Comparison of exclusive decay rates for $b \to u$ and $b \to c$ transitions

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We present a variety of estimates for nonleptonic and semileptonic few-body decays of the B meson which indicate that $b \to u$ transitions tend to produce multibody final states. This implies that channels such as $B \to \pi\pi$ or $B \to \pi\rho$ offer little chance to improve the current upper bound on the $b \to u$ Kobayashi-Maskawa matrix element. Also the leptons in $b \to u$ semileptonic transitions tend to be emitted together with massive multibody states rather than a single π or ρ. This makes the $b \to u$ spectrum somewhat softer and the task of distinguishing such leptons from $B \to Dl\nu$, say, more difficult.

I. INTRODUCTION

The ratio of Kobayashi-Maskawa (KM) matrix elements

$$r \equiv | V_{bu} / V_{bc} |^2$$

is of prime importance, and presently is bound by $r \leq 0.02$ (Ref. 1). The idea underlying efforts to improve this bound and eventually find $b \to u$ transitions is to use the special kinematics of the $b \to u$ decays of B mesons at $\Psi(4s)$. Two-body decays such as $B \to \pi\pi$, $\rho\pi$, $A_2\pi$ have characteristics allowing B reconstruction and efficient separation of background events.[2] The semileptonic decays $B \to \pi l \nu$, $B \to \rho l \nu$, and more generally $B \to X l \nu$, where X is a noncharmed, nonstrange, low-mass mesonic system, can yield leptons of maximal energy

$$E^u_{\max} = \left[\frac{m_B}{2} \right] - \left[\frac{(m_X)^2}{2m_B} \right] .$$

Such leptons extend beyond the end point of the dominant $B \to Dl\nu$, $D^* l \nu$ transitions with

$$E^c_{\max} = \frac{m_B}{2} - \frac{{m_D}^2}{2m_B} .$$

The $b \to u$ transitions could also produce very energetic pions from $B \to \pi X$ decays with X a low-mass hadronic system.

Let $P_u(f)$ denote the probability that a B decay proceeding via $b \to u$ leads to the specific final state f. The branching ratio for this final state is then given by[3]

$$B(B \to f) \approx 2rP_u(f) .$$

If $P_u(f)$ is known then an upper bound on $B(B \to f)$ translates into an upper bound on r:

$$\left| \frac{V_{bu}}{V_{bc}} \right|^2 \equiv r = B(B \to f)/2P_u(f) .$$

Strong upper bounds on V_{bu} or sensitive searches of $b \to u$ transitions require kinematically clean final channels which yield stringent experimental bounds on $B(B \to f)$. However, we also need to make sure that $P_u(f)$ is not dynamically suppressed.

In the following, we will compare $P_u(f)$ for $f = \pi\pi$, $\pi\rho$, $\pi l \nu$, $\rho l \nu$ with the corresponding quantity $P_c(f')$ for $f' = \pi D$, πD^*, $Dl\nu$, $D^* l \nu$. Our estimates of various kinds all indicate a suppression of the hadronization of the u quark into the low-lying states. This observation implies that the most straightforward signatures for $b \to u$ transitions are dynamically disfavored.

II. KINEMATICS

The basic feature affecting all subsequent estimates is the large mass ratio $m_b / m_u \geq 10$ involved in the $b \to u$ transition. In $b \to u l \nu$ or $b \to u \bar{u} d$ the average u-quark energy $\approx m_B / 3 = 1.75$ GeV considerably exceeds the typical light-quark energy in ground-state mesonic systems $E \approx m_u \approx 350-500$ MeV. In contrast, for charm ($c \to s$) or $b \to c$ transitions with $m_b / m_c \approx m_c / m_s \approx 3$, the energies of the secondary particles fit better into the low-lying states. Therefore, available results about charm decays such as $D \to \pi\pi$, $D \to K\pi$, or about $B \to D(D^*) + \pi$, $B \to D(D^*) + l\nu$ have to be used with caution as a guideline for the corresponding $b \to u$ transitions.

Following most workers[4,5] we adopt a simple spectator model for the b nonleptonic decays [see Fig. 1(a)]:

$$b \to q' + \bar{u}d, \quad q' = (u \text{ or } c) ,$$

with the other, "spectator" quark $\bar{q}$ in the B unaffected. Subsequently the four quarks (q', $\bar{u}d$, and $\bar{q}$) combine into the final-state hadrons.

Furthermore, following Ref. 4 we assume that the color-singlet clusters [$\bar{u}d$ from the virtual W, and u (or c) with $\bar{q}$] hadronize separately [see Fig. 1(b)]. With these assumptions the semileptonic and nonleptonic B decays are very similar. In particular, $B \to \pi\pi$, say, is analogous to $B \to \pi l \nu$ with $m^2(l\nu) \approx {m_\pi}^2 \approx 0$.

