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Final-photon polarization in the scattering of photons by high-energy electrons

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A general method for calculating the polarization of the outgoing photon beam in any reaction is presented. As an example the method is applied to the high-energy photon beam produced in Compton scattering of a laser beam by a high-energy electron beam. The Stokes parameters of the outgoing photon beam, relative to a unit vector normal to the photon momentum and including their dependence on the polarization of incident photon and electron beams, are obtained explicitly. It is expected that this method will be useful, both in photon production reactions and in the subsequent high-energy photon reactions.

I. INTRODUCTION

High-energy photon beams can be obtained in various ways, but one of the best methods is by backscattering of a laser by a high-energy electron beam. Since the laser and initial electron beams can be controlled to have any polarization, it is useful to consider what polarization the outgoing final photon will have. The polarized high-energy photon beam so produced can be used to investigate the electroweak theories through reactions such as $\gamma e \to W\nu$, $\gamma e \to Ze$, or γN scattering.

Compton scattering has been considered in the initial-electron rest frame when the incident photon beam is polarized linearly,[1,2] circularly,[3] or arbitrarily.[2,4] The polarization of the outgoing photon beam was considered in this frame in Refs. 2 and 4. The high-energy photon beam produced by Compton scattering of a laser beam by a high-energy electron beam was considered recently[5,6] and the final-photon polarization was calculated following the method given in Ref. 2.

In the early experiments the Compton backscattering between a laser beam of a few eV and a medium-energy electron beam was observed.[7,8] Recently[9] a photon beam of about 20 GeV was produced by backscattering 4.68-eV laser light from the SLAC 30-GeV electron beam and the inclusive photoproduction of strange baryons was measured. No doubt there will be higher-energy photon beams from the e^-e^+ colliders such as the Stanford Linear Collider, VLEPP, and CERN's LEP in the future.

The purpose of this paper is to introduce a general method for obtaining the polarization of the final photon beam produced in any reaction and to apply it to Compton scattering. We have obtained a general form for the Stokes parameters describing the polarization of the final photon beam, when the initial photon and electron beams are arbitrarily polarized and the final electron beam polarization is not observed. In the usual way, two of the Stokes parameters are defined relative to a unit vector normal to the photon beam, say $\hat{\mathbf{a}}$ for the initial beam and $\hat{\mathbf{b}}$ for the final. The results are obtained for arbitrary $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, allowing for several applications. The previous results for Compton scattering, obtained in Refs. 1–4, are recovered as special cases.

The polarization of high-energy photon beams will provide decisive information on many-particle interactions. The method given here is straightforward and we expect it will be valuable in studying various photon reactions.

II. PHOTON POLARIZATION

The eigenstates of the helicity operator $\mathbf{S}\cdot\hat{\mathbf{p}}$ of a spin-1 particle are

$$\psi_{\pm 1}=\frac{1}{\sqrt{2}}(\mp\hat{\mathbf{a}}-i\hat{\mathbf{p}}\times\hat{\mathbf{a}}),\quad \psi_0=\hat{\mathbf{p}}\ , \tag{1}$$

where $\hat{\mathbf{p}}$ is the unit vector along the momentum $\mathbf{p}$ of the particle and $\hat{\mathbf{a}}$ is an arbitrary unit vector perpendicular to $\mathbf{p}$. This is in the representation $(S^i)^{jk}=-i\epsilon^{ijk}$. The arbitrariness in $\hat{\mathbf{a}}$ corresponds to the arbitrary phase factor in ψ.

The real photon with momentum $\mathbf{k}$ is transversely polarized and its amplitude can be written as

$$\epsilon(\lambda)=\frac{1}{\sqrt{2}}(-\lambda\hat{\mathbf{a}}-i\hat{\mathbf{k}}\times\hat{\mathbf{a}})\ , \tag{2}$$

where λ is ± 1 for right/left-hand circular polarization. From this equation one obtains

$$\epsilon^i(\lambda)\epsilon^j(\lambda')^*=\frac{1}{2}\left\{(\delta^{ij}-\hat{k}^i\hat{k}^j)\delta_{\lambda\lambda'}-\frac{i}{2}(\lambda+\lambda')\epsilon^{ijk}\hat{k}^k-\frac{i}{2}(\lambda-\lambda')[\hat{a}^i(\hat{\mathbf{k}}\times\hat{\mathbf{a}})^j+\hat{a}^j(\hat{\mathbf{k}}\times\hat{\mathbf{a}})^i]+\tfrac{1}{2}(\lambda\lambda'-1)[\hat{a}^i\hat{a}^j-(\hat{\mathbf{k}}\times\hat{\mathbf{a}})^i(\hat{\mathbf{k}}\times\hat{\mathbf{a}})^j]\right\}\ . \tag{3}$$

This relation is useful in relating the density matrix elements of a photon beam, ρ^{ij} and $\rho_{\lambda\lambda'}$, between two different bases.

The photon states are described in the helicity state basis by the density matrix

$$\rho=\frac{1}{2}\begin{bmatrix} 1+\xi_2 & -\xi_3+i\xi_1 \\ -\xi_3-i\xi_1 & 1-\xi_2 \end{bmatrix},$$

which has components

$$\rho_{\lambda\lambda'}=\frac{1}{2}\left[\delta_{\lambda\lambda'}+\frac{i}{2}\xi_1(\lambda-\lambda')+\tfrac{1}{2}\xi_2(\lambda+\lambda') + \tfrac{1}{2}\xi_3(\lambda\lambda'-1)\right], \tag{4}$$

where ξ_1, ξ_2, and ξ_3 are the Stokes parameters. Here ξ_3 is the degree of linear polarization with respect to the $\hat{\mathbf{a}}$ direction and $\hat{\mathbf{k}}\times\hat{\mathbf{a}}$ direction defined as[10]

$$\xi_3=\frac{I(0)-I(90°)}{I}, \tag{5}$$

where I is the total intensity of the photon beam and $I(\beta)$ denotes the intensity transmitted by a Nicol prism oriented at an angle β with respect to the $\hat{\mathbf{a}}$ direction in the plane perpendicular to the photon wave vector $\mathbf{k}$. The parameter ξ_1 is the degree of linear polarization with respect to two orthogonal axes oriented at 45° to the $\hat{\mathbf{a}}$ direction,

$$\xi_1=\frac{I(45°)-I(135°)}{I}, \tag{6}$$

and ξ_2 is the degree of circular polarization defined as

$$\xi_2=\frac{I_+-I_-}{I}, \tag{7}$$

where I_+ and I_- are the intensities of light transmitted by polarization filters which fully transmit only photons with positive and negative helicity, respectively.

