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Nonlinear saturation of the longitudinal modes of the coasting beam in a storage ring

S. A. Bogacz and K-Y Ng
Accelerator Theory Department, Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510
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A simple nonlinear model of a coasting beam coupled to a sharp storage-ring impedance is formulated in the framework of the quasilinear Vlasov equation. Nonperturbative analytic treatment of the Vlasov equation allows us to study time evolution of a single coherent mode (an azimuthal harmonic of the density driven by the impedance) and the overall uniform-density distribution function. In the case of a Gaussian beam, this formalism simplifies to a pair of equations of motion which together with the dispersion relation fully describe the dynamics of the beam. Further numerical treatment reveals saturation of the mode growth which simultaneously provides a stabilizing mechanism (via Landau damping) for the overall distribution function. Some predictions about the energy overshoot and coherent-instability lifetime are made on the basis of the presented formalism.

INTRODUCTION

Various formalisms based on the linearized Vlasov equation are very useful in studying the stability of coherent modes arising in the beam due to its interaction with the self-field induced by the beam environment. Furthermore, the linear approach gives the correct analytic description of short-time evolution of coherent instabilities, e.g., in terms of the initial growth rate. However, this quantity fails to characterize longer time scales, i.e., when the growing instability can no longer be considered as a small fluctuation of the overall particle distribution in the beam. In order to go beyond short-time-evolution studies of collective modes, one has to develop a nonlinear description of the beam dynamics.

Following the arguments of Chin and Yokoya,[1] when the initial amplitude of the coherent mode is small and the instability does not develop too rapidly, one can assume that the nonlinearity modifies the particle distribution at a rate much smaller than the linear response of the system. Under this adiabaticity assumption one can formulate an instantaneous dispersion relation,[2] similar to the one employed in the linear theories.

Here we apply a *nonperturbative* approach to the Vlasov equation, describing an initially uniform Gaussian beam coupled to a sharp model impedance. The resulting formalism allows us to study the long-time behavior of driven coherent modes, their saturation due to the increasing Landau damping mechanism, and, finally, how they modify the uniform part of the density distribution. Some predictions about the energy-overshoot law and coherent-instability lifetime are made on the basis of analytically derived equations of motion.

THEORETICAL APPROACH

Consider an initially uniform distribution of particles inside a storage ring modeled by the following statistical density distribution function defined in the classical phase space as

$$f(\epsilon,\theta,t)=f^0(\epsilon,t)+\sum_{n\neq 0} h_n(\epsilon,t)e^{i\theta n} , \tag{1}$$

where θ is the azimuthal angle around the ring circumference and ϵ represents the energy deviation from its synchronous value E_0. Fourier series representation of the nonuniform part guarantees periodicity of the distribution, while the condition

$$h_{-n}(\epsilon,t)=h_n^*(\epsilon,t) , \tag{2}$$

assures that our distribution function defined by Eq. (1) is a real quantity. The Vlasov kinetic equation which governs $f(\epsilon,\theta,t)$ can be written as

$$\frac{\partial}{\partial t}f(\epsilon,\theta,t)+\omega\frac{\partial}{\partial\theta}f(\epsilon,\theta,t)+\dot\epsilon\frac{\partial}{\partial\epsilon}f(\epsilon,\theta t)=0 . \tag{3}$$

Here $\omega=\omega_0+k_0\epsilon$ and $k_0=-\eta\beta^{-2}\omega_0/E_0$ (revolution frequency dispersion, $\eta<0$ below the transition).

The beam environment is modeled by the wake-field impedance of the storage ring represented in frequency domain by $Z(\omega)$. In turn, impedance coupling through the nonuniform current induces an additional potential,[3] changing the energy of the beam by

$$\dot\epsilon=-e\omega_0\sum_{n\neq 0} Z_n\phi_n(t)e^{i\theta n} ,$$

where

$$\phi_n(t)=e\omega_0\frac{1}{2\pi}\int_{-\infty}^{\infty} d\epsilon\, h_n(\epsilon,t) \tag{4}$$

and

$$Z_n=Z(n\omega_0) .$$

We notice in passing that $Z_n^*=Z_{-n}$, since the wake function is real. This, together with Eq. (2), assures that $\dot\epsilon$ is also a real quantity. Substituting Eqs. (1) and (4) into Eq. (3) and using orthogonality of azimuthal plane waves, one can rewrite the Vlasov equation as a set of coupled equations of motion for individual azimuthal harmonics of the distribution function. The resulting equations fully describing the dynamics of the beam–storage-ring system are given by

