Comment on "Coherent states for the time-dependent harmonic oscillator"

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It is shown that the exact coherent states for the time-dependent harmonic oscillator constructed by Hartley and Ray are equivalent to the well-known squeezed states.

Some time ago Hartley and Ray,[1] in a very interesting paper, constructed exact coherent states for the time-dependent harmonic oscillator by making use of the Lewis-Riesenfeld quantum theory for the time-dependent harmonic oscillator.[2] According to Hartley and Ray, these "new" coherent states have all the properties of the coherent states for the time-independent oscillator[3] except that the uncertainty product of position and momentum is not minimum; i.e., they are not minimum-uncertainty states. The purpose of the present paper is to show that the "new" coherent states constructed by these authors are equivalent to the well-known squeezed states.[4-6]

In what follows I present a brief summary of Ref. 1. Consider the time-dependent harmonic oscillator

$$H(t)=\frac{p^2}{2}+\frac{\omega^2(t)}{2}q^2 \; , \tag{1}$$

where q,p are canonically conjugate and $\omega(t)$ is the time-dependent harmonic-oscillator frequency. An invariant for the Hamiltonian (1) is given by[1,2]

$$I(t)=\tfrac{1}{2}[(\rho p-\dot{\rho}q)^2+(q/\rho)^2] \; , \tag{2}$$

where $q(t)$ satisfies the harmonic-oscillator equation

$$\ddot{q}+\omega^2(t)q=0 \tag{3}$$

and $\rho(t)$ is any solution of the auxiliary equation

$$\ddot{\rho}+\omega^2(t)\rho=1/\rho^3 \; . \tag{4}$$

Now, by using the time-dependent operators[1,2]

$$b(t)=\left[\frac{1}{2\hbar}\right]^{1/2}[q/\rho+i(\rho p-\dot{\rho}q)] \; , \tag{5}$$

$$b^\dagger(t)=\left[\frac{1}{2\hbar}\right]^{1/2}[q/\rho-i(\rho p-\dot{\rho}q)] \; , \tag{6}$$

we can rewrite the invariant (2) as

$$I=\hbar[b^\dagger(t)b(t)+\tfrac{1}{2}] \; . \tag{7}$$

The operators $b(t)$ and $b^\dagger(t)$ have the properties

$$[b(t),b^\dagger(t)]=1 \; , \tag{8}$$

$$b(t)\,|\,n,t\,\rangle=n^{1/2}\,|\,n-1,t\,\rangle \; , \tag{9}$$

$$b^\dagger(t)\,|\,n,t\,\rangle=(n+1)^{1/2}\,|\,n+1,t\,\rangle \; , \tag{10}$$

where the states $|\,n,t\,\rangle$ are eigenstates of the invariant I, i.e.,

$$I\,|\,n,t\,\rangle=\hbar(n+\tfrac{1}{2})\,|\,n,t\,\rangle \; . \tag{11}$$

Then, using the Lewis-Riesenfeld theory,[2] Hartley and Ray constructed coherent states for the time-dependent oscillator (1). These states are given by[1]

$$|\,\alpha,t\,\rangle_S=e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}e^{i\alpha_n(t)}\,|\,n,t\,\rangle \; , \tag{12}$$

where α is a complex number and the phase functions $\alpha_n(t)$ are given by

$$\alpha_n(t)=-(n+\tfrac{1}{2})\int_0^t\frac{dt'}{\rho^2(t')} \; . \tag{13}$$

The subscript S in (12) indicates that the states evolve in time according to the Schrödinger equation.

The coherent states $|\,\alpha,t\,\rangle_S$ are eigenstates of the operator $b(t)$ with eigenvalue $\alpha(t)$:

$$b(t)\,|\,\alpha,t\,\rangle_S=\alpha(t)\,|\,\alpha,t\,\rangle_S \; , \tag{14}$$

where

$$\alpha(t)=\alpha e^{2i\alpha_0(t)} \; , \tag{15}$$

$$\alpha_0(t)=-\tfrac{1}{2}\int_0^t\frac{dt'}{\rho^2(t')} \; . \tag{16}$$

Now, by calculating the uncertainty in q and p in the state $|\,\alpha,t\,\rangle_S$, one finds

$$(\Delta q)^2=\frac{\hbar}{2}\rho^2 \; , \tag{17}$$

$$(\Delta p)^2=\frac{\hbar}{2}(\dot{\rho}^2+1/\rho^2) \; . \tag{18}$$

So, the uncertainty relation is expressed as

$$(\Delta q)(\Delta p)=\frac{\hbar}{2}(\dot{\rho}^2\rho^2+1)^{1/2} \; , \tag{19}$$

and, in general, does not attain its minimum value.