The covariant description of the incident photon density matrix can be obtained from

$$\rho_{\mu\nu}=\sum_{\lambda}\sum_{\lambda'}\rho_{\lambda\lambda'}\epsilon_\mu(\lambda)\epsilon_\nu^*(\lambda'). \tag{8}$$

Since the real photon can be described by

$$\epsilon_\mu(\lambda)=(0,\epsilon(\lambda)), \tag{9}$$

one obtains, from Eqs. (2)–(4), (8), and (9),

$$\rho_{00}=\rho_{0\nu}=\rho_{\mu 0}=0, \tag{10}$$

and

$$\begin{aligned}\rho_{ij}&=\sum_{\lambda}\sum_{\lambda'}\rho_{\lambda\lambda'}\epsilon_i(\lambda)\epsilon_j^*(\lambda')\\ &=\tfrac{1}{2}(\delta^{ij}-\hat{k}^i\hat{k}^j)+\tfrac{1}{2}\xi_1[\hat{a}^i(\hat{\mathbf{k}}\times\hat{\mathbf{a}})^j+\hat{a}^j(\hat{\mathbf{k}}\times\hat{\mathbf{a}})^i]\\ &\quad-\frac{i}{2}\xi_2\epsilon^{ijk}\hat{k}^k+\tfrac{1}{2}\xi_3[\hat{a}^i\hat{a}^j-(\hat{\mathbf{k}}\times\hat{\mathbf{a}})^i(\hat{\mathbf{k}}\times\hat{\mathbf{a}})^j].\end{aligned} \tag{11}$$

When a photon beam interacts with matter, the density matrix $\rho_{\mu\nu}$ must be used for the incident photon beam in place of $\epsilon_\mu\epsilon_\nu^*$ in the absolute square of the transition amplitude of the reaction.

The polarization of the outgoing photon beam, in a reaction where the Proca vector of the final photon ϵ_f^μ is included in the transition amplitude as

$$M=T_\mu\epsilon_f^{\mu*}(\lambda), \tag{12}$$

depends on this amplitude by way of

$$\begin{aligned}|M|^2(\lambda\lambda')&=T_\mu T_\nu^*\epsilon_f^{\mu*}(\lambda)\epsilon_f^\nu(\lambda')\\ &=(T_\mu T_\nu^*)\epsilon_f^\nu(\lambda')\epsilon_f^{\mu*}(\lambda).\end{aligned} \tag{13}$$

as will now be shown. The outgoing photon is described by

$$\phi_f^\mu=\sum_\lambda T_\alpha\epsilon_f^{\alpha*}(\lambda)\epsilon_f^\mu(\lambda), \tag{14}$$

and the density matrix of the outgoing photon beam is obtained as the ensemble average

$$\begin{aligned}\rho^{\mu\nu}&=\langle\phi_f^\mu\phi_f^{\nu*}\rangle\\ &=\sum_\lambda\sum_{\lambda'}\epsilon_f^\mu(\lambda)T_\alpha\epsilon_f^{\alpha*}(\lambda)T_\beta^*\epsilon_f^\beta(\lambda')\epsilon_f^{\nu*}(\lambda')/\mathrm{Tr}(|M|^2)\\ &=\sum_\lambda\sum_{\lambda'}\epsilon_f^\mu(\lambda)|M|^2(\lambda\lambda')\epsilon_f^{\nu*}(\lambda')/\mathrm{Tr}(|M|^2).\end{aligned} \tag{15}$$

In the helicity-state basis, the density matrix $\rho_{\lambda\lambda'}$ becomes

$$\begin{aligned}\rho_{\lambda\lambda'}&=\sum_\mu\sum_\nu\epsilon_{f\mu}^*(\lambda)\rho^{\mu\nu}\epsilon_{f\nu}(\lambda')\\ &=|M|^2(\lambda\lambda')/\mathrm{Tr}(|M|^2).\end{aligned} \tag{16}$$

Therefore, in order to obtain the density matrix of the outgoing photon beam, one must calculate $|M|^2(\lambda\lambda')$ in the form of Eq. (13) and obtain $\rho_{\lambda\lambda'}$ as Eq. (16) from it. When the result is compared with Eq. (4), the coefficients of $(i/2)(\lambda-\lambda')$, $\frac{1}{2}(\lambda+\lambda')$, and $\frac{1}{2}(\lambda\lambda'-1)$ give the Stokes parameters of the final photon in the reaction. The initial Stokes parameters appear within $|M|^2(\lambda\lambda')$ in the form of Eq. (11), as a consequence of the ensemble average.

III. COMPTON SCATTERING

Compton scattering is a pure QED process at least at the tree level. Usually the process is discussed in the rest frame of the initial electron. However in order to study a high-energy photon beam produced by the collision between a laser beam and a high-energy electron beam, it is appropriate to consider the process in an arbitrary frame. The transition amplitude for Compton scattering[1,2] is

$$M=\bar{u}(p_2,s_2)\left[(-i\not\epsilon_2^*)\frac{i}{\not p_1+\not k_1-m}(-i\not\epsilon_1) +(-i\not\epsilon_1)\frac{i}{\not p_1-\not k_2-m}(-i\not\epsilon_2^*)\right]u(p_1,s_1), \tag{17}$$

where k_1, k_2, p_1, and p_2 are momenta of the incident photon, final photon, incident electron, and final electron, respectively, and ϵ_1,ϵ_2 are the Proca vectors for incoming and outgoing photons.

