Green's function for the scalar field in the early Universe

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We derive the thermal Green's function for the scalar field in a de Sitter space-time and apply it to the problem of the early Universe. Field fluctuations relevant for inflation arise predominantly from wavelengths of the order of the inverse Hubble constant. Sufficient inflation is obtained in a Coleman-Weinberg model, provided the coupling constant is small enough. The results are insensitive to the choice of the vacuum of the field theory.

The new inflationary universe model, arising out of attempts to solve the major problems facing the original ideas of the old version,[1-3] appears to provide novel solutions to a number of outstanding difficulties of standard cosmology. Notable among these are the horizon problem, the flatness problem, and the primordial magnetic monopole problem.[4]

There are, however, some crucial problems which remain to be clarified before we can have a completely satisfactory treatment of the inflationary period. The important question is to find conditions for a field theory under which the inflationary scenario occurs and to ascertain whether such conditions are indeed realized in grand unified theories (GUT's). A related problem is to find the correct initial state. Although the assumption of thermal equilibrium at the beginning of de Sitter expansion seems likely, there are other plausible initial states as well.

The basic quantity of interest is the two-point Green's function for the scalar field, with appropriate initial conditions, which measures the quantum and thermal fluctuations.[2,5] In this note we obtain the free Green's function assuming an initial thermal equilibrium condition following the elegant real-time formalism.[6,7] This is the starting point of any perturbative treatment.

The thermal Green's function for the physical field, to the lowest order in the coupling constant, consists of two terms: the usual zero-temperature ($T=0$) expression and the $T\neq 0$ contribution. Each of the terms can be infrared singular, depending on the choice of the vacuum. The singularities cancel out mutually, independently of the definition of the vacuum. In the Coleman-Weinberg model,[8,2] we find that field fluctuations with a wavelength of the order of the inverse Hubble constant are most relevant for inflation. Sufficient inflation results in this class of theories with an upper bound on the coupling constant.

Consider a scalar field $\phi(x)$ with the action

$$S=\int d^4x\sqrt{-g}\left[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\tfrac{1}{2}(m_0{}^2+\xi R)\phi^2-\frac{\lambda}{4}\phi^4-V(0)\right]. \tag{1}$$

Here ξ represents a coupling between the scalar field and the gravitational field. $R(x)$ is the scalar curvature. $V(0)$ denotes the "false" (symmetric) vacuum energy density and is to be chosen to obtain a zero cosmological constant in the "true" (broken-symmetric) vacuum.

As usual we assume a Robertson-Walker universe, which is taken spatially flat for convenience:

$$ds^2=dt^2-a^2(t)d\mathbf{x}^2 . \tag{2}$$

The energy density ρ at temperature T is given by

$$\rho=\frac{\pi^2}{30}NT^4+V(0) , \tag{3}$$

with $N=N_b+\frac{7}{8}N_f$, $N_{b(f)}$ being the number of light (mass $\ll T$) bosons (fermions). The Einstein equation becomes

$$\left[\frac{\dot a}{a}\right]^2=\gamma^2GT^4+H^2 , \tag{4}$$

where

$$\gamma^2=\frac{4\pi^3N}{45} , \quad H^2=\frac{8\pi GV(0)}{3} . \tag{5}$$

The Universe starts undergoing a de Sitter expansion at a time t_0 when the false-vacuum energy exceeds the thermal energy density, the corresponding temperature T_0 being

$$T_0\simeq(M_PH/\gamma)^{1/2} , \quad G=M_P{}^{-2} , \tag{6}$$

and we get

$$a(t)=e^{Ht} . \tag{7}$$

Below we consider the evolution of the scalar field from t_0 until the time the scalar field has not shifted appreciably from its false-vacuum configuration.