As remarked by Hartley and Ray, all of their results for the Hamiltonian (1) reduce to the usual time-independent oscillator in the limit $\omega(t)\to\omega_0=\text{const}$ if we take the particular solution

$$\rho=\frac{1}{\omega_0^{1/2}} \; , \quad \omega(t)=\omega_0 \; , \tag{20}$$

for the auxiliary equation (4). In fact, in this case the operator $b(t)$ transforms into the ordinary annihilation operator for the time-independent oscillator [see Eq. (21) below]. Also, observe that relation (19) reduces to

$\Delta q \Delta p = \hbar/2$ as it should be.

Next we show that the coherent states $|\alpha,t\rangle_S$ are equivalent to the squeezed states. Consider the usual annihilation and creation operators for the time-independent oscillator defined by

$$a = \left[\frac{1}{2\hbar\omega_0}\right]^{1/2} (\omega_0 q + ip) , \quad (21)$$

$$a^\dagger = \left[\frac{1}{2\hbar\omega_0}\right]^{1/2} (\omega_0 q - ip) , \quad (22)$$

where $[a,a^\dagger]=1$. Then, using (21) and (22) we can write the operators $b(t)$ and $b^\dagger(t)$ in terms of a and $a^\dagger$ as

$$b(t)=\mu(t)a+\nu(t)a^\dagger , \quad (23)$$

$$b^\dagger(t)=\mu^*(t)a^\dagger+\nu^*(t)a , \quad (24)$$

where

$$\mu(t)=\left[\frac{1}{4\omega_0}\right]^{1/2}\left[\frac{1}{\rho}+\omega_0\rho-i\dot{\rho}\right] , \quad (25)$$

$$\nu(t)=\left[\frac{1}{4\omega_0}\right]^{1/2}\left[\frac{1}{\rho}-\omega_0\rho-i\dot{\rho}\right] . \quad (26)$$

Also, a straightforward calculation shows that the complex c numbers $\mu(t)$ and $\nu(t)$ satisfy the relation

$$|\mu|^2-|\nu|^2=1 . \quad (27)$$

Thus, from (14), (23), and (27) we see that the "new" coherent states $|\alpha,t\rangle_S$ constructed by Hartley and Ray,[1] by definition, are equal to the well-known squeezed states.[4-6] The properties of these states have been studied in detail by Yuen.[4] Here, we remark that for $\mu=1$ and $\nu=0$, $b(t)$ reduces to the annihilation operator a. Observe that this condition agrees with Eq. (20).

On the other hand, Hartley and Ray have pointed out that their states are not minimum-uncertainty states; i.e., they do not minimize the uncertainty relation (19). However, it is known that the uncertainty in q and p for a squeezed state is given by[4]

$$(\Delta q)^2=\frac{\hbar}{2\omega_0}|\mu-\nu|^2 , \quad (28)$$

$$(\Delta p)^2=\frac{\hbar\omega_0}{2}|\mu+\nu|^2 , \quad (29)$$

whence

$$(\Delta q)(\Delta p)=\frac{\hbar}{2}|\mu+\nu|\,|\mu-\nu| . \quad (30)$$

The uncertainty product (30) is minimized if $\mu=\delta\nu$, for δ real (see Ref. 4). Note that the relation (30) is equivalent to Eq. (19).

Therefore, from the arguments presented above we see that the "new" coherent states for the time-dependent harmonic oscillator constructed by Hartley and Ray[1] are equivalent to the well-known squeezed states.[4-7]

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[7]In quantum optics these states are also known as two-photon coherent states (see Refs. 4 and 6).