When the polarization of the final electron beam is not observed, the absolute square of the transition amplitude can be written as

$$|M|^2=|M|^2_{\rm unp}+|M|^2_{\rm pol}\ , \tag{18}$$

where $|M|^2_{\rm unp}$ and $|M|^2_{\rm pol}$ are independent of and dependent on the incident electron spin, respectively. Explicitly one finds

$$\begin{aligned}|M|^2_{\rm unp}=\frac{1}{8m^2}\Bigg[&2\frac{(k_1\cdot k_2)^2}{p_1\cdot k_1 p_1\cdot k_2}\epsilon_1\cdot\epsilon_1^*\epsilon_2\cdot\epsilon_2^*\\&-\left[\frac{1}{(p_1\cdot k_1)^2}+\frac{1}{(p_1\cdot k_2)^2}\right][(k_1\cdot k_2)^2|\epsilon_1\cdot\epsilon_2|^2+(p_1\cdot k_1+p_1\cdot k_2)^2|\epsilon_1\cdot\epsilon_2^*|^2\\&\qquad+2(p_1\cdot\epsilon_1 p_2\cdot\epsilon_1^*-p_1\cdot\epsilon_1^* p_2\cdot\epsilon_1)(p_1\cdot\epsilon_2 p_2\cdot\epsilon_2^*-p_1\cdot\epsilon_2^* p_2\cdot\epsilon_2)]\\&+4\left|\frac{p_1\cdot\epsilon_1 p_2\cdot\epsilon_2}{p_1\cdot k_1}-\frac{p_1\cdot\epsilon_2 p_2\cdot\epsilon_1}{p_1\cdot k_2}+\frac{(k_1\cdot k_2)^2}{2p_1\cdot k_1 p_1\cdot k_2}\epsilon_1\cdot\epsilon_2\right|^2\\&+4\left|\frac{p_1\cdot\epsilon_1 p_2\cdot\epsilon_2^*}{p_1\cdot k_1}-\frac{p_1\cdot\epsilon_2^* p_2\cdot\epsilon_1}{p_1\cdot k_2}-\frac{(p_1\cdot k_1+p_1\cdot k_2)^2}{2p_1\cdot k_1 p_1\cdot k_2}\epsilon_1\cdot\epsilon_2^*\right|^2\Bigg]\end{aligned} \tag{19}$$

and

$$\begin{aligned}|M|^2_{\rm pol}=-\frac{i}{4m}\Bigg\{&2A\cdot\epsilon_1 A\cdot\epsilon_1^*\langle\epsilon_2\epsilon_2^*k_2s_1\rangle+\frac{k_1\cdot s_1}{p_1\cdot k_1}(A\cdot\epsilon_1\langle\epsilon_1^*\epsilon_2\epsilon_2^*k_2\rangle+A\cdot\epsilon_1^*\langle\epsilon_1\epsilon_2\epsilon_2^*k_2\rangle)\\&+\frac{A\cdot k_2}{p_1\cdot k_2}(s_1\cdot\epsilon_2\langle\epsilon_1^*\epsilon_2^*k_1\rangle+s_1\cdot\epsilon_2^*\langle\epsilon_1\epsilon_1^*\epsilon_2k_1\rangle)\\&-\left[\frac{k_1\cdot s_1}{p_1\cdot k_1}-A\cdot s_1\right](B\cdot\epsilon_2\langle\epsilon_1\epsilon_1^*\epsilon_2^*k_1\rangle+B\cdot\epsilon_2^*\langle\epsilon_1\epsilon_1^*\epsilon_2k_1\rangle)\\&+\left[A\cdot\epsilon_2B\cdot\epsilon_2^*+A\cdot\epsilon_2^*B\cdot\epsilon_2-\frac{k_1\cdot k_2(p_1\cdot k_1+p_1\cdot k_2)^2}{(p_1\cdot k_1p_1\cdot k_2)^2}\epsilon_2\cdot\epsilon_2^*\right]\langle\epsilon_1\epsilon_1^*k_1s_1\rangle\\&-\frac{1}{p_1\cdot k_1}\left[A\cdot\epsilon_1s_1\cdot\epsilon_1^*+A\cdot\epsilon_1^*s_1\cdot\epsilon_1-\frac{p_1\cdot(k_1+k_2)}{p_1\cdot k_1p_1\cdot k_2}k_1\cdot s_1\epsilon_1\cdot\epsilon_1^*\right]\langle\epsilon_2\epsilon_2^*k_1k_2\rangle\\&+\frac{1}{p_1\cdot k_2}\left[A\cdot\epsilon_2s_1\cdot\epsilon_2^*+A\cdot\epsilon_2^*s_1\cdot\epsilon_2+\frac{p_1\cdot(k_1+k_2)}{p_1\cdot k_1}\left[\frac{k_1\cdot s_1}{p_1\cdot k_1}-A\cdot s_1\right]\epsilon_2\cdot\epsilon_2^*\right]\langle\epsilon_1\epsilon_1^*k_1k_2\rangle\Bigg\}\ .\end{aligned} \tag{20}$$

In Eq. (20), A, B, and $\langle abcd\rangle$ are defined as

$$A^\mu=\frac{p_1^\mu}{p_1\cdot k_1}-\frac{p_2^\mu}{p_1\cdot k_2}\ , \tag{21a}$$

$$B^\mu=\frac{p_2^\mu}{p_1\cdot k_1}-\frac{p_1^\mu}{p_1\cdot k_2}\ , \tag{21b}$$

and

$$\langle abcd\rangle=\epsilon_{\mu\nu\lambda\tau}a^\mu b^\nu c^\lambda d^\tau\ . \tag{21c}$$

One can see that the gauge transformation $\epsilon_i^\mu\to\epsilon_i^\mu+\text{const}\times k_i^\mu$ $(i=1,2)$, does not change $|M|^2$. Also $|M|^2_{\rm unp}$ can be considered as the value of $|M|^2$ spin averaged over the incident electron beam.

This is in agreement with the result of Stedman and Pooke[3] for scattering of circularly polarized photons and with the Klein-Nishina formula for plane polarized photons. One must specialize to the electron rest frame and consider $|M|^2_{\rm unp}$ only.

The polarization of the outgoing photon beam can be calculated in general from Eqs. (18)–(20) when the incident photon beam in the Compton scattering is arbitrarily polarized. In order to obtain $|M|^2(\lambda\lambda')$, $\epsilon_1^\mu\epsilon_1^{\nu *}$ contained in the equations is replaced by Eqs. (10) and (11) and Eq. (3) is used for $\epsilon_2^\mu(\lambda)\epsilon_2^\nu(\lambda')^*$ in Eqs. (19) and (20), except that a unit vector $\hat{\mathbf{b}}$ is used for the outgoing photon instead of $\hat{\mathbf{a}}$. Then the explicit form of $|M|^2(\lambda\lambda')$ becomes

$$|M|^2(\lambda\lambda')=|M|^2(\lambda\lambda')_{\rm unp}+|M|^2(\lambda\lambda')_{\rm pol}\ , \tag{22}$$

