affect the validity of the theorem so long as these terms respect the causal structure of the background spacetime. (Note that fully nonlinear gravitational perturbations will not, in general, have this property.[13]) Linearity is not necessary for any of the conditions (a), (b), (c).

(ii) As will be clear from the proof, the theorem will still hold if our assumptions (A1), (A2), and (A3) are valid only in an open subset of the region $I^-(\mathcal{S})$ whose closure in $\mathcal{M}$ contains $\mathcal{S}$.

(iii) Our only use of the vacuum assumption is in the Ricci identities involving Φ_{10} and Φ_{21}, which are ingredients in the proof[10,4] that (A2) permits setting the Newman-Penrose spin coefficient α to zero and thence permits specializing the tetrad from (3.1) to (3.4). Consequently, the theorem is also valid for a spacetime $(\mathcal{M},g)$ satisfying assumptions (A1)–(A4) with the exception that the stress-energy tensor T, instead of being zero, is assumed to only satisfy $T(\mathbf{l},\xi_i)=T(\mathbf{n},\xi_i)=0$ on $\mathcal{M}$, which will guarantee $\Phi_{10}=\Phi_{21}=0$.

(iv) We will formulate the genericity condition on the data for $\{Q^a\}$ on $\mathcal{N}$ in the course of the proof.

(v) The reader may find it helpful, when going through the details of the proof that follows, to carry along and look at the prototype example of a Killing-Cauchy horizon of type I discussed in the Introduction [Eqs. (1.1)–(1.9), and Figs. 1 and 3].

Proof of Theorem 1. By (A3) at least one of the ξ_i, which we can without loss of generality assume to be $\xi_1=\partial/\partial x$, degenerates to a null Killing vector on $\mathcal{S}$ and becomes orthogonal to ξ_2 since the (unique) null direction tangent to $\mathcal{S}=\{u=f\}$ is at the same time orthogonal to all vectors tangent to $\mathcal{S}$. This implies, putting $g_{ij}=g(\xi_i,\xi_j)$,

$$\lim_{u\to f} g_{11}=\lim_{u\to f} g_{12}=0 \; . \qquad (3.6)$$

On the other hand, throughout the open set $\mathcal{W}-\mathcal{S}$ on which strict plane symmetry holds we have $g(\mathbf{m},\mathbf{m})=0$, $g(\mathbf{m},\mathbf{m}^*)=1$ which reads

$$\frac{1}{F^2}g_{11}+\frac{2}{FG}g_{12}+\frac{1}{G^2}g_{22}=0 \; , \qquad (3.7a)$$

$$\frac{1}{F^*F}g_{11}+\left[\frac{1}{FG^*}+\frac{1}{F^*G}\right]g_{12}+\frac{1}{GG^*}g_{22}=1 \; . \qquad (3.7b)$$

Then, at least one of $\lim_{u\to f}F$ or $\lim_{u\to f}G$ has to vanish since otherwise by Eq. (3.7a) $\lim_{u\to f}g_{22}=0$ and it is impossible to satisfy Eq. (3.7b) in a neighborhood of $\mathcal{S}$ since $\lim_{u\to f}g_{11}=\lim_{u\to f}g_{12}=0$ by Eq. (3.6). Since by (A4) $g(\xi_i,\xi_i)$ are analytic functions in a neighborhood of $\mathcal{S}$, it is clear that F and G are regular in a neighborhood of $\mathcal{S}=\{u=f\}$, and hence by (A4) and (A3) (namely, that the Killing-Cauchy horizon $\mathcal{S}$ is of type I), we can express them as convergent power series in $(u-f)$ in a neighborhood of $\mathcal{S}$:

$$F(u,v)=\sum_{n=k}^{\infty}\alpha_n(v)(u-f)^n \; ,$$
$$G(u,v)=\sum_{n=l}^{\infty}\beta_n(v)(u-f)^n \; , \qquad (3.8)$$