$$\frac{\partial}{\partial t}f^0(\epsilon,t)-e\omega_0\sum_{n\neq 0}Z_n^*\phi_n^*(t)\frac{\partial}{\partial\epsilon}h_n(\epsilon,t)=0\ , \qquad (5)$$

and

$$\frac{\partial}{\partial t}h_n(\epsilon,t)+in(\omega_0+k_0\epsilon)h_n(\epsilon,t)-e\omega_0Z_n\phi_n(t)\frac{\partial}{\partial\epsilon}f^0(\epsilon,t)-e\omega_0\sum_{m\neq 0}Z_{n-m}\phi_{n-m}(t)\frac{\partial}{\partial\epsilon}h_m(\epsilon,t)=0\ . \qquad (6)$$

The simplest nontrivial storage-ring impedance can be modeled by a harmonic resonator described analytically by the formula

$$Z(\omega)=\frac{R}{1+iQ(\omega/\omega_c-\omega_c/\omega)}\ . \qquad (7)$$

Here, R is the so-called shunt resistance and Q denotes the quality factor of the resonator. The resonant frequency ω_c is tuned to the nth storage-ring mode so that $\omega_c=n\omega_0$. Graphical illustration of our model impedance is given in Fig. 1.

One can notice that for a sharply centered impedance, $Q\gg 1$, the real part of $Z(\omega)$ is peaked around a single harmonic, $n\sim 10^4$, with the imaginary part extending over several neighboring amplitudes, $n,\ldots,n+m$; $m\sim 10$ [even in the $Q\rightarrow\infty$ limit, the imaginary part of $Z(\omega)$ still retains a hyperbolic, $1/\omega$, tail]. This implies that the last term in Eq. (6) would couple pairs of modes h_{n+k} and h_{-k}, where $k=1,2,\ldots,m$. However, we see from Fig. 1 that Z_k is vanishingly small. Therefore modes with low k will not be driven by the impedance which justifies why the coupling term in Eq. (6) can be neglected for our particular impedance choice.

As we mentioned before, one can introduce an instantaneous coherent frequency, $\Omega_n(t)$, describing evolution of the nth mode within a small time interval (t,t') according to the formula

$$h_n(\epsilon,t')=e^{-i\Omega_n(t)(t-t')}h_n(\epsilon,t),\quad t\approx t'\ . \qquad (8)$$

We also require that $f^0(\epsilon,t)$ is a slowly varying function of time compared to rapidly oscillating coherent modes, $h_n(\epsilon,t)$. Therefore a simple adiabatic approximation is made

$$\frac{\partial}{\partial\epsilon}f^0(\epsilon,t)=\frac{\partial}{\partial\epsilon}f^0(\epsilon,t'),\quad t\approx t'\ . \qquad (9)$$

Including both assumptions, Eqs. (8) and (9), one can rewrite Eq. (6) as

$$h_n(t)=(e\omega_0)^2\frac{\partial}{\partial\epsilon}f^0(\epsilon,t)\frac{Z_n}{n\omega-\Omega_n}\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\epsilon' h_n(\epsilon',t)\ . \qquad (10)$$

As was pointed out by Landau,[2] an appropriate integration of Eq. (10) over the entire range of ϵ (including a detour contour extending into the complex ϵ plane) leads to the following dispersion relationship defining the coherent frequency $\Omega_n(t)$:

$$1=(e\omega_0)^2\frac{NZ_n}{2\pi nk_0}\frac{1}{2\pi i}\int_c d\epsilon\frac{\frac{\partial}{\partial\epsilon}g^0(\epsilon,t)}{\epsilon-\xi_n}\ . \qquad (11)$$

Here, $\xi_n=(\Omega_n/n-\omega_0)/k_0$ defines a pole in the complex ϵ plane, while C is the Landau contour[2] of integration chosen so that Ω_n is continuous while crossing the real axis. We also replaced $f^0(\epsilon,t)$ with a normalized to unity distribution function

$$g^0(\epsilon,t)\equiv\frac{2\pi}{N}f^0(\epsilon,t)\ , \qquad (12)$$

where N is the number of particles in the ring.