$$\begin{aligned}|M|^2(\lambda\lambda')_{\rm unp}=\frac{1}{2m^2}\Bigg[&\delta_{\lambda\lambda'}\left[1+\frac{(k_1\cdot k_2)^2}{2p_1\cdot k_1 p_1\cdot k_2}-2(1-\xi_3)r(1-r)-\xi_1 m^2\mathbf{A}\cdot\hat{\mathbf{a}}\,\mathbf{A}\cdot(\hat{\mathbf{k}}_1\times\hat{\mathbf{a}})-\xi_3 m^2(\mathbf{A}\cdot\hat{\mathbf{a}})^2\right]\\
&+\tfrac{1}{2}(\lambda\lambda'-1)\{-\xi_3+2(1-\xi_3)r(1-r)+\xi_1 m^2\mathbf{A}\cdot\hat{\mathbf{a}}\;\mathbf{A}\cdot(\hat{\mathbf{k}}_1\times\hat{\mathbf{a}})+\xi_3 m^2(\mathbf{A}\cdot\hat{\mathbf{a}})^2\\
&\qquad-(1-\xi_3)m^2(\mathbf{B}\cdot\hat{\mathbf{b}})^2+2\xi_3(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{J}}\cdot\hat{\mathbf{b}})^2+2\xi_1(\hat{a}\cdot\overleftrightarrow{\mathbf{J}}\cdot\hat{b})[(\hat{\mathbf{k}}_1\times\hat{\mathbf{a}})\cdot\overleftrightarrow{\mathbf{J}}\cdot\hat{\mathbf{b}}]\}\\
&+\tfrac{1}{2}(\lambda+\lambda')\xi_2(1-2r)\left[1+\frac{(k_1\cdot k_2)^2}{2p_1\cdot k_1 p_1\cdot k_2}\right]\\
&+\frac{i}{2}(\lambda-\lambda')\{\xi_1(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{J}}\cdot\hat{\mathbf{b}})[(\hat{\mathbf{k}}_1\times\hat{\mathbf{a}})\cdot\overleftrightarrow{\mathbf{J}}\cdot(\hat{\mathbf{k}}_2\times\hat{\mathbf{b}})]+\xi_1[\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{J}}\cdot(\hat{\mathbf{k}}_2\times\hat{\mathbf{b}})][(\hat{\mathbf{k}}_1\times\hat{\mathbf{a}})\cdot\overleftrightarrow{\mathbf{J}}\cdot\hat{\mathbf{b}}]\\
&\qquad-(1-\xi_3)m^2\mathbf{B}\cdot\hat{\mathbf{b}}\,\mathbf{B}\cdot(\hat{\mathbf{k}}_2\times\hat{\mathbf{b}})+2\xi_3(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{J}}\cdot\hat{\mathbf{b}})[\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{J}}\cdot(\hat{\mathbf{k}}_2\times\hat{b})]\}\Bigg]\ ,\end{aligned} \tag{23}$$

and

$$\begin{aligned}|M|^2(\lambda\lambda')_{\rm pol}=\frac{1}{4m}\Bigg\{&\xi_2\frac{k_1\cdot k_2}{p_1\cdot k_1 p_1\cdot k_2}\Bigg[\delta_{\lambda\lambda'}[s_1\cdot k_1(1-2r)+s_1\cdot k_2]\\
&\qquad+\tfrac{1}{2}(\lambda\lambda'-1)\left[(s_1\cdot k_2)\frac{p_1\cdot k_1}{p_1\cdot k_2}-(s_1\cdot k_1)(1-2r)\right]+\frac{i}{2}(\lambda\lambda')\frac{1}{p_1\cdot k_2}\langle s_1p_1k_1k_2\rangle\Bigg]\\
&+\tfrac{1}{2}(\lambda+\lambda')\frac{k_1\cdot k_2}{p_1\cdot k_1 p_1\cdot k_2}\Bigg[s_1\cdot k_1+s_1\cdot k_2(1-2r)-\frac{\xi_1}{p_1\cdot k_1}\langle s_1p_1k_1k_2\rangle\\
&\qquad+\xi_3\left[(s_1\cdot k_1)\frac{p_1\cdot k_2}{p_1\cdot k_1}-(s_1\cdot k_2)(1-2r)\right]\Bigg]\\
&+(\lambda\lambda'-1)\xi_2\mathbf{B}\cdot\hat{\mathbf{b}}\left[s_1\cdot k_1\mathbf{B}\cdot\hat{\mathbf{b}}+\frac{1}{p_1\cdot k_2}(k_1\cdot k_2\mathbf{s}_1\cdot\hat{\mathbf{b}}-s_1\cdot k_2\mathbf{k}_1\cdot\hat{\mathbf{b}})\right]\\
&-i(\lambda-\lambda')\xi_2\mathbf{B}\cdot\hat{\mathbf{b}}\left[s_1\cdot k_1\mathbf{B}\cdot(\hat{\mathbf{k}}_2\times\hat{\mathbf{b}})+\frac{1}{p_1\cdot k_2}[k_1\cdot k_2\mathbf{s}_1\cdot(\hat{\mathbf{k}}_2\times\hat{\mathbf{b}})-s_1\cdot k_2\mathbf{k}_1\cdot(\hat{\mathbf{k}}_2\times\hat{\mathbf{b}})]\right]\\
&+(\lambda+\lambda')\xi_3\,\mathbf{A}\cdot\hat{\mathbf{a}}\left[s_1\cdot k_2\,\mathbf{A}\cdot\hat{\mathbf{a}}+\frac{1}{p_1\cdot k_1}(k_1\cdot k_2\mathbf{s}_1\cdot\hat{\mathbf{a}}-s_1\cdot k_1\mathbf{k}_2\cdot\hat{\mathbf{a}})\right]\\
&+(\lambda+\lambda')\xi_1\,\mathbf{A}\cdot\hat{\mathbf{a}}\left[s_1\cdot k_2\,\mathbf{A}\cdot(\hat{\mathbf{k}}_1\times\hat{\mathbf{a}})+\frac{1}{p_1\cdot k_1}[k_1\cdot k_2\mathbf{s}_1\cdot(\hat{\mathbf{k}}_1\times\hat{\mathbf{a}})-s_1\cdot k_1\mathbf{k}_2\cdot(\hat{\mathbf{k}}_1\times\hat{\mathbf{a}})]\right]\Bigg\}\ ,\end{aligned} \tag{24}$$

where r and a tensor J are defined as

$$r=\frac{m^2k_1\cdot k_2}{2p_1\cdot k_1 p_1\cdot k_2} \tag{25}$$

and

$$J^{ij}=\delta^{ij}+\frac{p_1^ip_2^j}{p_1\cdot k_1}-\frac{p_1^jp_2^i}{p_1\cdot k_2}\ . \tag{26}$$

This is the general result. From this expression the polarization of the final photon can be obtained through the coefficients of $\delta_{\lambda\lambda'}$, $(i/2)(\lambda-\lambda')$, $\frac{1}{2}(\lambda+\lambda')$, and $\frac{1}{2}(\lambda\lambda'-1)$, which give the Stokes parameters of the final photon and their dependence on the polarizations of incident photon and electron beams.