where $k\geq 0, l\geq 0$ and $k+l\geq 1$; $\alpha_n(v)$, $\beta_n(v)$ are (not

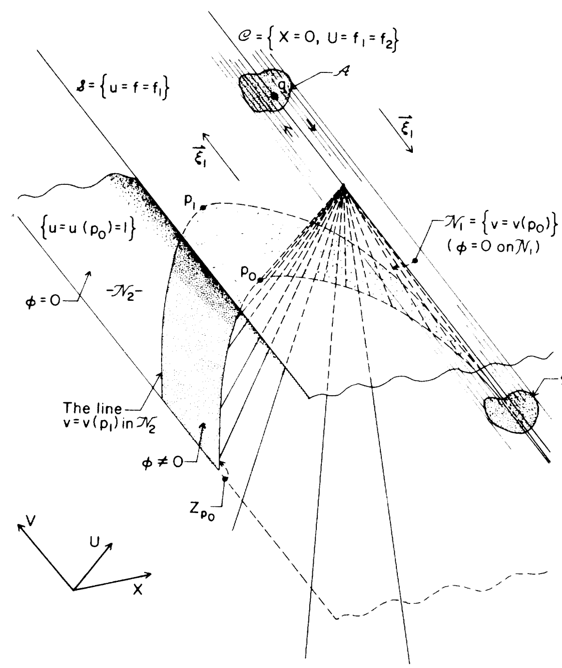

FIG. 3. The initial-value problem of Theorem 1 illustrated by the example of a plane-sandwich-wave spacetime with a type-I Killing-Cauchy horizon $\mathcal{S}$ [Eqs. (1.1)–(1.5) and Fig. 1]. As in Fig. 1, the case $f_1=f_2$ is depicted with the Y direction suppressed. The initial null boundary $\mathcal{N}$ consists of $\mathcal{N}_2$: the part of the null surface $\{u=u(p_0)=1\}$ lying above $\{v=v(p_0)\}$; and of $\mathcal{N}_1$: the piece of the null cone $\{v=v(p_0)=\text{const}\}$ lying above the surface $\{u=1\}$ with the exception of the single generator of this cone which lies in $\mathcal{S}$. $\mathcal{N}_1$ and $\mathcal{N}_2$ intersect on the spacelike two-surface Z_{p_0}. The initial data for the plane-symmetric scalar field ϕ are zero on $\mathcal{N}$ except on the dotted strip in $\mathcal{N}_2$ lying between Z_{p_0} and the line (two-surface) $v=v(p_1)$. If these data are generic, ϕ will be nonzero at some point q on the line $\mathcal{C}=\{X=0\}$ lying in the Killing-Cauchy horizon $\mathcal{S}=\{u=f_1\}$. If ϕ is smooth near $\mathcal{S}$, there will be an open neighborhood $\mathcal{A}$ in $\mathcal{S}$ around q where $\phi\neq 0$. Since the Killing vector ξ_1 is null and nonzero on $\mathcal{S}$ outside the line $\mathcal{C}$, it will transport this neighborhood $\mathcal{A}$ onto an infinite strip in $\mathcal{S}$ around $\mathcal{C}$ on which $\phi\neq 0$. Sufficiently far in the past, this strip will be neighboring the single null generator of the null cone $\{v=v(p_0)\}$ along $\mathcal{C}$ that does not belong to $\mathcal{N}_1$. In that region (labeled $\mathcal{B}$ in the figure), any neighborhood of this strip in spacetime will contain points that do not belong to either $J^+(\mathcal{N}_2)$ or $J^-(\mathcal{N}_2)$ and a smooth ϕ will therefore be incompatible with causal evolution.

necessarily analytic) complex functions with $\alpha_k(v)\neq 0$, $\beta_l(v)\neq 0$. Inserting Eqs. (3.8) into Eq. (3.5) we obtain that the asymptotic behavior of ρ to leading order as $u\rightarrow f$ is given by

$$\rho\sim\frac{k+l}{u-f}\quad(u\rightarrow f), \tag{3.9}$$

where $k+l\geq 1$.

Now consider a point $p_0\in I^-(\mathcal{S})$ lying in the region of strict plane symmetry $\mathcal{W}-\mathcal{S}$ with $u(p_0)<f$, and consider the two-surface Z_{p_0} obtained by sweeping the point p_0 with the Killing symmetry generators ξ_i; i.e., let Z_{p_0} be the Killing orbit of p_0. (See Fig. 3.) Clearly, the null geodesic generators of $\dot{J}^+(Z_{p_0})$ which have their past end points on Z_{p_0} will consist of those in the l direction on which $v=v(p_0)$, and those in the $\mathbf{n}$ direction on which $u=u(p_0)$. Since $R\neq 0$ on $\mathcal{S}=\{u=f\}$, the tangent vectors to the null geodesic generators of $\dot{J}^+(Z_{p_0})$ in the l direction which lie in the surface $\{v=v(p_0)\}$ and which are given by Rl have convergence $\hat{\rho}=R\rho$ which by Eq. (3.9) diverges to $-\infty$ as $u\rightarrow f$. This guarantees[8] that every null geodesic generator of $\dot{J}^+(Z_{p_0})$ having its past end point on Z_{p_0} has a conjugate point to Z_{p_0} along itself on the surface $\mathcal{S}$. We now claim that this actually corresponds to the null generators of the null surface $\{v=v(p_0)\}$ converging and intersecting each other in caustics on the Cauchy horizon $\mathcal{S}$. To see this, note that outside $\mathcal{S}$ the Killing vectors ξ_i generate translations on the set of null generators of the surface $\{v=v(p_0)\}$ by generating symmetries on their past end points in Z_{p_0}. On the other hand, if the null surface $\{v=v(p_0)\}$ intersects the null surface $\mathcal{S}$ transversally (i.e., not tangentially), then the intersection has to be a spacelike two-surface. But this is impossible since on $\mathcal{S}$ there does not exist a pair of spacelike linearly independent Killing vectors to generate translations on the set of null generators of $\{v=v(p_0)\}$ in this spacelike two-surface. Hence $\{v=v(p_0)\}$ intersects $\mathcal{S}$ nontransversally and as the convergence $\hat{\rho}$ of its generators diverges on $\mathcal{S}$, the intersection takes place either on a spacelike curve tangent to the Killing vector ξ_2 which is still spacelike on $\mathcal{S}$, in the case that only one of ξ_i (namely, ξ_1) becomes null; or on a single point, in the case that both ξ_1 and ξ_2 become null on $\mathcal{S}$ (Fig. 3). Note that in the first case, when ξ_2 is still spacelike on $\mathcal{S}$, it generates translations on the set of generators of $\{v=v(p_0)\}$ along the curve in $\mathcal{S}$ on which these null generators converge and intersect each other, while the vector ξ_1 which is null on $\mathcal{S}$ has to vanish on this curve. In the second case (the case depicted in Figs. 1 and 3) both ξ_1 and ξ_2 have to vanish at the point in $\mathcal{S}$ on which the generators of $\{v=v(p_0)\}$ intersect each other, since they must not generate any translations on the set of these generators at that point.

