On a stationary asymptotically flat solution of the Ernst equation

S. Bonanos
Nuclear Research Centre, "Demokritos," Aghia Paraskevi, Attiki, 15310 Greece

E. Kyriakopoulos
Department of Physics, National Technical University of Athens, Zografou Campus, Athens 15772, Greece
(Received 3 April 1987)

It is pointed out that the stationary, axisymmetric and asymptotically flat vacuum solution of Das [Phys. Rev. D **27**, 322 (1983)] is isometric to the "extreme" Kerr solution.

Several approaches to the problem of finding solutions of the Ernst[1] equation have been studied recently.[2] One method of finding such solutions was formulated by Herlt[3] and used to derive, from the van Stockum metric, the Kerr solution and a new class of axially symmetric static electrovacuum solutions. Herlt's method was used by Das[4] to derive explicitly the Ernst potential for a stationary vacuum and a static electrovacuum solution of Einstein's equations. He also used the method of Israel and Wilson[5] and Perjes[6] to derive explicitly a stationary axisymmetric solution of the Einstein-Maxwell equations. Das claims that the vacuum solution he obtained is "new" and "no doubt constitutes an addition to the list of stationary gravitational solutions." In this paper we point out that Das's vacuum solution is isometric to the "extreme" Kerr solution.

In applying Herlt's method, Das[4] chooses the following solution of the Laplace equation in prolate spheroidal coordinates x,y:

$$\psi = A/(x+y) - iB(xy+1)/(x+y)^3 , \tag{1}$$

where A and B are arbitrary real constants. Then the function χ of Herlt's method, which is a solution of the equation $\chi_{\rho\rho}+\chi_{zz}-(1/\rho)\chi_\rho=0$ is given by the relation $\chi_\rho=\rho\psi$. Calculating χ which comes from the above ψ and substituting in Herlt's formulas, one obtains the following expressions for the real and imaginary parts of the Ernst potential $E=f+i\phi$:

$$f=\frac{A^2(x+y)^4-B^2(x^2-1)(1-y^2)}{(x+y)^2[A(x+y)+B]^2+B^2(xy+1)^2} , \tag{2}$$

$$\phi=-\frac{2B^2(xy+1)(x+y)}{(x+y)^2[A(x+y)+B]^2+B^2(xy+1)^2} . \tag{3}$$

These are the expressions given by Eqs. (3.3)–(3.5) of Das[4] if we put $\beta=-B$, so that $f\to 1$ as $x\to\infty$ (instead of $\beta=-1$ as Das claims) and replace B by A in the denominator of Das's Eq. (3.4). We observe that the solution depends on a single arbitrary parameter which we call $a\,(=B/A)$. From the above expressions we can calculate the potential ξ of Ernst.[1] We get

$$\xi=(1+E)/(1-E)=(x+y)/a-i(xy+1)/(x+y) . \tag{4}$$

Integrating the remaining Einstein equations[7] for the other metric functions, we find that the full metric for this solution takes the form

$$ds^2=\frac{P}{Q}\left[dt-2\frac{R}{P}d\varphi\right]^2-\frac{Q}{P}(x^2-1)(1-y^2)d\varphi^2 -\frac{Q(x^2-y^2)}{(x+y)^4}\left[\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right] , \tag{5}$$

where

$$\begin{aligned}P&=(x+y)^4-a^2(x^2-1)(1-y^2) ,\\ Q&=(x+y)^2(x+y+a)^2+a^2(xy+1)^2 ,\\ R&=a^2(x+y+a)(x^2-1)(1-y^2) .\end{aligned} \tag{6}$$

Now the Kerr metric in Boyer-Lindquist coordinates is[2]

$$ds^2=\frac{\Delta-a^2\sin^2\theta}{\Sigma}\left[dt-\frac{2amr\sin^2\theta}{\Delta-a^2\sin^2\theta}d\varphi\right]^2 -\frac{\Sigma\Delta\sin^2\theta}{\Delta-a^2\sin^2\theta}d\varphi^2-\Sigma\left[\frac{dr^2}{\Delta}+d\theta^2\right] , \tag{7}$$

where $\Delta=r^2+a^2-2mr$ and $\Sigma=r^2+a^2\cos^2\theta$. Setting $m=a$ and performing the coordinate transformation[8] $r=x+y+a$, $\cos\theta=(xy+1)/(x+y)$, we obtain (5), i.e., the metric implied by Das's solution. Thus, Das's "new" vacuum solution is equivalent to the "extreme" Kerr solution.

The equivalence of the two solutions can also be seen from the fact that both the ξ potential given by Eq. (4) and the ξ potential of the "extreme" Kerr solution take the form[9]

$$\xi=kr-i\cos\theta,\quad k=\text{const} , \tag{8}$$

after appropriate coordinate transformations. To obtain (8) from Eq. (4) we make the coordinate transformation

$$x = \{r + [r^2 + 4c(c - z)]^{1/2}\}/2c ,$$
$$y = \{r - [r^2 + 4c(c - z)]^{1/2}\}/2c , \quad (9)$$

where $z = r\cos\theta$ and $c = 1/ak$. To bring $\xi_{\mathrm{Kerr}} = px - iqy$, $p = (1 - q^2)^{1/2}$, $q = a/m$, to the form (8) we let

$$x = [(r^2 + 2mzp + m^2p^2)^{1/2} + (r^2 - 2mzp + m^2p^2)^{1/2}]/2mp ,$$
$$y = [(r^2 + 2mzp + m^2p^2)^{1/2} - (r^2 - 2mzp + m^2p^2)^{1/2}]/2mp , \quad (10)$$

and then take the limit $q \to 1$, obtaining (8) with $k = 1/m$.

---

[1]F. J. Ernst, Phys. Rev. **167**, 1175 (1968).

[2]D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980).

[3]E. Herlt, Gen. Relativ. Gravit. **9**, 711 (1978).

[4]K. C. Das, Phys. Rev. D **27**, 322 (1983).

[5]W. Israel and G. A. Wilson, J. Math. Phys. **13**, 865 (1972).

[6]Z. Perjes, Phys. Rev. Lett. **27**, 1668 (1971).

[7]See Ref. 2, Eq. (17.40).

[8]This transformation was pointed out by W. Kinnersley (private communication) to S. Bonanos, who (S.B.) had obtained the same solution (5) by a different method (unpublished). For simplicity, an unimportant constant factor with the dimensions of length has been set equal to unity.

[9]This form of the Kerr potential was independently derived by one of us (E.K.) and used to generate a family of asymptotically flat solutions of the Ernst equation.