Instability of the translationally invariant vacuum of a system of fermions coupled to a chiral field

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It is shown, in the case of fermions coupled to a chiral field, that for every value of the coupling constant the energy of the $B=0$ sector is lower for a localized chiral field than for a translationally invariant (uniform) field. An expansion is given which converges for fields localized in a region of space smaller than the inverse mass of the fermions and which in lowest order explains the vacuum deformation. Exact energies are calculated for a hedgehog field. The instability of the uniform vacuum is found to increase with the winding number of the chiral field.

I. INTRODUCTION

The σ model,[1] in its linear and nonlinear versions, is used in many contexts in field theory with the assumption that its ground state, the physical vacuum, is translationally invariant. MacKenzie, Wilczek, and Zee[2] discussed the possibility that heavy fermions could deform the vacuum in the $B=0$ sector. Their analysis was based on the gradient expansion. The instability however occurs for small size deformations of the vacuum which involves strong gradients of the fields. For such deformations the validity of the gradient expansion breaks down. In this paper we investigate this question further, in an exact fermion one-loop calculation, without relying on the gradient expansion. We show that when fermions are coupled to an SU(2) chiral field which is localized in space, an instability of the translationally invariant vacuum occurs in the $B=0$ sector for any value of the coupling constant. We give an expansion which converges for chiral fields localized in a region of space of size R smaller than the Compton wavelength of the heavy fermions and for which the gradient expansion breaks down. The expansion is compared to an exact calculation of the fermion one-loop energy in the case of a chiral field with a hedgehog shape. The lowest-order term of our expansion is able to describe the deformation of the vacuum. The instability of the translationally invariant vacuum is found to increase with the winding number of the chiral field.

In Secs. II and III we explain the exact calculation of the fermion one-loop contribution to the energy in the case where the chiral field has a hedgehog shape. In Sec. IV we give an expansion for the energy which converges for chiral fields localized in small regions of space and we use this expansion to prove that the energy (measured relative to the translationally invariant system) is always negative for sufficiently small sizes irrespective of the shape of the chiral field and for any value of the coupling constant. Our work is similar to that of Soni[3] who noticed the instability of the translationally invariant vacuum by analyzing some of our previous calculations for chiral solitons.[4]

We consider the nonlinear σ model with fermions ψ coupled to a chiral field U. The Lagrangian density is

$$\mathcal{L}=\frac{f^2}{16}\mathrm{tr}(\partial_\mu U)(\partial^\mu U^\dagger)+\bar{\psi}(-i\partial\!\!\!/+gfU)\Phi \ , \tag{1.1a}$$

where U is an SU(2) chiral field:

$$U=e^{i\theta\cdot\tau\gamma_5} \ . \tag{1.1b}$$

The partition function is, formally,

$$\mathrm{Tr}e^{-\beta H}=\int \mathcal{D}(\bar{\psi})\mathcal{D}(\psi)\mathcal{D}(U)\exp\left[-\int d_4x\,\mathcal{L}\right] . \tag{1.2}$$

We work in a Euclidean metric in which

$$x^\mu=x_\mu=(\tau,\mathbf{r}),\quad \gamma^\mu=\gamma_\mu=(i\beta,\boldsymbol{\gamma})=-\gamma^{\mu\dagger} \ ,$$
$$\gamma_5=\gamma_5^\dagger,\quad \int d_4x=\int_0^\beta d\tau\int d_3x \ . \tag{1.3}$$

In the fermion one-loop approximation we integrate the quadratic form of the fermion fields. This leads to the effective Lagrangian

$$\mathcal{L}(U)=\frac{f^2}{16}\mathrm{tr}(\partial_\mu U)(\partial^\mu U^\dagger)-\mathrm{tr}\ln(-i\partial\!\!\!/+gfU) \ . \tag{1.4}$$