Therefore there is a null two-surface $\mathcal{P}$ (a null curve $\mathcal{C}$) in $\mathcal{S}$ which is the union of all spacelike curves (points) in $\mathcal{S}$ on which generators of the surfaces $\{v=v_0\}$ converge as v_0 ranges from $-\infty$ to $+\infty$, in the case ξ_2 is spacelike on $\mathcal{S}$ (in the case both ξ_1,ξ_2 are null on $\mathcal{S}$). Moreover, this two-surface $\mathcal{P}$ (curve $\mathcal{C}$) is generated by the past endless null generators of $\dot{J}^+(Z_{p_0})$. (This can be seen by noting that the local chart (u,v,x,y) is regular on $I^-(\mathcal{S})$, thus all points in $I^-(\mathcal{S})$ with $v<v(p_0)$ are outside $J^+(Z_{p_0})$ and therefore, as $\dot{J}^+(Z_{p_0})$ is edgeless,[8,14] $\mathcal{P}$ ($\mathcal{C}$) must be generated—in $\overline{J^-[\{v=v(p_0)\}]}$—by null geodesic generators of $\dot{J}^+(Z_{p_0})$ along $\mathcal{S}$ which are past endless and which intersect the generators of $\{v=v(p_0)\}$ at their focal points on $\mathcal{S}$.) The Killing vector ξ_1 (both ξ_1 and ξ_2) vanishes on this surface $\mathcal{P}$ (curve $\mathcal{C}$) and since by (A3) and (A4) ξ_1 (ξ_1,ξ_2) is a null vector not identically vanishing on $\mathcal{S}$ whose components in some coordinate frame are analytic functions, it has to be nonzero outside $\mathcal{P}$ ($\mathcal{C}$) on any open neighborhood in $\mathcal{S}$ of $\mathcal{P}$ ($\mathcal{C}$), generating symmetries along the null generators of $\mathcal{S}$.

We now show that it is sufficient to prove the theorem only for the case where $\{Q^a\}$ is a single scalar field ϕ. Let $\partial_\mu,\mu=1,2,3,4$, denote, respectively, the local coordinate basis fields $\partial/\partial u$, $\partial/\partial v$, $\partial/\partial x$, $\partial/\partial y$. Then for an arbitrary multicomponent (contravariant) tensor field $\{Q^a\}$, the Lie derivative along ξ_i of the inner product of Q with the $(p,0)$ tensor basis elements is given by

$$\xi_i[g(Q,\,\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_p})]=g\left[Q,\,\sum_{k=1}^{p}\partial_{\mu_1}\otimes\cdots\otimes\mathcal{L}_{\xi_i}\,\partial_{\mu_k}\otimes\cdots\otimes\partial_{\mu_p}\right]+g(\mathcal{L}_{\xi_i}\,Q,\,\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_p})\,,$$

where we have made use of the fact that ξ_i are Killing vectors hence $\mathcal{L}_{\xi_i}g\equiv 0$. But the first term is zero as $\mathcal{L}_{\xi_i}\partial_\mu=[\xi_i,\partial_\mu]\equiv 0$, thus

$$\xi_i[g(Q,\,\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_p})]=g(\mathcal{L}_{\xi_i}\,Q,\,\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_p})\,.$$