Let the "decay" quark q' from $b \to q' + W^-$, $q' = u$, or c, carry energy ηm_b. The invariant mass of the $\bar{q}q'$ system then is

$$m^2(\bar{q}q') = (m_{\bar{q}})^2 + 2m_b \eta m_{\bar{q}} + ({m_u}^2 \text{ or } {m_c}^2) . \quad (1)$$

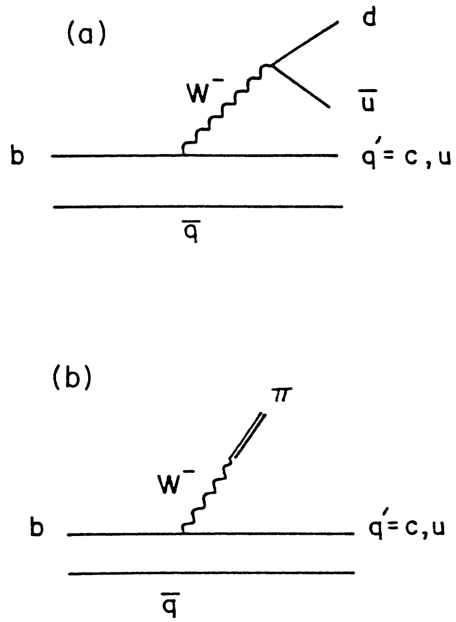

FIG. 1. (a) Spectator-model diagram for nonleptonic B decays. (b) Spectator-model diagram for $B \to \pi\pi$ with color clustering.

In $B \to \pi\pi$ (or $B \to \pi D$), $\eta \approx \frac{1}{2}$, whereas in the semileptonic case we expect an average $\bar{\eta} \approx \frac{1}{3}$.

Using $m_{\bar{q}} = m_u = 0.35$ GeV, $m_c = 1.5$ GeV, and $m_b = 4.9$ GeV we find, for the semileptonic case,

$$m(\bar{q}u) \approx 1.2 \text{ GeV}, \quad m(\bar{q}c) \approx 1.9 \text{ GeV} , \tag{2a}$$

and, for the two-body nonleptonic case,

$$m(\bar{q}u) \approx 1.4 \text{ GeV}, \quad m(\bar{q}c) \approx 2.0 \text{ GeV} . \tag{2b}$$

Thus, $m(\bar{q}c)$ is the D mass or not very far above it—consistent with the dominance of $B \to Dl\nu$, $D^* l\nu$ and the reasonable branchings for $B \to D\pi$, $D^*\pi$.

This is to be contrasted with the invariant masses in the $b \to u$ case. Here $m(\bar{q}u)$ is definitely above the π,ρ region and for the two-body decay even above the p-wave band $[A_1, A_2, f_0, B(1235)]$. As a consequence we expect that $P_u(f) < P_c(f')$ where f and f' denote analogous few-body final states involving only the low-lying hadrons.

III. NONLEPTONIC CHANNELS

We now present estimates to the effect of the kinematical differences between $b \to c$ and $b \to u$ on the nonleptonic few-body decay rates of the B meson.

A. Rudimentary parton model

The expectation that D, D^* mesons are a more likely end product of the $b \to c$ transition than the π or ρ mesons are for the $b \to u$ transition is related to the concept of "hard fragmentation" of heavy quarks introduced in Ref. 6. Fast charmed quarks fragment into jets, which typically contain a fast D or D^* meson carrying $\approx 80\%$ of the total energy, whereas u,d quarks fragment into softer leading pions or ρ mesons.

Thus it is more likely for the $c\bar{q}$ "jet" resulting from the $b \to c$ transition to fragment into a D or D^* only, than it is for the $u\bar{q}$ "jet" to fragment into a π or a ρ only.

The same conclusion is reached by considering the infinite-momentum frame distribution functions $c_D(x)[u_\pi(x)]$ for the $c(u)$ valence parton in the $D(\pi)$ meson. For the u-quark distribution (in the π) we take the conventional parametrization

$$u_\pi(x) = 6x(1-x) , \tag{3}$$

which is symmetric under $x \to 1-x$ and peaks at $x = \frac{1}{2}$. In contrast, the c-quark distribution $c_D(x)$ should be peaked near the point x_0 given by the quark mass ratio:

$$x_0 = m_c/(m_c + m_u) \approx 0.8 . \tag{4}$$

An example for $c_D(x)$ is shown in Fig. 2 which has the property that the probability to find $x \le x_0$ is approximately equal to x_0.

We now use the qualitative difference between the distribution functions depicted in Fig. 2 to estimate for the nonleptonic two-body decays the ratio $P_u(\pi\pi)/P_c(\pi D)$ [or $P_u(\pi\rho)/P_c(\pi D^*)$]. Using the boost-invariant quantity (energy + longitudinal momentum) to estimate the x value x' of the "decay" quark which eventually has to fit into