From here on in, we will confine our discussion to a single harmonic mode h_n driven by the model impedance, Eq. (7), and therefore the index n will be suppressed throughout the rest of the paper ($Z\equiv Z_n$, $h\equiv h_n$, $\xi\equiv\xi_n$). Furthermore, the summation over all modes in Eq. (5) reduces to two terms only (n and $-n$). This, combined with the symmetry condition given by Eq. (3), yields

$$\frac{\partial}{\partial t}g^0(\epsilon,t)-\frac{2\pi}{N}e\omega_0 2\,\mathrm{Re}\left[Z^*\phi^*(t)\frac{\partial}{\partial\epsilon}h(\epsilon,t)\right]=0\ . \qquad (13)$$

We can easily generalize the above result to the case of the coupling impedance extending over several ΔN azimuthal harmonics. Simply replacing the summation over n in Eq. (5) by integration, carrying it out and retaining only the leading order in $\Delta N/N$, one obtains Eq. (13) with Z replaced by $Z\,\Delta N$. The last expression is obviously proportional to the area under the coupling impedance peak which assures the correct scaling of our result.

Now, we make use of the fact that the distribution function $g^0(\epsilon,t)$ is uniquely defined by an infinite set of its moments with respect to ϵ. Introducing

$$G_k(t)\equiv\int_{-\infty}^{\infty}d\epsilon\, g^0(\epsilon,t)\epsilon^k$$

and (14)

$$H_k(t)\equiv\int_{-\infty}^{\infty}d\epsilon\, h(\epsilon,t)\epsilon^k\ ,$$

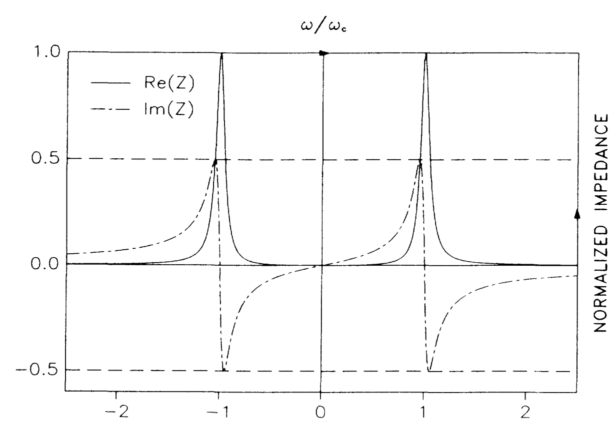

FIG. 1. Model impedance of a harmonic Q cavity ($Q=10$) normalized to unity ($R=1$). The dimensionless frequency is defined by ω/ω_c.

one can rewrite Eq. (13) as the following set of equations of motion for ϵ moments, $G_k(t)$:

$$\frac{\partial}{\partial t}G_0(t)=0,\quad G_0=1\quad \text{(normalization)}\ ,$$
$$\frac{\partial}{\partial t}G_m(t)-\frac{m}{N}(e\omega_0)^2 2\,\mathrm{Re}[Z^* H_0^*(t)H_{m-1}(t)]=0,\quad m\geq 1\ . \tag{15}$$

We observe that integrating Eq. (10) along the Landau contour C leads to the desired recursion formula for H_m:

$$H_{m-1}=(e\omega_0)^2\frac{NZ}{2\pi n k_0}\frac{1}{2\pi i}\int_c d\epsilon\,\epsilon^{m-1}\frac{\frac{\partial}{\partial\epsilon}g^0(\epsilon,t)}{\epsilon-\xi}H_0\ . \tag{16}$$

The dispersion relationship, Eq. (11), applied to the above expression, Eq. (16), after simple algebra and integration by parts yields

$$H_{m-1}=\left\{\xi^{m-1}-(e\omega_0)^2\frac{NZ}{2\pi n k_0}\frac{1}{2\pi i}\times\sum_{k=1}^{m-2}k\xi^{m-k-2}G_{k-1}\right\}H_0\ . \tag{17}$$