To obtain the thermal Green's function to lowest order in λ, we consider the terms quadratic in the field $\phi(x)$ only. We now work in arbitrary $(D-1)$ spatial dimensions rather than in 3. It will serve as a regularization parameter for a divergent integral. It is well known[9,10] that the quadratic terms in S can be set in a flat (Minkowski) space form. One introduces the so-called conformal time η by

$$d\eta = \frac{dt}{a(t)}\ , \qquad (8)$$

so that

$$H\eta = -e^{-Ht}\ . \qquad (9)$$

The metric (2) then becomes

$$ds^2 = \frac{1}{(H\eta)^2}(d\eta^2 - d\mathbf{x}^2)\ . \qquad (10)$$

Define

$$\psi(x) = (H\eta)^{(2-D)/2}\phi(x)\ , \qquad (11)$$

and the action for the free Lagrangian becomes

$$S_0 = \int d^{D-1}\mathbf{x}\, d\eta \tfrac{1}{2}\psi \left[-\Box - \left[\frac{m^2}{H^2} - \frac{D(D-2)}{4} \right] \Big/ \eta^2 \right] \psi\ , \qquad (12)$$

where

$$m^2 = m_0{}^2 + \xi D(D-1)H^2\ . \qquad (13)$$

The path-integral formalism for the partition function in the absence of interaction ($\lambda=0$) and in presence of a source $j(x)$ is[6,7]

$$Z[j] = N \int \mathscr{D}\psi_c \exp\left[i \int d^{D-1}\mathbf{x}\, d\eta \tfrac{1}{2} \left\{ \psi \left[-\Box - \left[\frac{m^2}{H^2} - \frac{D(D-2)}{4} \right] \Big/ \eta^2 \right] \psi + j\psi \right\} \right]\ , \qquad (14)$$

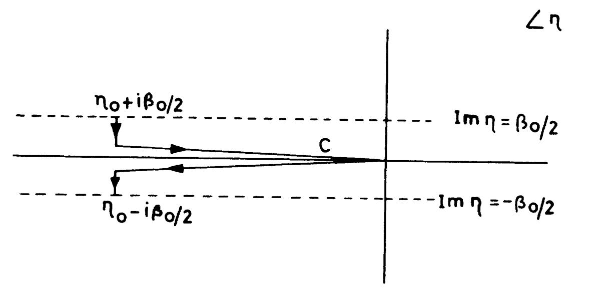

FIG. 1. The contour c in the η plane. The dashed lines mark the domain of analyticity of the Green's functions.

where the η integration runs over the contour c shown in Fig. 1. With this choice of contour,[10] $Z[j]$ generates real, analytic n-point functions, also when the Hamiltonian is not constant in time, as is the case here. To obtain the thermal Green's function, we require the path integral to go only over all periodic paths:

$$\psi(\mathbf{x}, \eta_0 + \tfrac{1}{2}i\beta_0) = \psi(\mathbf{x}, \eta_0 - \tfrac{1}{2}i\beta_0)\ . \qquad (15)$$

In this formalism the system is assumed to be in thermal equilibrium at time η_0 at a temperature $T_0 = H\eta_0/\beta_0$. Here and below the subscript c denotes functions on the contour.[6]

The two-point Green's function satisfies

$$\left[-\Box - \left[\frac{m^2}{H^2} - \frac{D(D-2)}{4} \right] \Big/ \eta^2 \right] G^\beta(x,x') = \delta^{D-1}(\mathbf{x}-\mathbf{x}')\delta_c(\eta-\eta')\ . \qquad (16)$$

Because of spatial translation invariance

$$G^\beta(x,x') = \int \frac{d^{D-1}\mathbf{x}}{(2\pi)^{D-1}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} G_k^\beta(\eta,\eta')\ , \qquad (17)$$

where $G_k^\beta(\eta,\eta')$ satisfies

$$\left[-\frac{d^2}{d\eta^2} - \mathbf{k}^2 - \left[\frac{m^2}{H^2} - \frac{D(D-2)}{4} \right] \Big/ \eta^2 \right] G_k^\beta(\eta,\eta') = \delta_c(\eta-\eta')\ . \qquad (18)$$