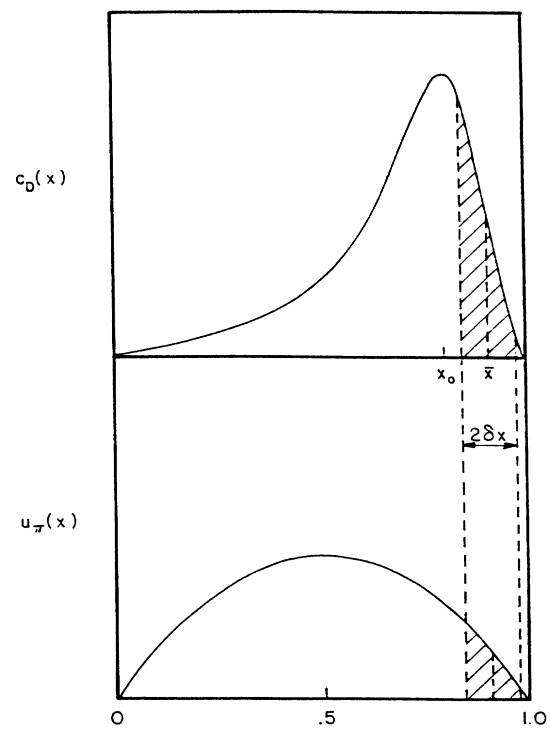

FIG. 2. Typical shapes for, respectively, the c quark distribution inside the D meson and the u-quark distribution inside the pion. For details see Sec. III A.

$c_D(x)$ or $u_\pi(x)$, we get for both $b\to c$ and $b\to u$ that

$$x'=\frac{m_b-k\cos\theta}{m_B} . \tag{5}$$

Here k is a typical momentum of the spectator quark with polar angle θ relative to the decay quark. The angle is assumed to be distributed uniformly in $\cos\theta$. Thus, x' is uniformly distributed in the interval $(\bar{x}-\delta x,\ \bar{x}+\delta x)$, where

$$\bar{x}=\frac{m_b}{m_B}=\frac{4.9}{5.27}=0.93,\quad \delta x=\frac{k}{m_B}=\frac{0.3}{5.27}\approx 0.06 . \tag{6}$$

Clearly the magnitude of $\bar{x}$ makes the formation of a D meson more likely than the formation of a π meson. As a measure for this probability we take the area within $\bar{x}\pm\delta x$ under each of the respective distribution functions. We then get

$$\tfrac{1}{3}\le P_u(\pi\pi)/P_c(\pi D)\le\tfrac{1}{2} ,$$

somewhat depending on the particular form used for c_D but always in this range as long as x_0 approximately measures the probability for x to be below x_0.

B. Potential models

A large suppression of $B\to\pi\pi$ relative to $B\to D\pi$ is indicated in potential models. Let $\Psi_B(\mathbf{r})$, $\Psi_D(\mathbf{r})$, and $\Psi_\pi(\mathbf{r})$ be nonrelativistic wave functions describing the B, D, and π mesons, respectively. The overlap integrals arising for $B\to\pi\pi$ or $B\to D\pi$ are

$$F_{B\to D}=\int d\mathbf{r}\,\overline{\Psi}_D(\mathbf{r})\exp\left[i\frac{m_u}{m_u+m_c}\mathbf{q}_D\cdot\mathbf{r}\right]\Psi_B(\mathbf{r}) ,$$
$$F_{B\to\pi}=\int d\mathbf{r}\,\overline{\Psi}_\pi(\mathbf{r})e^{i(1/2)\mathbf{q}_\pi\cdot\mathbf{r}}\Psi_B(\mathbf{r}) , \tag{7}$$

where $\mathbf{q}_D$ and $\mathbf{q}_\pi$ are the momenta of the decay products in the rest frame of the B meson. The suppression of $F_{(B\to\pi)}$ relative to $F_{(B\to D)}$ stems mainly from the fact that the first form factor is evaluated at a momentum transfer

$$Q_{B\to\pi}=\tfrac{1}{2}q_\pi=1.32\text{ GeV} ,$$

much higher than

$$Q_{B\to D}=\frac{m_u}{m_u+m_c}q_D=0.2q_D\approx 0.46\text{ GeV} .$$

Assuming the pole form $F\approx(Q^2+m_\rho{}^2)^{-1}$ we find

$$\left|\frac{F_{B\to\pi}}{F_{B\to D}}\right|^2\approx\left|\frac{m_\rho{}^2+Q_{B\to D}{}^2}{m_\rho{}^2+Q_{B\to\pi}{}^2}\right|^2\approx 0.1 . \tag{8}$$

A pion of momentum 2.6 GeV is clearly relativistic and this simple approach is questionable. It suggests, however, that $P_u(\pi\pi)$ and $P_c(\pi D)$ can differ by an order of magnitude.

C. Empirical estimates

The small branching for the decay of a massive object into a particular two-body final state has to do with the large number of competing channels. From this point of view the transitions $B\to m_1+m_2$, $\psi\to m_1+m_2$, $\Upsilon\to m_1+m_2$, where m_1 and m_2 are light hadrons, share a common feature. We propose to use empirical information available for these processes to estimate the probability $P_u(f)$ for $f=\pi\pi$ and $\pi\rho$. In the spectator model the $B\to\pi\pi$ and $B\to\rho\pi$ decays proceed through the hadronization of a two-quark–two-antiquark system. We therefore expect comparisons with ψ and Υ decays where the hadrons emerge from a multigluon system to be reasonable.