IV. DISCUSSION

In Compton scattering considered in the rest frame of an initial electron, the scattering occurs in a plane and usually one uses one unit vector normal to the scattering plane instead of two vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ to define Stokes parameters of initial and final photon beams. However, the general Compton scattering does not occur in a plane and the Stokes parameters of the outgoing photon beam depends on $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and the Stokes parameters of initial photon beam ξ_i $(i=1,2,3)$ as well as the polarization of incident electron beam s_1^μ. One can choose the unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ arbitrarily except that they must be normal to $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively. But the Stokes parameters of the photon beams change according to the choice of $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

It is noted that, if $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are chosen as

$$\hat{\mathbf{a}}=\frac{\mathbf{k}_1\times\mathbf{A}}{|\mathbf{k}_1\times\mathbf{A}|}\ , \tag{27}$$

$$\hat{\mathbf{b}}=\frac{\mathbf{k}_2\times\mathbf{B}}{|\mathbf{k}_2\times\mathbf{B}|}\ , \tag{28}$$

the following relations hold:

$$\mathbf{A}\cdot\hat{\mathbf{a}}=\mathbf{B}\cdot\hat{\mathbf{b}}=0\ , \tag{29a}$$

$$\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{J}}\cdot(\hat{\mathbf{k}}_2\times\hat{\mathbf{b}})=(\hat{\mathbf{k}}_1\times\hat{\mathbf{a}})\cdot\overleftrightarrow{\mathbf{J}}\cdot\hat{\mathbf{b}}=0\ , \tag{29b}$$

$$\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{J}}\cdot\hat{\mathbf{b}}=1\ , \tag{29c}$$

$$(\hat{\mathbf{k}}_1\times\hat{\mathbf{a}})\cdot\overleftrightarrow{\mathbf{J}}\cdot(\hat{\mathbf{k}}_2\times\hat{\mathbf{b}})=1-2r\ , \tag{29d}$$

and one obtains a rather concise covariant result for $|M|^2(\lambda\lambda')$ from Eqs. (22)–(24) as

$$\begin{aligned}|M|^2(\lambda\lambda')=\frac{1}{2m^2}\Bigg\{&\delta_{\lambda\lambda'}\left[1+\frac{(k_1\cdot k_2)^2}{2p_1\cdot k_1 p_1\cdot k_2}-2(1-\xi_3)r(1-r)+\xi_2\frac{r}{m}[s_1\cdot k_1(1-2r)+s_1\cdot k_2]\right]\\&+\frac{i}{2}(\lambda-\lambda')\left[\xi_1(1-2r)+\xi_2\frac{r}{mp_1\cdot k_2}\langle s_1p_1k_1k_2\rangle\right]\\&+\tfrac{1}{2}(\lambda+\lambda')\left[\frac{r}{m}[s_1\cdot k_1+s_1\cdot k_2(1-2r)]-\xi_1\frac{r}{mp_1\cdot k_1}\langle s_1p_1k_1k_2\rangle\right.\\&\qquad\left.+\xi_2(1-2r)\left[1+\frac{(k_1\cdot k_2)^2}{2p_1\cdot k_1p_1\cdot k_2}\right]+\xi_3\frac{r}{m}\left[(s_1\cdot k_1)\frac{p_1\cdot k_2}{p_1\cdot k_1}-s_1\cdot k_2(1-2r)\right]\right]\\&+\tfrac{1}{2}(\lambda\lambda'-1)\left[2(1-\xi_3)r(1-r)+\xi_3-\xi_2\frac{r}{m}\left[s_1\cdot k_1(1-2r)-(s_1\cdot k_2)\frac{p_1\cdot k_1}{p_1\cdot k_2}\right]\right]\Bigg\}\ .\end{aligned} \tag{30}$$

One can obtain the Stokes parameters of the final photon beam through the coefficients of $\delta_{\lambda\lambda'}$, $(i/2)(\lambda-\lambda')$, $\frac{1}{2}(\lambda+\lambda')$, and $\frac{1}{2}(\lambda\lambda'-1)$. But it is noted that they are specified by the unit vector $\hat{\mathbf{b}}$ given by Eq. (28). It is seen that the covariant expression, Eq. (30), is more compact than the general results, Eqs. (22)–(24), but at the expense of being limited to the special values of $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

When the polarization of the outgoing photon beam in Compton scattering is detected and it is specified by some specific Stokes parameters ξ_i^f $(i=1,2,3)$ relative to unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ which satisfy Eqs. (29a)–(29d), the differential cross section can be obtained by multiplying the matrix form of Eq. (30), $|M|^2(\lambda\lambda')$, by

$$\rho^f=\frac{1}{2}\begin{bmatrix}1+\xi_2^f & -\xi_3^f+i\xi_1^f\\ -\xi_3^f-i\xi_1^f & 1-\xi_2^f\end{bmatrix}\ , \tag{31}$$

and then taking trace of it. Explicitly it is

$$\begin{aligned}\frac{d\sigma}{d\Omega}=\frac{\alpha^2(\omega_2)^2}{2(p_1\cdot k_1)^2}\Bigg\{&1+\frac{(k_1\cdot k_2)^2}{2p_1\cdot k_1p_1\cdot k_2}-2(1-\xi_3)r(1-r)+\xi_2\frac{r}{m}[s_1\cdot k_1(1-2r)+s_1\cdot k_2]\\&+\xi_1^f\left[\xi_1(1-2r)+\xi_2\frac{r}{mp_1\cdot k_2}\langle s_1p_1k_1k_2\rangle\right]\\&+\xi_2^f\left[\frac{r}{m}[s_1\cdot k_1+s_1\cdot k_2(1-2r)]-\xi_1\frac{r}{mp_1\cdot k_1}\langle s_1p_1k_1k_2\rangle+\xi_2(1-2r)\left[1+\frac{(k_1\cdot k_2)^2}{2p_1\cdot k_1p_1\cdot k_2}\right]\right.\\&\qquad\left.+\xi_3\frac{r}{m}\left[s_1\cdot k_1\frac{p_1\cdot k_2}{p_1\cdot k_1}-s_1\cdot k_2(1-2r)\right]\right]\\&+\xi_3^f\left[\xi_3+2(1-\xi_3)r(1-r)-\xi_2\frac{r}{m}\left[s_1\cdot k_1(1-2r)-s_1\cdot k_2\frac{p_1\cdot k_1}{p_1\cdot k_2}\right]\right]\Bigg\}\ .\end{aligned} \tag{32}$$

This result is given in Ref. 6 [in particular, Eq. (B2) of Ginzburg *et al.*].