Final substitution of Eq. (17) into the equations of motion, Eqs. (15), allows us to rewrite them in a form particularly convenient for further discussion:

$$\frac{\partial}{\partial t}G_1(t)+2\pi e\omega_0 I_0\,|H_0/N|^2 2\,\mathrm{Re}(Z)=0\ ,$$
$$\frac{\partial}{\partial t}G_2(t)+4\pi e\omega_0 I_0\,|H_0/N|^2 2\,\mathrm{Re}(\xi Z^*)=0\ ,$$
$$\frac{\partial}{\partial t}G_3(t)+6\pi e\omega_0 I_0\,|I_0/N|^2 2\,\mathrm{Re}(\xi^2 Z^*)=0\ ,$$
$$\frac{\partial}{\partial t}G_4(t)+8\pi e\omega_0 I_0\,|H_0/N|^2 2\,\mathrm{Re}\left\{\xi^3 Z^*-\frac{\xi e\omega_0 I_0|Z|^2}{2\pi n k_0}\right\}=0\ , \tag{18}$$
$$\ldots,$$
$$\frac{\partial}{\partial t}G_m(t)+2m\pi e\omega_0 I_0\,|H_0/N|^2 2\,\mathrm{Re}\left\{\xi^{m-1}Z^*-\frac{e\omega_0 I_0|Z|^2}{2\pi n k_0}[\xi^{m-3}+2\xi^{m-4}G_1(t)+3\xi^{m-5}G_2(t)+\cdots]\right\}=0\ .$$

Here $I_0=Ne\omega_0/2\pi$ represents the current in the storage ring and $\xi\equiv\xi_n$ is defined implicitly by the dispersion relation, Eq. (11). We also notice that the time evolution of a dimensionless quantity $A(t)\equiv|H_0(t)/N|^2$ is governed by the coherent frequency $\Omega(t)$ through the equation

$$\frac{\partial}{\partial t}A(t)-2\,\mathrm{Im}[\Omega(t)]A(t)=0\ , \tag{19}$$

which is the immediate consequence of Eq. (8).

One can summarize our scheme by realizing that an infinite system of coupled equations, Eqs. (18), together with Eq. (19) and the dispersion relation, Eq. (11), form a closed set of equations which will be used as a starting point for the detailed discussion of the Gaussian beam dynamics in the next section.

GAUSSIAN BEAM DYNAMICS

For the purpose of our model calculation, we will start with a Gaussian beam coasting in a storage ring of impedance given by Eq. (7). We assume that the distribution maintains its initial Gaussian shape with the time-dependent dimensionless parameters $M=M(t)$ and $S=S(t)$:

$$g^0(u,t)=(\alpha/\pi)^{1/2}\exp[-\alpha(u-M)^2]\ , \tag{20}$$

where $\alpha=1/2S^2$ and $u=\epsilon/E_0$.

The above assumption will be further justified by studying the skewness of the distribution. The vanishing of this characteristic and all its higher analogs (to be defined shortly) guarantees that the beam, indeed, retains its initial shape. The obvious identification

$$G_1/E_0=M$$
and
$$G_2/E_0{}^2=S^2+M^2\ , \tag{21}$$

allow us to use the second and third equations of motion, Eqs. (18), to fully describe the evolution of the Gaussian beam. Introducing the convenient dimensionless quantities

$$\mathcal{Z}=eI_0Z/E_0$$
and
$$x\equiv\xi/E_0\ , \tag{22}$$

these equations can be rewritten as

$$\frac{\partial}{\partial t}M(t)+2\pi\omega_0 A(t)2\,\mathrm{Re}(\mathcal{Z})=0$$
and
$$\frac{\partial}{\partial t}S^2(t)+4\pi\omega_0 A(t)2\,\mathrm{Re}[(x-M)\mathcal{Z}^*]=0\ . \tag{23}$$

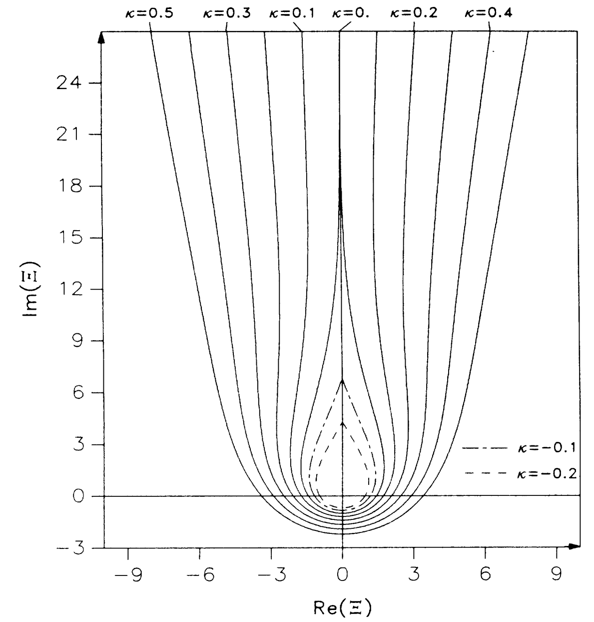

FIG. 2. Stability diagram for a Gaussian beam. The reduced growth rate is defined by $\kappa = \mathrm{Im}(x/S)\sqrt{2}$.