1. $B\to\rho\pi$ versus $\psi\to\rho\pi,\Upsilon\to\rho\pi$

Let us conjecture that the respective probabilities for the hadronization into $\pi\rho$ derive from a common power law, i.e.,

$$P(\pi\rho)=C_{\pi\rho}\left[\frac{1}{m}\right]^{d_{\pi\rho}} , \tag{9}$$

where m is either the B, the ψ, or the Υ mass. Since $m_B\approx(m_\psi m_\Upsilon)^{1/2}$ the conjecture implies

$$P_u(\pi\rho)\approx[B(\psi\to\rho\pi)B(\Upsilon\to\rho\pi)]^{1/2}\le 10^{-3} , \tag{10}$$

where $B(\Upsilon\to\rho\pi)\le 10^{-4}$ has been used.[7] From the measured branching ratios for $B\to D^*\pi$ or $B\to D\pi\pi$ the probability $P_c(D\rho+D^*\pi)$ is expected to be on the one percent level.[8] The above result for $P_u(\pi\rho)$ therefore implies a suppression by about a factor of 10, when comparing $P_u(\pi\rho)$ to $P_c(D\rho+D^*\pi)$.

Of course, the basis upon which Eq. (9) has been derived is debatable. We cannot solidly underpin the assumption that the constants $c_{\pi\rho}$ and $d_{\pi\rho}$ are roughly the same for the four-quark and the three-gluon systems.

2. Extrapolation from $D\to\pi\pi$ to $B\to\pi\pi$

We are concerned here with the question of how $P_u(\pi\pi)$ scales with the mass of the b quark or the B meson. Once this is settled we can estimate $P_u(\pi\pi)$ by substituting m_B for m_D and using the experimental results about $D\to\pi\pi$.

Since the spectator model diagram for $B\to\pi\pi$ [Fig. 1(b)] includes the extra dimensional coupling f_π, dimensional arguments suggest that the rate for $B\to\pi\pi$ is proportional to $m_B{}^3$ as opposed to the characteristic $m_B{}^5$ behavior of the inclusive rate. One then expects

$$P_u(\pi\pi)\sim 1/m_B{}^2 \tag{11}$$

and

$$P_u(\pi\pi)\approx\frac{m_D{}^2}{m_B{}^2}\frac{1}{|V_{cd}|^2}B(D\to\pi\pi)\approx 0.8\% . \tag{12}$$

To obtain the 0.8% figure, we have taken the experimental value $B(D^0\to\pi^+\pi^-)\approx 0.18\%$ (Ref. 9) and assumed that the $\pi^0\pi^0$ channel contributes an equal amount.

Since the $D\pi$ branching ratios are now below 0.5% (Ref. 8) the estimate (12) does not support the expectation, that $P_u(\pi\pi)$ is suppressed. We believe, however, that Eq. (12) is an overestimate. The reason is that it does not take into account the price to pay for the formation of a single π at the lower branch in Fig. 1(b) instead of a multipion system. Thus the above reasoning applies to the comparison of the fictitious processes $b \to \pi u$ and $c \to \pi d$ rather than the real process with two pions in the final state.

As Fig. 1(b) indicates, this deficiency of the estimate Eq. (12) should be corrected by including the m_B dependence of $F_{B\to\pi}(q^2=0)$, the form factor of the $B \to \pi l \nu$ decay at $q^2=0$. The precise dependence of $F_{B\to\pi}(q^2=0)$ on m_B is not known. The type of considerations which will be made in Sec. IV, which is devoted to the semileptonic decay, indicate that anything between $m_B{}^{-1/2}$ or $m_B{}^{-3/2}$ could be true. Thus an additional suppression of a factor from 3 to 30 should be expected making $P_u(\pi\pi)$ up to 10 times smaller than $P_c(\pi D)$.

IV. SEMILEPTONIC CHANNELS

In the limit of zero lepton masses the $B \to 0^- l\nu$ decay amplitude ($0^- = \pi$ or D), is described by

$$A(B \to 0^- l\nu) = -\sqrt{2} G_F F_{B\to 0^-}(p_B + p_{0^-})_\mu \bar{l} \gamma^\mu \frac{1-\gamma_5}{2} \nu \ , \tag{13}$$

where $F_{B\to 0^-}$ is a form factor depending on

$$q^2 = (p_B - p_{0^-})^2 = m_B{}^2 - 2m_B E_{0^-} + m_{0^-}{}^2 \ . \tag{14}$$

A. B^*-dominance estimate for $B \to \pi l \nu$

Parametrizing $F_{B\to\pi}$ by the $B^*(1^-)$ pole we have

$$F_{B\to\pi}(q^2) = \frac{-\mathrm{Res}}{q^2 - m_{B^*}{}^2} \approx \frac{\mathrm{Res}}{2m_B(E_\pi + \Delta)} \ ,$$
$$\Delta = m_{B^*} - m_B = 50 \text{ MeV} \ . \tag{15}$$