This equation tells us that each component of a multicomponent tensor field $\{Q^a\}$ in the basis frame field $(\partial_u,\partial_v,\partial_x,\partial_y)$ behaves exactly like a scalar field under Lie transport by ξ_i since (as $\xi_i=\partial/\partial x^i$ are Killing vectors) $\xi_i[g(\partial_\mu,\,\partial_\nu)]\equiv 0$. For spinor fields, by the same argument, the components of an arbitrary spinor field in the spin basis corresponding to the null tetrad (3.1) or (3.4) will behave like scalar fields under Lie transport by the ξ_i. Clearly, this is also true for the components of the arbitrary tensor or spinor field in any local basis field that is Lie parallel along the ξ_i, or in the spin basis that corresponds to any null tetrad that is Lie parallel along the ξ_i. Therefore, despite the obvious fact that these basis fields themselves will in general develop singularities on the Killing-Cauchy horizon $\mathcal{S}$, precisely the following arguments by which we prove the singularity result of the theorem for a scalar field ϕ will prove the same result for an arbitrary field $\{Q^a\}$ (after constructing a suitable basis field Lie parallel along the ξ_i for each such field $\{Q^a\}$) when the initial data satisfy the conditions of the theorem.

Now consider a characteristic initial-value problem for the scalar field ϕ (Fig. 3) in which the initial null boundary is given by $\mathcal{N}=\mathcal{N}_1\cup\mathcal{N}_2$, $\mathcal{N}_2=\{u=u(p_0)<f\}$, $\mathcal{N}_1=\{v=v(p_0)\}$, $\mathcal{N}_1\cap\mathcal{N}_2=Z_{p_0}$, and the initial data have the form $\phi\equiv 0$ on $\mathcal{N}_1$, and $\phi=\phi(v)$, generic, nonzero, plane-symmetric ("sandwich") data on $\mathcal{N}_2$ vanishing for $v\geq v(p_1)>v(p_0)$ and for $v\leq v(p_0)$, and satisfying the constraint equations (when there are any). The well posedness of this problem is clear from conditions (a), (b), (c) of the theorem. By condition (c), the evolution in $D^+(\mathcal{N})$ will have the full Killing symmetries: $\mathcal{L}_{\xi_i}\phi=\xi_i(\phi)\equiv 0$ throughout spacetime. We formulate the following notion of genericity for the data on $\mathcal{N}$.

Initial data for ϕ on $\mathcal{N}$ of the above class are generic if the solution is nonzero somewhere on the surface $\mathcal{P}$ (curve $\mathcal{C}$) in $\mathcal{S}$. For a multicomponent field we similarly demand that the solutions evolving from generic initial data take nonzero tensor (or spinor) values at some points of the surface $\mathcal{P}$ (curve $\mathcal{C}$) on $\mathcal{S}$. Note that, by "the solution on $\mathcal{P}$ ($\mathcal{C}$) in $\mathcal{S}$" we mean the limit of the solution on $I^-(\mathcal{S})$ as the field point approaches the plane $\mathcal{P}$ (the curve $\mathcal{C}$) lying in $\mathcal{S}$. Hence, more precisely, initial data for ϕ are generic if either this limit does not exist or it exists and is nonzero somewhere in $\mathcal{P}$ ($\mathcal{C}$) on $\mathcal{S}$. If the limit does not exist, then the field ϕ is singular near the horizon $\mathcal{S}$ and the theorem is proved.

Now let us assume that this limit does exist and the field ϕ obtained by evolving the above data on $\mathcal{N}$ is smooth in a neighborhood of $\mathcal{S}=\{u=f\}$. (This assumption will produce a contradiction thereby implying that ϕ cannot be smooth—the conclusion of our theorem.) Then, since ϕ is smooth and not identically zero on $\mathcal{P}$ ($\mathcal{C}$), it will be nonzero on some open subset in $\mathcal{S}$ intersecting $\mathcal{P}$ ($\mathcal{C}$) in the region on which $\phi\neq 0$. But as the Killing vector ξ_1 (ξ_1,ξ_2) generates symmetries along the null generators of $\mathcal{S}$ everywhere near $\mathcal{P}$ ($\mathcal{C}$) except on $\mathcal{P}$ ($\mathcal{C}$) itself, and since $\xi_1(\phi)\equiv 0$ [$\xi_i(\phi)\equiv 0$] on $\mathcal{S}$ as this holds prior to $\mathcal{S}$ and ϕ and ξ_1 (ξ_i) are smooth, ξ_1 (ξ_i) will carry this region on which ϕ is nonzero arbitrarily down into the past along the generators of $\mathcal{S}$. But when we move a sufficiently large affine distance into the past along these generators we clearly enter the region $J^-[\{v=v(p_0)\}]$ in which the generators along $\mathcal{P}$ ($\mathcal{C}$) are past endless generators of $\dot{J}^+(Z_{p_0})$ and hence of $\dot{J}^+(\mathcal{N}_2)$. Therefore any neighborhood in $\mathcal{M}$ of $\mathcal{P}$ ($\mathcal{C}$) in this region intersects a piece of $\mathcal{M}$ not contained in $J^{\pm}(\mathcal{N}_2)$ (Fig. 3). But again by the smoothness of ϕ and as $\xi_1(\phi)\equiv 0$ [$\xi_i(\phi)\equiv 0$], ϕ will be nonzero at all points of $\mathcal{P}$ ($\mathcal{C}$) in this region and thereby be nonzero in a neighborhood in $\mathcal{M}$ of any point of $\mathcal{P}$ ($\mathcal{C}$) there, contradicting condition (b) of the theorem. Thus the assumption that ϕ is smooth near $\mathcal{S}=\{u=f\}$ is contradictory and must be false, and the field ϕ must develop singularities on $\mathcal{S}$ proving the theorem. □