Writing

$$G^\beta(x,x') = G^{\beta>}(x,x')\theta_c(\eta-\eta') + G^{\beta<}(\eta,\eta')\theta_c(\eta'-\eta)\ , \qquad (19)$$

the boundary condition (15) leads to the familiar Kubo-Martin-Schwinger periodicity condition:

$$G^{\beta>}\left[\eta_0 - \frac{i}{2}\beta_0, \eta'; \mathbf{x}-\mathbf{x}' \right] = G^{\beta<}\left[\eta_0 + \frac{i}{2}\beta_0, \eta'; \mathbf{x}-\mathbf{x}' \right]\ . \qquad (20)$$

To construct the Green's function, we need the solutions of the homogeneous equation

$$\left[\frac{d^2}{d\eta^2} + \mathbf{k}^2 + (\tfrac{1}{4} - \nu^2)/\eta^2 \right] f(\eta) = 0\ ,$$
$$\nu^2 = \left[\frac{D-1}{2} \right]^2 - \frac{m^2}{H^2}\ . \qquad (21)$$

A set of two linearly independent solutions are $(k\eta)^{1/2}H_\nu^{(1)}(k\eta)$ and $(k\eta)^{1/2}H_\nu^{(2)}(k\eta)$ where $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are the Hankel functions of the first and second kind.[11] However, to leave the choice of the vacuum state of the field theory open,[12] we consider the set

$$f_k^{(1)}(\eta) = (\pi k\eta/2)^{1/2}[c_1^*(k)H_\nu^{(1)}(k\eta) + c_2^*(k)H_\nu^{(2)}(k\eta)]\ , \qquad (22)$$

and its complex conjugate

$$f_k^{(2)}(\eta) = (\pi k\eta/2)^{1/2}[c_1^*(k)H_\nu^{(2)}(k\eta) + c_2^*(k)H_\nu^{(1)}(k\eta)]\ . \qquad (23)$$

Each choice of the functions $c_1(k)$ and $c_2(k)$ subject to

$$|c_1|^2 - |c_2|^2 = 1 \qquad (24)$$

gives rise to a different vacuum. $G_k^\beta(\eta,\eta')$ is now given by[13]

$$G_k^\beta(\eta,\eta') = \frac{1}{2ik}[f_k^{(2)}(\eta)f_k^{(1)}(\eta')\theta_c(\eta-\eta') + f_k^{(2)}(\eta')f_k^{(1)}(\eta)\theta_c(\eta'-\eta) + A_k f_k^{(2)}(\eta)f_k^{(1)}(\eta') + B_k f_k^{(1)}(\eta)f_k^{(2)}(\eta')]\ , \qquad (25)$$

where

$$A_k = B_k^* = f_k^{(2)}\left[\eta_0 - i\frac{\beta_0}{2}\right] \Big/ \left[f_k^{(2)}\left[\eta_0 + i\frac{\beta_0}{2}\right] - f_k^{(2)}\left[\eta_0 - i\frac{\beta_0}{2}\right]\right] . \tag{26}$$

The Gibbs ensemble average of the field operators is related to the Green's function as

$$\langle T\psi(x)\psi(x')\rangle = iG^{\beta}(x,x') , \tag{27}$$

where T denotes η ordering on the contour. We now let $x \to x'$ and a measure of the quantum and thermal fluctuations is given by

$$\langle \phi^2(\eta)\rangle = (H\eta)^{D-2}\langle \psi^2(\eta)\rangle = \frac{(H\eta)^{D-2}}{\Gamma\left[\frac{D-1}{2}\right](4\pi)^{(D-1)/2}} \int_0^\infty dk\, k^{D-3}(1+2\,\mathrm{Re}A_k)\,|f_k^{(1)}(\eta)|^2 . \tag{28}$$

It is implied that any term divergent in the limit $D \to 4$ has to be removed by a corresponding counterterm in the Lagrangian.