Because of the proximity of the B^* pole Eq. (15) causes a strong variation of $F_{B\to\pi}$ across the physical region. With

$$(m_{B^*}{}^2 - q^2)\,|_{\min} \approx 2m_B(m_\pi + \Delta) \approx 2 \text{ GeV}^2 \ ,$$
$$(m_{B^*}{}^2 - q^2)\,|_{\max} \approx 2m_B \left[\frac{m_B}{2} + \Delta \right] \approx 28 \text{ GeV}^2 \ , \tag{16}$$

$|F_{B\to\pi}|^2$ would decrease by a factor of 200 from the maximal to the minimal q^2. This is to be contrasted with the situation for $B \to D l \nu$. The dominance of the lowest $(b\bar{c})$ (1^-) state only leads to a variation of $|F_{B\to D}|^2$ by a factor of 2. Thus the q^2 dependence of the form factors is of main concern for an estimate of the $b \to u$ transition, but of relatively modest importance for the $b \to c$ case.

The residue in Eq. (15) is the product of the B^*-vector-current coupling conventionally written as $m_{B^*}{}^2/g_{B^*}$ and the $B^*B\pi$ coupling, i.e.,

$$\mathrm{Res} = \frac{1}{\sqrt{2}} \frac{m_{B^*}{}^2}{g_{B^*}} g_{B^*B\pi} \ . \tag{17}$$

Like the B-decay constant f_B, the B^*-vector-current coupling $m_{B^*}{}^2/g_{B^*}$ involves the spatial wave function at the origin times an appropriate power of mass. Assuming the spatial wave functions of B^* and B to be equal (at the origin) we can write

$$\frac{m_{B^*}{}^2}{g_{B^*}} \approx (m_{B^*} m_B)^{1/2} f_B \ . \tag{18}$$

The decay constant f_B in turn is estimated by assuming that the behavior $f_B \sim m_B{}^{-1/2}$, which is expected to hold in the heavy-quark limit, remains valid in an extrapolation down to the s-quark mass. This yields

$$f_B \approx \left[\frac{m_K}{m_B} \right]^{1/2} f_K \ . \tag{19}$$

For the $B^*B\pi$ coupling we resort to a PCAC (partial conservation of axial-vector current) consideration similar to the one leading to the Goldberger-Treiman relation, with the B and B^* mesons replacing the nucleons. From this we find

$$g_{B^*B\pi} \approx \frac{m_b}{f_\pi} \ . \tag{20}$$

This result also has some experimental support. The measured branching ratio of $D^* \to D\pi$ can be converted into an absolute value for $\Gamma(D^* \to D\pi)$ if we assume that the electromagnetic transition rate can be calculated reliably.[10] The width then yields a value for $g_{D^*D\pi}$ which is near to the prediction m_c/f_π.

Combining Eqs. (17)–(20) we arrive at

$$\mathrm{Res} \approx \frac{1}{\sqrt{2}} (m_{B^*} m_K)^{1/2} m_B \frac{f_K}{f_\pi} \approx 7.3 \text{ GeV}^2 \ , \tag{21}$$

where $f_K = 158$ MeV and $f_\pi = 131$ MeV have been used. With these specifications the B^* dominance model yields[11]

$$\frac{\Gamma(\bar{B}^0 \to \pi^+ l\nu)}{\Gamma(b \to u l \nu)} = 2 \frac{\Gamma(B^- \to \pi^0 l \nu)}{\Gamma(b \to u l \nu)} \approx 3\% \ . \tag{22}$$

The lepton spectrum observed in B decays is well described by assuming that the $b \to c$ transition essentially leads to the formation of a D or D^* meson only, with proportions ranging between 1:2 and 1:3 (Ref. 12). The 3% figure for the formation of a pion therefore means a substantial suppression of $P_u(\pi l \nu)$ relative to $P_c(D l \nu)$.

We stress that due to the uncertainties involved, the above calculation can only be considered as providing an estimate for the suppression rather than definitively establishing its magnitude. In particular, the assumption that the q^2 dependence is dominated by the B^* pole all the way down to $q^2=0$ may be criticized. Note, however, that the model has the feature that $F_{B\to\pi}$ at $q^2=0$ behaves like $m_B{}^{-1/2}$ as a function of m_B. This is a weaker decrease than we find otherwise. The consideration where the spectator quark is made collinear to the u quark by an exchange of a hard gluon or a consideration based on the properties of the infinite-momentum-frame wave functions for the B and π (see the next section) most naturally lead to powers like m_B^{-1} or $m_B^{-3/2}$.

B. Estimating $B \to Dl\nu$, $B \to \pi l\nu$ from normalization at maximal q^2

Most calculations of exclusive semileptonic decays (e.g., Ref. 4), rely on an estimate of the form factors at $q^2=0$. At $q^2=0$ the lepton and neutrino momenta are parallel and the $D(\pi)$ is recoiling with maximal energy.

Instead of $q^2=0$ we prefer to work at $q^2=(m_B-m_D)^2$ or $q^2=(m_B-m_\pi)^2$, respectively, where no recoil is imparted to the final hadron (D or π) in the B meson rest frame.