In the rest frame of the initial electron, Compton scattering occurs in a plane, and $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ can be chosen equally to be the normal direction to the scattering plane for convenience. Then Eqs. (29a)–(29d) are satisfied and Eq. (30) can be used to find the polarization of the outgoing photon. Therefore, in the case of initial electron at rest, the Stokes parameters ξ_i' $(i=1,2,3)$ which characterize the polarization of the outgoing photon through Eq. (4) become

$$\xi_1'=\frac{1}{A}\left[2\xi_1\cos\theta-\xi_2\frac{\omega_1}{m}(1-\cos\theta)\sin\theta\,\mathbf{s}_1\cdot\hat{\mathbf{a}}\right],$$

$$\xi_2'=\frac{1}{A}\left[\xi_2\left[\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}\right]\cos\theta-\frac{1}{m}(1-\cos\theta)[(\omega_1+\omega_2)\cos\theta\,\mathbf{s}_1\cdot\hat{\mathbf{k}}_2-\xi_1\omega_2\sin\theta\,\mathbf{s}_1\cdot\hat{\mathbf{a}}+(\omega_1+\omega_2\xi_3)\sin\theta\,\mathbf{s}_1\cdot(\hat{\mathbf{k}}_2\times\hat{\mathbf{a}})]\right], \quad (33)$$

$$\xi_3'=\frac{1}{A}\left[\sin^2\theta+\xi_3(1+\cos^2\theta)+\xi_2\frac{\omega_1}{m}(1-\cos\theta)\sin\theta\,\mathbf{s}_1\cdot(\hat{\mathbf{k}}_1\times\hat{\mathbf{a}})\right],$$

where θ is the angle between incoming and outgoing photons, i.e., $\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2=\cos\theta$, and A is defined as

$$A=\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}-(1-\xi_3)\sin^2\theta$$
$$-\xi_2\frac{1}{m}(1-\cos\theta)(\cos\theta\,\mathbf{s}_1\cdot\mathbf{k}_1+\mathbf{s}_1\cdot\mathbf{k}_2)\ . \quad (34)$$

When the initial electrons are unpolarized, the Stokes parameters of the outgoing photon beam can be obtained[2] from Eqs. (33) and (34) just by setting $\mathbf{s}_1=0$.

Now consider two special cases of the Compton scattering between a polarized laser beam and a polarized high-energy electron beam. In the first case the outgoing photon beam is in the direction of the incident electron beam, and in the second case[5] the incident photon beam makes a head-on collision with the incident electron beam but the outgoing photon beam is not necessarily in the direction of the incident electron beam. In both cases the scattering occurs in a plane and Eqs. (29a)–(29d) are satisfied with $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ in the $\mathbf{k}_1\times\mathbf{k}_2$ direction. Therefore Eq. (30) can be used to find the polarization of the outgoing photon beam.

In general the outgoing photon beam has the energy

$$\omega_2=\omega_1\frac{1-(|p_1|/E_1)\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}_1}{1-(|p_1|/E_1)\hat{\mathbf{p}}_1\cdot\hat{\mathbf{k}}_2+(\omega_1/E_1)(1-\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)}, \quad (35)$$

where $|p_1|$ indicates the length of the three-vector $\mathbf{p}_1$. If the outgoing photon beam in the process is in the direction of incident electron beam, i.e., $\mathbf{k}_2//\mathbf{p}_1$, Eq. (35) becomes

$$\omega_2=\omega_1\frac{1+2|p_1|\cos^2\frac{1}{2}\Theta/(E_1-|p_1|)}{1+2\omega_1\cos^2\frac{1}{2}\Theta/(E_1-|p_1|)}, \quad (35a)$$

where Θ is the angle between $-\mathbf{k}_1$ and $\mathbf{p}_1$ (or $\mathbf{k}_2$), i.e., $\Theta=\pi-\theta$. The dependence of ω_2 on Θ is illustrated in Fig. 1(a). It is appropriate for high-energy electron scattering to consider small m/E_1 for arbitrary values of x and y, where x and y are defined as

$$x=\frac{4\omega_1E_1}{m^2}, \quad (36)$$

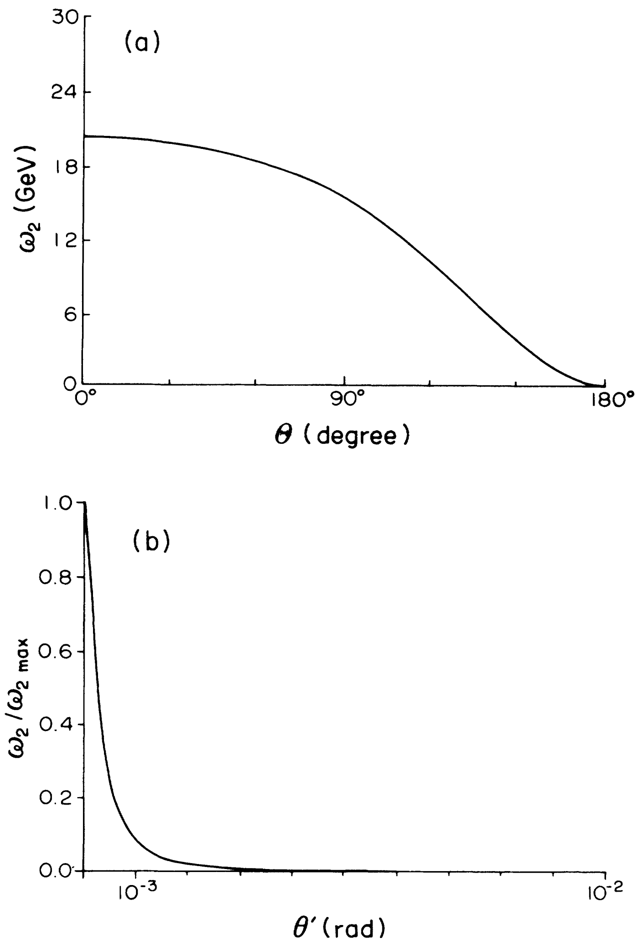

FIG. 1. (a) Output photon energy ω_2 as a function of input photon angle Θ, according to Eq. (35a). The incident electron energy E_1 is 30 GeV and the incident photon energy ω_1 is 4.68 eV. (b) Output photon energy ω_2 as a function of output photon angle θ', according to Eq. (35c),

$$\frac{\omega_2}{\omega_{2_{\max}}}\simeq\frac{1+x}{1+x+(\gamma\theta')^2}\ .$$

Here also E_1 is 30 GeV and ω_1 is 4.68 eV.