The singularity of ϕ on $\mathcal{S}$ will in most cases be of the form $\phi\neq 0$ on $\mathcal{P}$ ($\mathcal{C}$) for $v(p_0)<v<v(p_1)$ (bounded or unbounded) whereas $\phi\equiv 0$ on $\mathcal{S}$ outside $\mathcal{P}$ ($\mathcal{C}$), with possibly an added smooth background field on $\mathcal{S}$ which satisfies $\phi^B(p)=0$ $\forall p\in\mathcal{P}$ ($\mathcal{C}$). Thus even though the field itself might be bounded near $\mathcal{S}$, some of its derivatives will diverge on the two-surface $\mathcal{P}$ (curve $\mathcal{C}$) in $\mathcal{S}$. However, if the field equations are linear, exactly the same argument we will use in proving Theorem 2 will imply (as ξ_1 or ξ_1,ξ_2 vanish on the surface $\mathcal{P}$ or the curve $\mathcal{C}$ in $\mathcal{S}$) that ϕ actually diverges on the set $\mathcal{P}$ (or on $\mathcal{C}$) in $\mathcal{S}$.

IV. INSTABILITY OF HORIZONS OF TYPE II

Theorem 2. Let $(\mathcal{M},g)$ be a plane-symmetric spacetime with a Killing-Cauchy horizon $\mathcal{S}$ of type II where strict plane symmetry holds on the intersection $\mathcal{W}$ of a neighborhood of $\mathcal{S}$ with $I^-(\mathcal{S})$. Let $\{Q^a\}$ denote a field satisfying an arbitrary set of evolution equations such that (a) the equations are linear, (b) there is a consistent (noncharacteristic) initial-value formalism for the field $\{Q^a\}$ and the evolution equations it satisfies, with local existence and uniqueness holding for both the general and the plane-symmetric initial-value problems.

If these conditions are satisfied, then there exists a spacelike partial Cauchy surface Σ in $I^-(\mathcal{S})$ such that the evolution of any generic, plane-symmetric initial data for $\{Q^a\}$ on Σ results in singularities on the Killing-Cauchy horizon $\mathcal{S}$.

Remarks

(i) As will be clear from the proof, the assumptions of the theorem need only hold on $I^-(\mathcal{S})\cup\mathcal{S}$ in $\mathcal{M}$.

(ii) The condition of genericity for the initial data on Σ will be formulated in the proof.

(iii) When studying the proof the reader may find it helpful to carry along and look at the prototype example of a type-II horizon discussed in the Introduction [Eqs. (1.10) – (1.16)].

Proof of Theorem 2. We can set up the canonical local tetrad (3.1) on $(\mathcal{M},g)$ in which the metric will be of the form

$$g=-\frac{1}{R(u,v)}du\,dv+A(u,v)du^2+B(u,v)dv^2+M^2(u,v)dx^2+N^2(u,v)dy^2+K(u,v)dx\,dy+L_i(u,v)du\,dx^i+J_i(u,v)dv\,dx^i\,, \tag{4.1}$$

where $R(u,v)$ is positive, bounded, and nonzero on $\mathcal{S}$. Put ${}^{(2)}g=-du\,dv+RA\,du^2+RB\,dv^2$. Find local functions $t(u,v)$, $z(u,v)$ such that $t=0$ on $\mathcal{S}$ and

$${}^{(2)}g=P(dz^2-dt^2)\,, \tag{4.2}$$

where $P(>0)$ is the conformal factor. [This can be done, for example, by solving the initial-value problem $\{t=0$ on $\mathcal{S}$, ${}^{(2)}\Box t=0\}$ which in general has nonunique solutions, and then finding a "conjugate" $z(u,v)$ such that (4.2) is satisfied.[15]] Then the metric (4.1) becomes

$$g=-\frac{1}{\hat{R}(t,z)}(dt^2-dz^2)+F^2(t,z)dx^2+G^2(t,z)dy^2+K(t,z)dx\,dy+\hat{L}_i(t,z)dt\,dx^i+\hat{J}_i(t,z)dz\,dx^i\,, \tag{4.3}$$

where $\hat{R}(t,z)$ is again positive and bounded on $\mathcal{S}$. In both coordinate systems (4.1) and (4.3) the Killing vectors are $\xi_i = \partial/\partial x^i$. Our definition of t guarantees that $\mathscr{S}$ is given by $\{t=0\}$; and by choosing ξ_1 to be the Killing vector that becomes null and tangent to $\mathscr{S}$ on $\mathscr{S}$, we find that $F(t=0,z)=0$. Note that $\nabla t = -\hat{R}\,\partial/\partial t$ is a timelike vector field which blows up on $\mathscr{S}$ while at the same time becoming tangent to $\mathscr{S}$. It will not be necessary in what follows to fix the coordinate (gauge) freedom further than that of Eq.(4.3).