For $k > K$ where K is sufficiently large, the mode functions $f_k^{(1)}(\eta)$ $[f_k^{(2)}(\eta)]$ are dominated by negative [positive] frequencies if we set

$$c_1(k) = 1 , \quad c_2(k) = 0 , \quad k > K . \tag{29}$$

Then for $k > K$, we have the conventional vacuum of the field theory in Minkowski space. For $k < K$, there is, however, no such unique vacuum for any choice of c_1 and c_2. Now if we choose (29) to continue to values $k < K$, we see, by a small argument expansion of the Hankel functions, that both the integrals in (28) are separately infrared divergent[12] at the physical $D = 4$. However, it has been shown[14] that the coefficients c_1 and c_2 may be chosen to remove the infrared divergence of the terms in the two-point function, the coefficients themselves being divergent in that limit. For example, if we augment (29) with

$$c_1(k) = (k/K)^{-p} , \quad c_2(k) = [(k/K)^{2p} - 1]^{1/2} , \quad p > 0, k < K , \tag{30}$$

it satisfies (24) and

$$|f_k^{(1)}(\eta)|^2 \sim (k\eta)^{-2\nu}(k/K)^{2p} \tag{31}$$

as $k \to 0$. The two integrals in (28) are now separately convergent for $m = 0$ $[\nu = (D-1/2)]$ at the lower end.

Our discussion of the very early Universe is based on (28). Thus we assume the Universe to be in thermal equilibrium at t_0 at a temperature $T_0 = e^{-Ht}/\beta_0$ given by (6). Further we assume

$$M_P \gg T_0 \gg H , \tag{32}$$

as is the case in GUT's. Then

$$\beta_0/\eta_0 = H/T_0 \ll 1 . \tag{33}$$

A natural approximation for $\mathrm{Re}A_k$ suggests itself if we consider the regions $k < k_1$ and $k > k_1$ separately, where

$$k_1\eta_0 \simeq 1 , \quad \text{or } k_1 \sim He^{Ht_0} . \tag{34}$$

Then for $k < k_1 (k > k_1)$ we may use the small (large) argument expansion of the Hankel functions appearing in A_k and we get $(K \simeq k_1)$

$$\mathrm{Re}A_k = -\tfrac{1}{2} + O((k/k_1)^{2(\nu-p)}) , \quad k < k_1 \tag{35}$$

and

$$\mathrm{Re}A_k = (e^{\beta_0 k} - 1)^{-1}\left[1 + \tfrac{1}{2}(\nu^2 - \tfrac{1}{4})(1 - e^{-\beta_0 k})^{-1}\frac{H}{T_0}\frac{k_1}{k} + O\left[\left[\frac{k_1}{k}\right]^2\right]\right] , \quad k > k_1 . \tag{36}$$

Keeping terms to leading order, (28) simplifies to[15]

$$\langle \phi^2(\eta)\rangle = \frac{(H\eta)^{D-2}\eta}{8(4\pi)^{(D-3)/2}\Gamma\left[\frac{D-1}{2}\right]} \int_{He^{Ht_0}}^\infty dk\, k^{D-2}\left[1 + \frac{2}{e^{\beta_0 k} - 1}\right] H_\nu^{(1)}(k\eta)H_\nu^{(2)}(k\eta) . \tag{37}$$

We note, in particular, the mutual cancellation of the infrared divergence of the two terms in (28) (Ref. 16). Mathematically, the role of the coefficients c_1 and c_2 is to define separately each of the two integrals in (28), at the lower end when m^2 is zero or negative. It is clear that the cancellation is independent of any particular choice of c_1 and c_2.

Our result resembles the one obtained by Guth and Pi.[5] The difference lies in the manner that the low-momentum region is treated. While we are led to a cutoff $\sim H$ at the lower end of integration, they obtain an effective coupling-constant-dependent cutoff by explicit introduction of a coupling-constant-, and temperature-dependent mass term in the Lagrangian.