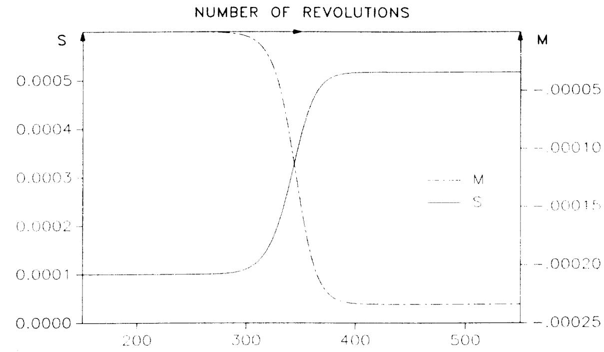

FIG. 4. Saturation of the longitudinal momentum spread and the energy shift caused by nonlinear Landau damping.

The skewness of the distribution is defined as $Q \equiv G_3/E_0{}^3 - M(M^2+3S^2)$, where Q obeys the equation of motion

$$\frac{\partial}{\partial t}Q(t)+6\pi\omega_0 A(t)2\,\mathrm{Re}\{[(x-M)^2-S^2]Z^*\}=0 \qquad (24)$$

obtained directly from Eqs. (18). Obviously for the Gaussian distribution, $Q(t)$ should vanish, which will set the validity probe of our assumption. Similarly, one can introduce the so-called squareness of the distribution

$$K \equiv G_4/E_0{}^4 - 3S^4 - 6M^2S^2 - M^4$$

and the analogous higher-order measures of Gaussian stability to formulate a more refined justification.

Now our problem is reduced to a self-consistent solution of Eqs. (19) and (23) with the coherent frequency x defined implicitly by the integral dispersion formula, Eq. (11). This system of nonlinear differential-integral equations is no longer tractable analytically; nevertheless, its time evolution can be easily iterated numerically assuming the initial condition of our system, $M(t=0)=0$, $S(t=0)=S_0$, and the appropriate time step.

ENERGY OVERSHOOT

As a simple application of our formalism, one can study nonlinear saturation effects contributing to the overshoot of the energy spread. The initial state of the beam is defined so that at $t=0$ the intrinsic nonuniform nth mode is unstable, $\mathrm{Im}(\Omega_n)>0$. To visualize it one can represent the dispersion relation, Eq. (11), in the form of the so-called stability diagram[4] by mapping the coherent frequency into a complex plane of reduced impedance defined by $\Xi \equiv Z/2\pi n S^2$. Figure 2 illustrates a family of dispersion curves of constant growth rate, calculated numerically for a Gaussian beam from Eq. (11). The drop-like curve corresponding to $\kappa=0$ divides the complex Ξ plane into stable (inside the curve) and unstable (outside the curve) regions. The initial condition is chosen for our

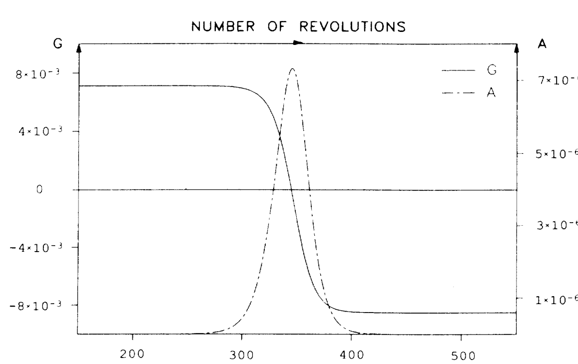

FIG. 3. Nonlinear Landau damping mechanism, single-mode coupling. The growth rate is defined as $G \equiv \mathrm{Im}(\Omega/\omega_0)$.