Now we claim that since $\mathscr{S}$ is a Killing-Cauchy horizon of type II, some null generators of $\mathscr{S}$ must have future end points on $\mathscr{S}$. Null generators of $\mathscr{S}$ by definition have no past end points; if they do not have any future end points either, then one can globally express $\mathscr{S}$ in the form $\{f(u,v)=\text{const}\}$ where ∇f is perfectly smooth and everywhere nonzero on $\mathscr{S}$, contradicting our assumption that $\mathscr{S}$ is of type II. To see this, assume null generators of $\mathscr{S}$ have no end points. Take a spacelike two-dimensional section Z of $\mathscr{S}$, and take a smooth field of (spacelike) basis fields $\mathbf{k}_1$, $\mathbf{k}_2$ on Z in $\mathscr{S}$. (This can be done since by plane symmetry the spacelike sections of $\mathscr{S}$ will not have spherical topology.) Propagate $\mathbf{k}_i$ along null generators $\mathbf{l}$ of $\mathscr{S}$ by parallel transport to all of $\mathscr{S}$. Since generators of $\mathscr{S}$ are both past and future complete in $\mathcal{M}$ by definition, $\mathbf{k}_i$ will be smooth on $\mathscr{S}$ and will have smooth extensions to a neighborhood of $\mathscr{S}$ in $\mathcal{M}$. Construct a null vector field $\mathbf{n}$ on $\mathscr{S}$ satisfying $g(\mathbf{n},\mathbf{k}_i)=0$, $g(\mathbf{n},\mathbf{l})=-1$. $\mathbf{n}$ will be a smooth vector field on $\mathscr{S}$ and will have a smooth extension (as a null vector) to a neighborhood of $\mathscr{S}$ in $\mathcal{M}$. Then take f to be the affine parameter along geodesics in the $\mathbf{n}$ direction, so that $\mathscr{S}=\{f=0\}$ and ∇f on $\mathscr{S}$ is equal to $-\mathbf{l}$ and hence is smooth, null, and everywhere nonzero on $\mathscr{S}$. By choosing $\mathbf{l}$, hence $\mathbf{n}$ and the (now not necessarily affine) parameter f such that f is constant on a family of parallel null surfaces near $\mathscr{S}$, ∇f will retain these properties over a neighborhood of $\mathscr{S}$ in $\mathcal{M}$. Finally, by the same argument as we gave just before the statement of Theorem 1, f can be chosen to be a function of only u and v.

Therefore, there is a nonempty subset $\mathscr{C}$ of $\mathscr{S}$ which consists of the end points of null generators of $\mathscr{S}$. (As $\mathscr{S}$ is achronal and edgeless it is a closed set and must contain these end points.) Now our Killing field ξ_1 becomes null and tangent to $\mathscr{S}$ on $\mathscr{S}$, pointing along its null generators. But since $\mathscr{S}$ is a Killing horizon, the convergence and shear of its null geodesic generators must identically vanish on $\mathscr{S}$, and since $\mathscr{S}$ has no edge[8,14] the only way these generators can have end points on $\mathscr{S}$ is by intersecting other non-neighboring geodesic generators. Therefore at any point in $\mathscr{C}$, there are at least two distinct null directions pointing to the past along two distinct generators of $\mathscr{S}$. Then, as ξ_1 is smooth and parallel to these generators on $\mathscr{S}$, it has to vanish at all points in $\mathscr{C}\subset\mathscr{S}$. (This is also expected because the set $\mathscr{C}$ represents an isolated set of points with a special geometric property that would be left invariant under the action of ξ_1 if it were nonzero on $\mathscr{C}$.) Thus, we have a nonempty subset $\mathscr{C}$ of $\mathscr{S}$ on which the Killing field ξ_1 vanishes (that is, $\mathscr{C}$ is the bifurcation set for the Killing horizon $\mathscr{S}$).

We now note that, as before we only need to prove the theorem in the case $\{Q^a\}$ is a scalar field ϕ. As each component of a multi-index field $\{Q^a\}$ in the basis frame field $(\partial_t,\partial_z,\partial_x,\partial_y)$ or in the spin basis corresponding to the tetrad (3.1) (and similarly in any local basis field Lie parallel along the ξ_i or in the spin basis corresponding to any null tetrad Lie parallel along the ξ_i so that the argument we gave in the proof of Theorem 1 applies without modification) behaves like a scalar field under Lie transport by ξ_i, exactly the same arguments which prove the singularity of ϕ on $\mathscr{S}$ will prove the singularity of an arbitrary field $\{Q^a\}$ (by constructing a suitable basis field Lie parallel along the ξ_i for each such field $\{Q^a\}$) when the initial data satisfy the conditions of the theorem.