$$y = \tfrac{1}{2}(1+\cos\Theta) = \cos^2\frac{\Theta}{2} \ . \qquad (37)$$

If one neglects $m^2/E_1{}^2$ and higher order, one obtains

$$\omega_2 \simeq \frac{xy}{1+xy} E_1 \qquad (38)$$

$$r \simeq 1 \ , \qquad (39)$$

and if ω_1/m is also neglected, Eq. (30) becomes

$$|M|^2(\lambda\lambda') = \frac{1}{2m^2}\left\{\delta_{\lambda\lambda'}\left[1+\frac{(xy)^2}{2(1+xy)} - \xi_2\frac{xy(2+xy)}{2(1+xy)}\hat{\mathbf{p}}_1\cdot\hat{\mathbf{s}}_1\right] - \frac{i}{2}(\lambda-\lambda')\xi_1 + \tfrac{1}{2}(\lambda\lambda'-1)\xi_3 + \tfrac{1}{2}(\lambda+\lambda')\left[-\xi_2\left[1+\frac{(xy)^2}{2(1+xy)}\right] + \frac{xy(2+xy)}{2(1+xy)}\hat{\mathbf{p}}_1\cdot\hat{\mathbf{s}}_1\right]\right\} \qquad (40)$$

where $\hat{\mathbf{s}}_1$ is the polarization three-vector, $s_1^\mu = (0, \hat{\mathbf{s}}_1)$, in the electron rest frame.

If the incident photon beam makes a head-on collision with the incident electron beam, the energy of the scattered photon is, from Eq. (35),

$$\omega_2 \simeq 2\omega_1 \frac{1-m^2/4E_1^2}{1-\cos\theta' + (\omega_1/E_1)(1+\cos\theta') + (m^2/2E_1{}^2)\cos\theta'} \ , \qquad (35b)$$

where θ' is the angle between the incident electron beam and the outgoing photon beam. It is seen that ω_2 is sensitive to θ', i.e., if θ' is larger than $\gamma^{-1} = m/E_1$, ω_2 becomes small [see Fig. 1(b)]. In order to have high-energy photon beam, θ' must be small, i.e., $\theta' \le \gamma^{-1}$, and ω_2 becomes

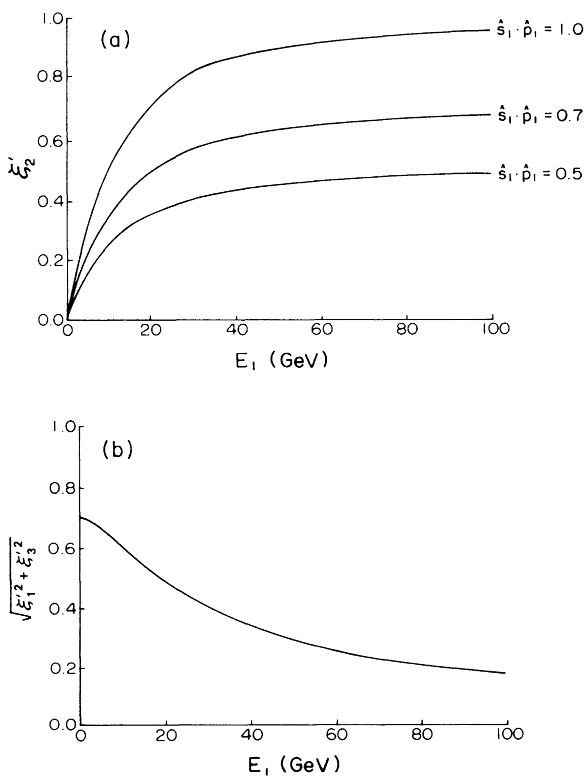

FIG. 2. When the incident photon beam is partially linearly polarized, i.e., $\xi_2 = 0$ and $(\xi_1{}^2+\xi_3{}^2)^{1/2} = 0.7$, (a) degree of circular polarization of outgoing photon beam ξ_2' vs incident electron energy E_1 and (b) degree of linear polarization of outgoing photon beam $(\xi_1'^2+\xi_3'^2)^{1/2}$ vs incident electron energy E_1, when incident electron beam is linearly polarized, $\hat{\mathbf{s}}_1\cdot\hat{\mathbf{p}}_2 = 1$, $\hat{\mathbf{s}}_1\cdot\hat{\mathbf{p}}_1 = 0.7$, and $\hat{\mathbf{s}}_1\cdot\hat{\mathbf{p}}_1 = 0.5$.