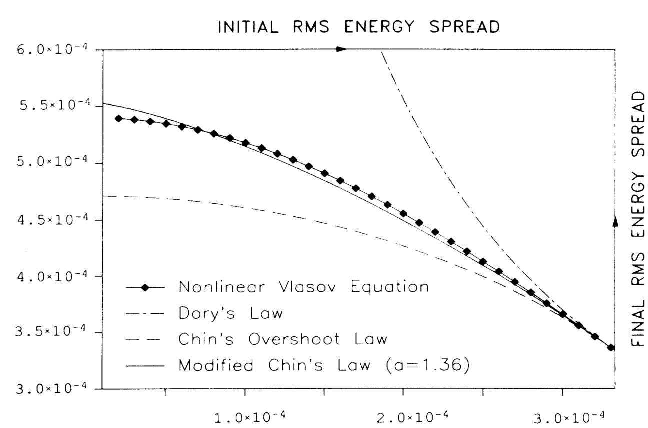

FIG. 5. Nonlinear-Vlasov-equation approach to the energy-overshoot phenomenon. Comparison with the existing results.

calculation as a point on the positive part of the real axis and within the unstable region.

Assumed beam–storage-ring parameters are collected in the table below.

I_0	E_0	Z/n	n
10^{-3} A	100 GeV	10^8 Ω	10^4

The above values fix the instability threshold at $S_{th}=3.33\times10^{-4}$. Obviously, in order to start with an initially growing nonuniform mode, one has to select S_0 below S_{th}. The intrinsic amplitude of the nth mode, A_0, is assigned an arbitrary small value of 10^{-18} which sets the level of nonuniform Schottky noise in the system. The time step is equal to one revolution period and therefore our treatment is equivalent to beam tracking. The result for $S_0=10^{-4}$ is illustrated in Figs. 3 and 4. One can see that the coherent mode of an arbitrarily small amplitude A_0 is growing initially very fast, according to Eq. (19). Its growth, in turn, causes an increase of the energy spread S and a negative shift of the distribution mean value M (energy losses due to the resistive storage-ring impedance) governed by Eqs. (23). This affects the coherent frequency through the dispersion relation, Eq. (11); the new values of S and M correspond to stronger Landau damping which results in a successive decrease of the growth rate $\mathrm{Im}(\Omega)$. Finally, the coherent frequency crosses into the stable region $\mathrm{Im}(\Omega)<0$, which triggers rapid decay of the driving nonuniform coherent mode. This eventually stabilizes all characteristics S, M, and G since the amplitude A goes exponentially back to zero. After saturation, both Eqs. (23) approach asymptotically their stationary solutions S_∞ and M_∞. We notice in passing that the choice of intrinsic small amplitude A_0 has very little influence on the curves presented in Figs. 3 and 4 (as long as $A_0 \ll A_{max}$). Going to smaller values of A_0 does not change the shape of presented beam characteristics, S, M, G, and A; it only shifts them in time (it takes longer for the instability to develop).

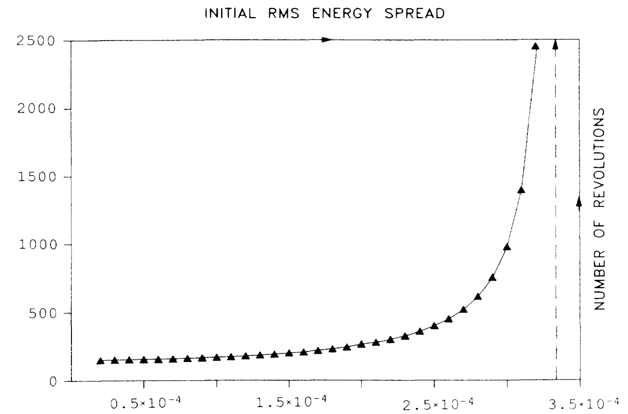

FIG. 6. Coherent-instability lifetime as a function of initial width of the beam.

The correlation between S_0 and S_∞ has been studied before, first by Dory[5] by computer simulations and later by Chin and Yokoya[1] by approximated analytic treatment of the Vlasov equation. Applying our formalism, the stationary values S_∞ were calculated numerically for several values of S_0. The resulting energy-overshoot law is compared with the ones previously formulated by Dory and Chin (Fig. 5). Using the least-squares-fit criterion we realize that by replacing the "square" exponent in the formula by Chin and Yokoya ($S_\infty{}^2+S_0{}^2=2S_{th}{}^2$) by the exponent $a=1.36$, one achieves a good fit to our numerical result.

Finally, the lifetime of coherent instability measured by the width of the A peak is plotted as a function of S_0. The resulting curve, Fig. 6, obviously diverges at $S_0=S_{th}$ since the mode is no longer unstable above the threshold.

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