Now consider the spacelike partial Cauchy surface $\Sigma=\{t=-c\}$ in $I^-(\mathscr{S})$ where $c>0$ is sufficiently small so that Σ lies within the region of strict plane symmetry $\mathcal{W}$ (Fig. 4). Since $\mathscr{S}$ has past endless null generators and $\mathscr{S}=H^+(\Sigma)$, Σ has no edge; i.e., it is infinite in the Killing ξ_i directions.[8] Consider generic, plane-symmetric initial data for our scalar field ϕ on Σ. We will adopt the following notion of genericity.

Plane-symmetric initial data for ϕ on Σ are generic, if we can find an arbitrarily large number L and coordinate values $x=a$, $x=b$ with $b-a=L$ such that if we cutoff the data for ϕ on Σ except on the portion of Σ between $x=a$ and $x=b$ (thereby breaking the plane symmetry), then the solution $\phi^{(L)}$ to the initial-value problem with data $\{\phi^{(L)}=0,\ \dot\phi^{(L)}=0$ *on* Σ, *except on the strip between* $x=a$ *and* $x=b$ *where they are equal to the data of* $\phi\}$ will be nonzero at least on some points of the subset $\mathscr{C}$ on $\mathscr{S}$. [Note that, even though the data for $\phi^{(L)}$ on Σ are

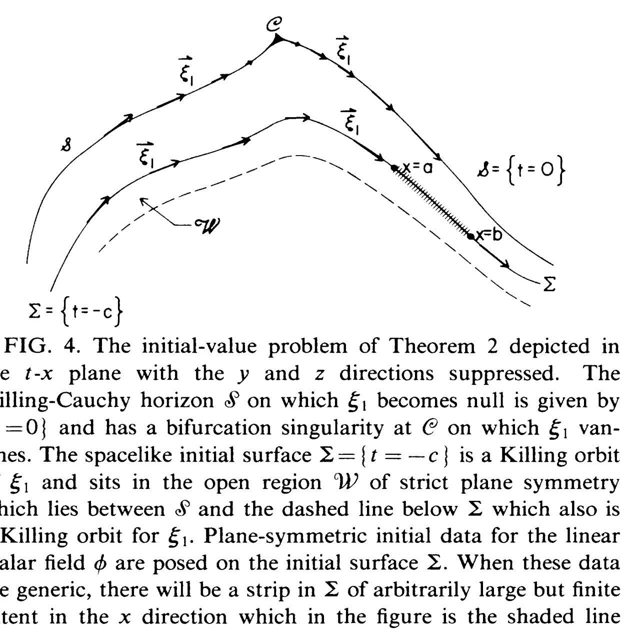

FIG. 4. The initial-value problem of Theorem 2 depicted in the t-x plane with the y and z directions suppressed. The Killing-Cauchy horizon $\mathscr{S}$ on which ξ_1 becomes null is given by $\{t=0\}$ and has a bifurcation singularity at $\mathscr{C}$ on which ξ_1 vanishes. The spacelike initial surface $\Sigma=\{t=-c\}$ is a Killing orbit of ξ_1 and sits in the open region $\mathcal{W}$ of strict plane symmetry which lies between $\mathscr{S}$ and the dashed line below Σ which also is a Killing orbit for ξ_1. Plane-symmetric initial data for the linear scalar field ϕ are posed on the initial surface Σ. When these data are generic, there will be a strip in Σ of arbitrarily large but finite extent in the x direction which in the figure is the shaded line segment lying between the points $x=a$ and $x=b$. This strip has the property that if the initial data on Σ everywhere outside it are replaced with zero, then the solution corresponding to these truncated initial data (which, even though cut off in the x direction, still extend infinitely far in the other Killing y direction) will take nonzero values somewhere on the subset $\mathscr{C}$ of the horizon $\mathscr{S}$.

cut off in the x direction, they still extend infinitely far in the other Killing (y) direction.] In the case of a multicomponent field $\{Q^a\}$, plane-symmetric initial data on Σ are called generic if there is an arbitrarily large $L=b-a$ so that the solution developing from the truncation of these initial data in the manner described above takes nonzero tensor (or spinor) values at some points on the subset $\mathscr{C}$ in $\mathscr{S}$. As before, the values of the solution $\phi^{(L)}$ at the points on the subset $\mathscr{C}$ in $\mathscr{S}$ are defined as the limiting values of the solution on $I^-(\mathscr{S})$ as the field points approach the set $\mathscr{C}$ in $\mathscr{S}$. Again to be more precise, we will call the initial data for ϕ on Σ generic if either this limit does not exist for $\phi^{(L)}$, or it does exist and is nonzero somewhere on the subset $\mathscr{C}$ in $\mathscr{S}$. In the case this limit does not exist, the solution ϕ is clearly singular (and divergent) on the horizon $\mathscr{S}$ and the theorem is proved. Therefore, in the following we will assume that this limit does exist for $\phi^{(L)}$ and takes nonzero values somewhere in the subset $\mathscr{C}$ of $\mathscr{S}$.