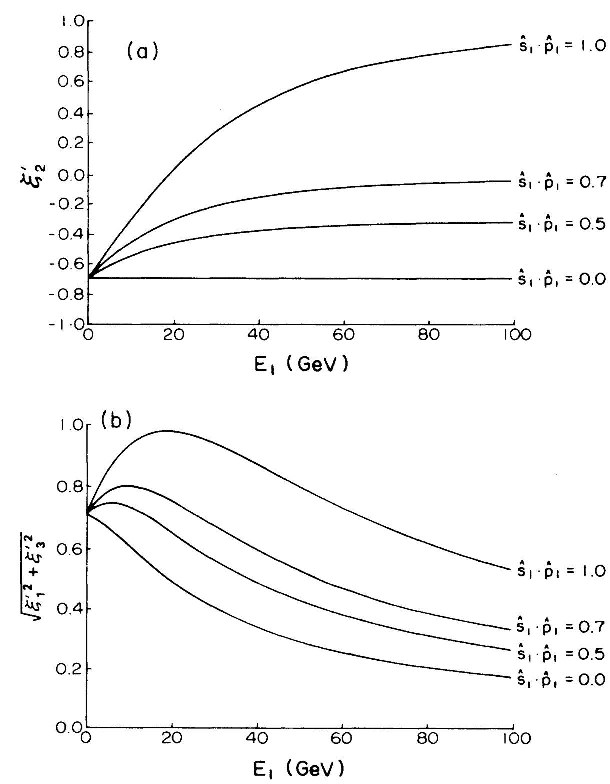

FIG. 3. When the incident photon beam is polarized, $\xi_2 = 0.7$ and $(\xi_1{}^2+\xi_3{}^2)^{1/2} = 0.7$, (a) degree of circular polarization of outgoing photon beam ξ_2' vs incident electron energy E_1, and (b) degree of linear polarization of outgoing photon beam $(\xi_1'^2+\xi_3'^2)^{1/2}$ vs incident electron energy E_1, for various incident electron polarization, $\hat{\mathbf{s}}_1\cdot\hat{\mathbf{p}}_2 = 1$, $\hat{\mathbf{s}}_1\cdot\hat{\mathbf{p}}_1 = 0.7$, $\hat{\mathbf{s}}_1\cdot\hat{\mathbf{p}}_1 = 0.5$, and $\hat{\mathbf{s}}_1\cdot\hat{\mathbf{p}}_1 = 0$.

$$\omega_2 \simeq \frac{xE_1}{1+x+(\gamma\theta')^2} \tag{35c}$$

and $\omega_{2_{\max}}$ becomes $xE_1/(1+x)$ as one can also get from Eq. (38). For $E_1 = 50$ GeV, x is about 3.6 and γ^{-1} is about 10^{-5}.

Therefore, for small θ', $|M|^2(\lambda\lambda')$ becomes

$$\begin{aligned}|M|^2(\lambda\lambda') = &\frac{1}{2m^2(1+u^2)^2(1+x+u^2)} \\ &\times \Big[\delta_{\lambda\lambda'}\{1+(1+x)^2+(2+x^2)u^2+2(1+x)u^4+2u^6+4\xi_3(1+x+u^2)u^2 \\ &\qquad -\xi_2[x(1-u^2)(2+x+2u^2)\hat{\mathbf{p}}_1\cdot\hat{\mathbf{s}}_1+2xu(1+u^2)\hat{\mathbf{s}}_1\cdot(\hat{\mathbf{p}}_1\times\hat{\mathbf{a}})]\} \\ &\quad -\frac{i}{2}(\lambda-\lambda')2(1+x+u^2)[\xi_1(1-u^4)+\xi_2 xu\hat{\mathbf{s}}_1\cdot\hat{\mathbf{a}}] \\ &\quad +\tfrac{1}{2}(\lambda+\lambda')\{x(2+x+xu^2+2u^4)\hat{\mathbf{p}}_1\cdot\hat{\mathbf{s}}_1+2xu(1-u^2)\hat{\mathbf{s}}_1\cdot(\hat{\mathbf{p}}_1\times\hat{\mathbf{a}}) \\ &\qquad +2\xi_1 u(1+u^2)\hat{\mathbf{s}}_1\cdot\hat{\mathbf{a}}-\xi_2(1-u^2)[1+(1+x)^2+2(2+x)u^2+2u^4] \\ &\qquad +2\xi_3 xu[2u\hat{\mathbf{p}}_1\cdot\hat{\mathbf{s}}_1-(1-u^2)\hat{\mathbf{s}}_1\cdot(\hat{\mathbf{p}}_1\times\hat{\mathbf{a}})]\} \\ &\quad +\tfrac{1}{2}(\lambda\lambda'-1)\cdot 2(1+x+u^2)[2u^2+\xi_3(1+u^4)-\xi_2 xu\hat{\mathbf{s}}_1\cdot(\hat{\mathbf{p}}_1\times\hat{\mathbf{a}})]\Big] ,\end{aligned} \tag{41}$$

where u and $\hat{\mathbf{a}}$ are defined as

$$u = \gamma\theta' = \frac{E_1\theta'}{m} , \tag{42}$$

$$\hat{\mathbf{a}} = \frac{\hat{\mathbf{k}}_1\times\hat{\mathbf{k}}_2}{|\hat{\mathbf{k}}_1\times\hat{\mathbf{k}}_2|} = -\frac{\hat{\mathbf{p}}_1\times\hat{\mathbf{k}}_2}{\sin\theta'} . \tag{43}$$

This result agrees with Eqs. (5a) and (5b) of Grinchishin's second paper[5] when his angle θ is small. One can obtain the Stokes parameters of the outgoing photon beam in both cases from Eqs. (40) and (41).

In the limit $\theta' \simeq 0$ in Eq. (41) or $\Theta \simeq 0$ in Eq. (40), i.e., when ω_2 has the maximum value, they become

$$\xi_1' = \frac{-(1+x)\xi_1}{1+x+\frac{1}{2}x^2-\xi_2(x+\frac{1}{2}x^2)\hat{\mathbf{p}}_1\cdot\hat{\mathbf{s}}_1} , \tag{44a}$$

$$\xi_2' = \frac{-(1+x+\frac{1}{2}x^2)\xi_2+(x+\frac{1}{2}x^2)\hat{\mathbf{p}}_1\cdot\hat{\mathbf{s}}_1}{1+x+\frac{1}{2}x^2-\xi_2(x+\frac{1}{2}x^2)\hat{\mathbf{p}}_1\cdot\hat{\mathbf{s}}_1} , \tag{44b}$$

$$\xi_3' = \frac{(1+x)\xi_3}{1+x+\frac{1}{2}x^2-\xi_2(x+\frac{1}{2}x^2)\hat{\mathbf{p}}_1\cdot\hat{\mathbf{s}}_1} . \tag{44c}$$

The Stokes parameters of the outgoing photon beam depend on the energy of the incident electron beam and the Stokes parameters of the incident photon beam as well as the polarization of the initial electron beam as shown in Figs. 2 and 3. If the incident electron is unpolarized, $\hat{\mathbf{s}}_1 = 0$ and the result becomes the same as that given in Ref. 5. Equations (44a)–(44c) do not depend on the transverse polarization of the incident electron beam. Since the energy of the outgoing photon beam increases with the energy of the incident electron beam as given by Eq. (35a) and x increases according to Eq. (36), the linear polarization becomes smaller as the unpolarized incident electron energy becomes larger. One can see from Eq. (44b) that, when the incident electron beam is unpolarized, the circular polarization of the outgoing photon beam is the same as that of the incident photon beam, but the polarization of the incident electron beam induces circular polarization of the final photons even though the incident photon beam is not polarized. In the latter case the electron polarization does not influence the linear polarization of the outgoing photon beam.

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