Consider first $B \to Dl\nu$ where both hadrons contain a massive quark. Since $m_B-m_D \approx m_b-m_c$, the decay at $q^2=(m_B-m_D)^2$ can be viewed as transforming the b quark in the B meson into a c quark without changing its momentum. Consequently gluon emission from the heavy quark is not likely to occur. Also, since the remaining components in the wave function (the spectator quark and possibly gluons or $q\bar{q}$ pairs) see the same color source before and after the weak transition very little adjustment is necessary for the formation of the D meson. The form factor at $q^2=(m_B-m_D)^2$ is therefore expected to have a value almost independent of dynamical details.

Strictly speaking, the B and D mesons contain quarks, antiquarks, and/or gluons with Fourier components $\mathbf{k}$ in the range $m_b \geq k \geq m_c$, which can probe the difference between the initial b and the final c quark. Since for such k values the strong coupling constant $\alpha_{\rm QCD}$ is small, these effects are calculable perturbatively. They amount to changing the bare vector-current vertex γ_μ into the effective vertex

$$\Gamma_\mu = F_1(q^2)\gamma_\mu - iF_2(q^2)\frac{\sigma_{\mu\nu}q^\nu}{m_b+m_c} - F_3(q^2)q_\mu \; . \tag{23}$$

To one loop the form factors F_i at $q^2=(m_b-m_c)^2$ are (F_3 does not contribute in the limit $m_l=0$)

$$F_1 = 1+\frac{\alpha_{\rm QCD}(\mu)}{3\pi}\left[-4+2\frac{m_b+m_c}{m_b-m_c}\ln\frac{m_b}{m_c}\right] ,$$
$$F_2 = \frac{\alpha_{\rm QCD}(\mu)}{3\pi}\frac{m_b+m_c}{m_b-m_c}\ln\frac{m_b}{m_c} \; . \tag{24}$$

These results were first given in Ref. 13 and reproduced by us in an independent calculation. Taking $\alpha_{\rm QCD}=0.2$ one gets $F_1=1.008$ and $F_2=0.05$, which lead to a short distance correction of 2% in the form factor $F_{B\to D}$. Since this correction is small we will not explicitly display it in the following.

We now state our main result about the value of the form factor $F_{B\to D}$ at $q^2=(m_B-m_D)^2$ which is

$$F_{B\to D}(q^2=(m_B-m_D)^2) \approx \frac{1}{2}\frac{m_B+m_D}{(m_B m_D)^{1/2}} \; . \tag{25}$$

In light of the preceding remarks this result is expected to have the status of a theorem in the hypothetical world in which m_B and m_D (i.e., m_b and m_c) can be made sufficiently large. We have derived Eq. (25) in two ways which will be outlined below.

The first derivation is in the realm of nonrelativistic quantum mechanics. This may seem inappropriate so far as the spectator quark is concerned, but since only very general properties are playing a role we feel justified to do so. By a straightforward calculation, one finds

$$F_{B\to D}(q^2=(m_B-m_D)^2) = \frac{1}{2}\frac{m_B+m_D}{(m_B m_D)^{1/2}}\left[\langle \Psi_B \mid \Psi_D\rangle + O\left[\frac{m_u}{m_b},\frac{m_u}{m_c}\right]\right] , \tag{26}$$

where $\langle \Psi_B \mid \Psi_D\rangle$ is the overlap integral between the bound-state wave functions of the B and D meson. Because of flavor symmetry the overlap is complete, i.e., $\langle \Psi_B \mid \Psi_D\rangle=1$ in the limit where the b- and c-quark masses become large (keeping $m_{\bar{q}}$ fixed). The wave functions only depend on the reduced masses which in this limit become equal to the common spectator mass. If the deviation of the ratio of reduced masses from 1 is ϵ, the correction to $\langle \Psi_B \mid \Psi_D\rangle=1$ is found to be quadratic in ϵ. For the case at hand with $m_u=0.35$ GeV, $m_c=1.5$ GeV, and $m_b=4.9$ GeV, ϵ is of the order of 0.2. Therefore, even for the modest value of the c-quark mass $\langle \Psi_B \mid \Psi_D\rangle$ can deviate from 1 by at most a few percent only.[14]

The second derivation uses the infinite-momentum-frame formalism of quantum field theory. Hence, no restriction to a nonrelativistic motion of the spectator quark or on the degree of compositeness of the B and D meson is made here.

One starts out describing the mesons by its various Fock-space components such as

$$\mid \Psi_B\rangle = \{\Psi_B^{(2)}(x,q), \Psi_B^{(3)}(x_1,x_2,q_1,q_2), \ldots\} \; . \tag{27}$$

Here $\Psi_B^{(2)}$ is the $b\bar{q}$ wave function with x denoting the longitudinal-momentum fraction carried by the spectator quark and q being a relative transverse momentum. Similarly $\Psi_B^{(3)}$ is the $b\bar{q}g$ component with longitudinal-momentum fractions x_1,x_2 for, respectively, the spectator and the gluon, etc. By considering a particular component of the $b\to c$ vector current one then derives a representation for $F_{B\to D}(q^2)$ in terms of overlap integrals between the Fock components of the B and D meson [analog of Eq. (26)]. This representation yields the result of Eq. (25) if the Fock-space components for $m=m_b(m_c)\to\infty$ have the following two properties: (i) scaling,

$$\Psi_m^{(i)}(x_1,\ldots,q_1,\ldots) \approx m^{(i-1)/2}\Psi^{(i)}(mx_1,\ldots,q_1,\ldots) \; ; \tag{28a}$$