But now consider the action of the symmetry group generated by ξ_1, given by $\mathcal{G}_L$: $(x,y,z,t) \rightarrow (x+L,y,z,t)$. By assumption (b) of the theorem, if we Lie transport the initial data truncated in the manner of the preceding paragraph with the Killing vector field ξ_1, then the solution will be Lie transported by ξ_1. But ξ_1 vanishes on $\mathscr{C}$ and $\mathcal{L}_{\xi_1}\phi=\xi_1(\phi)$; therefore the action of ξ_1 leaves the value of $\phi^{(L)}$ on $\mathscr{C}$ invariant. However, by the linearity of the field equations, the solution for the original plane-symmetric initial data will be

$$\phi=\sum_{n=-\infty}^{\infty} \mathcal{G}_L{}^n\,(\phi^{(L)})\ ;$$

hence on $\mathscr{C}$, since $\mathcal{G}_L\,(\phi^{(L)})\,(\mathscr{C})=\phi^{(L)}\,(\mathscr{C})$,

$$\phi(\mathscr{C})=\sum_{n=-\infty}^{\infty}\phi^{(L)}\,(\mathscr{C})=\phi^{(L)}\,(\mathscr{C})\left[\sum_{n=-\infty}^{\infty}1\right],$$

and thus ϕ diverges on $\mathscr{C}$ as $\phi^{(L)}\,(\mathscr{C})\neq 0$ by genericity; and the theorem is proved. □

V. CONCLUSIONS

We have shown the instability of Killing-Cauchy horizons in plane-symmetric spacetimes to arbitrary plane-symmetric perturbations satisfying reasonable genericity conditions. Although it remains to be shown that our genericity criteria follow, under suitable restrictions, from the more general and standard notions of genericity employed by mathematicians,[16] it seems intuitively clear to us that they agree quite naturally with a physicist's notion of genericity. Accepting this, then it is clear that if initial data whose evolution is a plane-symmetric spacetime containing a Killing-Cauchy horizon are slightly perturbed in some "generic" plane-symmetric direction, the horizon will be destroyed. Therefore, we conclude that the type-II Killing-Cauchy horizons present in the new Chandrasekhar-Xanthopoulos solutions[3] of colliding plane-wave spacetimes are probably isolated features and will not be present in a generic colliding plane wave solution. This conclusion, as was mentioned in the Introduction, is in accord with the simultaneous and independent work by Chandrasekhar and Xanthopoulos showing that null dust or a fluid with pressure = (energy density), when inserted into their spacetime, destroys the horizon.

It is intriguing to note that, despite this nongeneric horizon behavior, the Chandrasekhar-Xanthopoulos solutions are more general than the previously known exact solutions for colliding plane waves with parallel polarizations—more general in the same sense as the Kerr solution is more general than the Schwarzschild solution. Nevertheless, the previously known exact solutions for colliding plane waves possess the generic plane-symmetric causal structure (no Killing-Cauchy horizons), while the Chandrasekhar-Xanthopoulos solutions do not.

It is also interesting to note that, the occurrence of timelike singularities in a plane-symmetric spacetime would imply the existence of a Killing-Cauchy horizon if in the vicinity of such a singularity at least one of the Killing vectors which generate plane symmetry becomes timelike. Even though this is the case for the presently known solutions[3] with timelike singularities, a satisfactory argument to the effect that in any plane-symmetric spacetime with sufficiently "strong" timelike curvature singularities[17] at least one of the plane-symmetry generating Killing vectors must be timelike near the singularity is unavailable to the author. If such an argument could be provided (possibly with some weak assumption of genericity, e.g., under the restriction that the spacetime has no Killing symmetries other than plane symmetry), then the results of the present paper would indicate that the singularities in a "generic," plane-symmetric spacetime cannot be timelike (in the sense of Penrose[18]); and this would constitute an interesting verification of the cosmic censorship hypothesis[18,17] in the restricted domain of plane-symmetric spacetimes. A (possibly) stronger result which would be sufficient to reach this last conclusion rigorously would be the formulation and proof of a theorem to the effect that whenever the evolution of "generic," plane-symmetric Cauchy data for the gravitational and matter fields on an initial surface Σ results in the formation of a Cauchy horizon $\mathscr{S}$ for Σ, $\mathscr{S}$ is also a Killing horizon for at least one of the plane-symmetry generating Killing vectors on $D^+(\Sigma)$.

ACKNOWLEDGMENTS

The author wishes to thank Kip S. Thorne for a critical reading of the first draft of this paper and for guidance in its subsequent revision. This work was supported by the National Science Foundation under Grant No. AST 85-14126.

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