An approximate evaluation of (37) follows again by dividing the k-integration range at k_2 into two parts:

$$k_2\eta = (k_2/H)e^{-Ht} \sim 1 . \tag{38}$$

Then approximating the Hankel functions by small (large) argument expansions for $k < k_2$ $(k > k_2)$ as before, we readily get

$$\langle\phi^2(t)\rangle = \left[\frac{2^\nu\Gamma(\nu)}{2\pi}\right]^2\frac{H^2}{\pi}\left[\left[\frac{T_0}{H}\right]^{3-2\nu} e^{-(3-2\nu)H(t-t_0)} I_{2-2\nu} + \frac{1}{2(3-2\nu)}\left(1-e^{-(3-2\nu)H(t-t_0)}\right)\right]$$
$$+e^{-2H(t-t_0)}\left[\frac{T_0{}^2}{12}-\frac{T_0{}^2}{2\pi^2}I_1-\frac{HT_0}{2\pi^2}\right]-\frac{H^2}{8\pi^2}\ , \tag{39}$$

where

$$I_\alpha = \int_{H/T_0}^{(H/T_0)e^{H(t-t_0)}} dx \frac{x^\alpha}{e^x-1}\ . \tag{40}$$

For the Coleman-Weinberg case, $m^2/H^2 \ll 1$. Consider the time development after sufficient Hubble damping has taken place:

$$(H/T_0)e^{H(t-t_0)} \gtrsim 1\ . \tag{41}$$

We then get, from (39),

$$\langle\phi^2(t)\rangle = \frac{HT_0}{2\pi^2}e^{-(2m^2/3H)(t-t_0)} + \frac{3H^4}{8\pi^2 m^2}\left(1-e^{-(2m^2/3H)(t-t_0)}\right)+O(H^2)\ . \tag{42}$$

The $O(H^2)$ terms arise from the "transients" (i.e., terms undergoing Hubble damping due to the factor $e^{-H(t-t_0)}$). Considering times not too large, $t-t_0<3H/2m^2$, (42) simplifies to

$$\langle\phi^2(t)\rangle \simeq \frac{HT_0}{2\pi^2}+\frac{H^3}{4\pi^2}(t-t_0)+O(H^2)\ . \tag{43}$$

The period of inflation is easy to determine.[2] As $\langle\phi^2(t)\rangle$ grows, the effective mass squared of the field acquires an extra negative contribution of $-3\lambda\langle\phi^2\rangle$ and this in turn makes the growth of $\langle\phi^2\rangle$ exponentially fast. The period of inflation stops when this stage is reached. For the Universe to expand more than exp(70) times, we find from (6) and (43) that

$$\lambda \le \frac{\pi^2}{70}\frac{H}{T_0} \simeq 2\times 10^{-5}\ , \tag{44}$$

if we use $H=10^{10}$ GeV and $N\sim 200$ as typical values in GUT's.

From (39) and (43) it is clear that at $t=t_0$, $\langle\phi^2\rangle$ starts out with a large value $\sim T_0{}^2/12$. After sufficient Hubble damping, it drops to $\sim HT_0$. Only then it starts slowly to increase to the large "true-"vacuum value $\sim HM_P/\sqrt{\lambda}$. We see that $\langle\phi^2\rangle$ does not at all settle to the latter value during Hubble damping,[17] thus invalidating the objection raised by Mazenko, Unruh, and Wald.[18]

We now compare our results with those stated by Linde,[2,4] who initiated this approach to the problem. The first term in (42) and (43) which dominates $\langle\phi^2\rangle$ is missing in the corresponding expression of Linde. Inspection of integrals in (37) show that field fluctuations at wavelengths $\sim H^{-1}$ contribute dominantly to $\langle\phi^2\rangle$, while Linde claims fluctuations at very long wavelengths $\sim H^{-1}e^{H(t-t_0)}$ to be dominant. Indeed these wavelengths are completely absent from (37). Finally the upper bound on λ (44) is about 2 orders of magnitude lower than that given by Linde.

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[15] Note that the first nonleading term in (36) is insignificant under condition (33).
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