(ii) $x \leq O(1/m)$,

$$\lim_{m\to\infty}\int_0^m d\xi_1\cdots\int dq_1\cdots \mid \Psi^{(i)}(\xi_1,\ldots,q_1,\ldots)\mid^2 = \text{finite} \; . \tag{28b}$$

The scaling property means that for large m the various components reduce to universal functions $\Psi^{(i)}$ dependent on m only through the momentum xm. The explicit m dependence in front of $\Psi^{(i)}$ is dictated by the property (28b) which guarantees that the light constituents only carry momentum fractions of order $1/m$. In fact, without this feature the scaling property would not make sense because factors such as $(1-x)$ would invalidate it.

At present we cannot give a rigorous justification for either (28a) or (28b). They are, however, natural assumptions as they substitute for the property of the nonrelativistic wave function to become independent of the heavy mass and to fall off sufficiently fast in momentum space.

Knowing the normalization of $F_{B\rightarrow D}$ at $q^2=(m_B-m_D)^2$ we can now determine the semileptonic rate $B\rightarrow Dl\nu$. With a q^2 dependence of $F_{B\rightarrow D}$ as given by the dominance of the lowest $(b\bar{c})\,(1^-)$ state we get

$$\frac{\Gamma(B\rightarrow Dl\nu)}{\Gamma(b\rightarrow cl\nu)}\approx 20\% \ . \tag{29}$$

It was pointed out in Ref. 13 that the similarity between the nonrelativistic wave functions for the D and D^* mesons allows us to relate the $B\rightarrow D^*l\nu$ form factors at

$$q^2=(m_B-m_D^*)^2\approx(m_B-m_D)^2$$

to $F_{B\rightarrow D}$. Assuming a common q^2 dependence for the form factors appearing in either $B\rightarrow Dl\nu$ and $B\rightarrow D^*l\nu$ they obtained

$$\frac{\Gamma(B\rightarrow D^*l\nu)}{\Gamma(B\rightarrow Dl\nu)}\approx 2 \ . \tag{30}$$

Both (29) and (30) are consistent with the experiment.

To gain some idea on the corresponding quantities for $b\rightarrow u$ we take the seemingly bold step and utilize Eq. (25) also for the $b\rightarrow u$ transition (m_π substituting for m_D). This extrapolation does not look so dramatic when the size of the short-distance corrections and the lack of complete overlap between Ψ_B and Ψ_π are considered. For the limiting cases of a harmonic or a Coulomb potential, $\langle\Psi_B\,|\,\Psi_\pi\rangle$ is down from 1 by 25% only. The short-distance correction for $\alpha_{\rm QCD}\approx 1/2$ has about the same magnitude but goes into the opposite direction. Numerically, the estimate

$$F_{B\rightarrow\pi}(q^2=(m_B-m_\pi)^2)\approx\frac{1}{2}\left[\frac{m_B}{m_\pi}\right]^{1/2} \tag{31}$$

supplemented by a q^2 dependence as given by B^* dominance yields

$$\frac{\Gamma(\bar{B}^0\rightarrow\pi^+l\nu)}{\Gamma(b\rightarrow ul\nu)}=2\frac{\Gamma(B^-\rightarrow\pi^0l\nu)}{\Gamma(b\rightarrow ul\nu)}\approx 2.5\% \ . \tag{32}$$

The fact that essentially the result in Eq. (22) is reproduced can be understood from the following approximate equalities:

$$\frac{\text{Res}}{m_B{}^2-(m_B-m_\pi)^2}\approx\frac{1}{2}\left[\frac{m_B}{m_\pi}\right]^{1/2}\left[\frac{m_K}{2m_\pi}\right]^{1/2}\frac{f_K}{f_\pi}\approx\frac{1}{2}\left[\frac{m_B}{m_\pi}\right]^{1/2} . \tag{33}$$

In the spirit of extending $b\rightarrow c$ results to $b\rightarrow u$ transitions we may also assume the relationship between the D and D^* rates to be extendable to the π-ρ case. Using the expression of Ref. 13 with m_u substituting for m_c one obtains a ρ to π ratio which is near 1. In conjunction with the above figure for the π rate, this result would imply that $P_u(\rho l\nu)$ is disfavored by about one order of magnitude compared to $P_c(D^*l\nu)$. Closer examination of the model shows that the near equality between the π and ρ rates depends critically on the fact, that a common u-quark mass $m_u=0.35$ GeV has been used for both π and ρ. To obtain an idea about how sensitive the result is to this assumption we have used the expressions of Ref. 13 with the physical masses m_B, m_ρ, m_π appropriately substituted for the quark masses. This causes the ρ to π ratio to increase by about a factor of 10 thereby making $P_u(\rho l\nu)$ almost comparable to $P_c(D^*l\nu)$.

The general conclusion which we draw from our investigation of form-factor models is that the semileptonic branching into π,ρ is subject to large uncertainties from the form factors. The π rate seems to be suppressed but the situation is less clear for the ρ where three different form factors play a role, whose magnitude and q^2 dependence are only partially known.

The properties laid out in Eq. (28) have direct bearing on the m_B dependence of $F_{B\rightarrow\pi}$ at either $q^2=(m_B-m_\pi)^2$ or $q^2=0$. The typical overlap integrals at these values of q^2 are

$$F_{B\rightarrow\pi}((m_B-m_\pi)^2)=\int dx\,dq\,\bar{\Psi}_\pi^{(2)}(x,q)\times\Psi_B^{(2)}\left[x\frac{m_\pi}{m_B},q\right]+\cdots , \tag{34a}$$

$$F_{B\rightarrow\pi}(0)=\int dx\,dq\,\bar{\Psi}_\pi^{(2)}(x,q)\Psi_B^{(2)}(x,q)+\cdots . \tag{34b}$$

Using (28) one immediately recognizes that (34a) is proportional to $m_B{}^{1/2}$ confirming therefore the m_B behavior from Eq. (31). It also becomes obvious that for (34b) the important region is near $x=0$. Assuming that $\Psi_B^{(2)}$ satisfies an integrability condition analogous to the square-integrability condition of (28b) and assuming that near $x=0$ $\Psi_\pi^{(2)}$ is proportional to x^{d_π} one concludes that

$$F_{B\rightarrow\pi}(q^2=0)\sim 1/m_B^{\,d_\pi+1/2} \ . \tag{35}$$

The parton distribution function u_π from Fig. 2 suggests $d_\pi=\frac{1}{2}$ and, thus, $F_{B\rightarrow\pi}(0)\sim m_B{}^{-1}$. It is also conceivable that $\Psi_\pi(x)$ vanishes linearly at $x=0$ resulting in $F_{B\rightarrow\pi}(0)\sim m_B{}^{-3/2}$.

V. SUMMARY

We have used a variety of approaches to estimate how the kinematic differences between $b\rightarrow u$ and $b\rightarrow c$ transitions affect the branchings of the B meson into few-body final states. Information of this kind has important bearing on the extraction of the KM ratio $r=|V_{bu}|^2/|V_{cu}|^2$ from measurements of exclusive B decays. Since exclusive processes depend on long-distance physics which is not easily brought under control, a phenomenological attitude had to be taken. Some of our esti-

TABLE I. Summary of our estimates for $P_u(f)/P_c(f')$ where $P_u(f)$ and $P_c(f')$ are the probabilities, that a $b \to u$ or $b \to c$ transition leads to a particular final state f or f'. The approach used for the individual estimates is indicated by keywords.

f/f'	$P_u(f)/P_c(f')$	Type of consideration
$\pi\pi/\pi D$	4×10^{-1}	Rudimentary parton model
$\pi\pi/\pi D$	10^{-1}	Potential model
$\pi\rho/(D\rho+D^*\pi)$	10^{-1}	Interpolation of $\psi,\Upsilon\to\pi\rho$
$\pi\pi/\pi D$	10^{-1} to 1	Extrapolation of $B(D\to\pi\pi)$
$\pi l\nu/Dl\nu$	2×10^{-1}	B^* dominance
$\pi l\nu/Dl\nu$	10^{-1}	Normalization at maximal q^2
$\rho l\nu/D^* l\nu$	10^{-1} to 5×10^{-1}	Normalization at maximal q^2

mates are rather *ad hoc;* some rely on extrapolating a functional dependence into regions where it need no longer be valid. In spite of this we believe to have provided some evidence that the exclusive few-body decays offer little chance to improve the current upper bound for r derived from the semileptonic lepton spectrum. As can be seen from Table I the nonleptonic and semileptonic few-body channels for $b\to u$ transitions are disfavored by almost one order of magnitude compared to the corresponding $b\to c$ decay channels.

While this work was underway, two comprehensive studies of exclusive B decays have appeared in the literature.[15,16] Reference 15 uses the relativistic formalism of the infinite-momentum frame supplemented by a particular model for the two-particle component of the wave functions in Eq. (27), whereas Ref. 16 relies on a nonrelativistic picture with wave functions of Gaussian type. As far as the semileptonic rates are concerned there is overall agreement on the order of magnitude, but individual numbers from either Refs. 15, 16, or this paper can differ by factors of 2. A systematic feature is that the results of Ref. 15 exhibit less of a suppression for the $b\to u$ transitions than found either in Ref. 16 or by us.

In our opinion, the spread between the individual results has to be viewed as a measure of the precision to which these rates can be calculated at present.

Evidently, the various theoretical estimates will not discourage the efforts to find a clear exclusive signal for $b\to u$ transitions. All that the present work suggests is that when such a signal is found, the $b\to u$ mixing $V_{bu}{}^2$ is in fact at least twice and most likely even 5–10 times larger than the value that a more naive estimate neglecting the dynamic $(b\to u/b\to c)_{\text{exclusive}}$ suppression may suggest.

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