Our reference state is the translationally invariant vacuum defined by $U=1$. We subtract from (1.4) the constant $\mathcal{L}(U=1)$. The effective Lagrangian becomes

$$\mathcal{L}_{\mathrm{eff}}(U)=\frac{f^2}{16}\mathrm{tr}(\partial_\mu U)(\partial^\mu U^\dagger)$$
$$-\mathrm{tr}\ln(-i\partial\!\!\!/+gfU)+\mathrm{tr}\ln(-i\partial\!\!\!/+gf) \ . \tag{1.5}$$

One further subtraction is required to eliminate an ultraviolet logarithmic divergence. For an SU(2) field U we have

$$\mathrm{tr}\ln(-i\partial\!\!\!/+gfU)=\mathrm{tr}\ln(+i\partial\!\!\!/+gfU^\dagger)$$
$$=\tfrac{1}{2}\mathrm{tr}\ln[-\partial^2+g^2f^2+igf(\partial\!\!\!/U)] \ . \tag{1.6}$$

The second line of (1.5), which is the fermion contribution $\mathcal{L}_F$ to the action, is thus equal to

$$\mathcal{L}_F(U) = -\tfrac{1}{2}\text{tr}\ln\left[1+\frac{1}{-\partial^2+g^2f^2}igf(\partial\!\!\!/ U)\right] . \quad (1.7)$$

Since (∂U) is odd in γ matrices, only even powers of g contribute to the expansion of the logarithm. All except the second-order terms are ultraviolet convergent. The divergent part of (1.7), and therefore of the Lagrangian (1.5), is equal to

$$\frac{g^2f^2}{4}\text{tr}\left[\frac{1}{-\partial^2+g^2f^2}\right]^2(\partial U)(\partial U^\dagger)$$
$$= -\frac{g^2f^2}{4}\text{tr}\left[\frac{1}{-\partial^2+g^2f^2}\right]^2(\partial_\mu U)(\partial^\mu U^\dagger)$$
$$= -\text{tr}(\partial_\mu U)(\partial^\mu U^\dagger)\frac{g^2f^2}{4\beta\Omega}\int d_4k\frac{1}{k^2+g^2f^2} , \quad (1.8)$$

where the trace is evaluated on the plane-wave basis (4.4) normalized to a volume Ω. No other terms appear in the expansion of the logarithm which are quadratic in the derivatives of the field U. Thus a subtraction of the divergent term (1.8) from the Lagrangian (1.5) is a renormalization of the decay constant f. The renormalized effective Lagrangian we shall work with is

$$\mathcal{L}_{\text{eff}}(U) = \frac{f^2}{16}\text{tr}(\partial_\mu U)(\partial^\mu U^\dagger) - \text{tr}\ln(-i\partial\!\!\!/+gfU)$$
$$+\text{tr}\ln(-i\partial\!\!\!/+gf)$$
$$-\frac{g^2f^2}{4}\text{tr}\left[\frac{1}{-\partial^2+g^2f^2}\right]^2(\partial U)(\partial U^\dagger) . \quad (1.9)$$

II. CALCULATION OF THE ENERGY FOR A HEDGEHOG CHIRAL FIELD

In this section we explain how the exact energy was calculated numerically for a chiral field with a hedgehog shape. In Sec. IV we will show that the instability of the translationally invariant vacuum occurs for any localized shape of the chiral field.

A stationary state, for which $(\partial_\tau U)=0$, has an energy equal to

$$E = \frac{1}{\beta}\int d_4x\,\mathcal{L}(U) \quad (2.1)$$

which is evaluated with the Lagrangian (1.9). In the Euclidean metric we have

$$-i\partial\!\!\!/+gfU = \beta(\partial_\tau+h) , \quad (2.2)$$

where h is the Dirac Hamiltonian the eigenstates of which are the fermion orbitals $|\lambda\rangle$:

$$h = \frac{\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}{i}+gf\beta U,\quad h|\lambda\rangle = e_\lambda|\lambda\rangle,\quad \langle\lambda|\lambda\rangle = 1 . \quad (2.3)$$

Similarly for the translationally invariant case $U=1$ we have $-i\partial\!\!\!/+gf=\beta(\partial_\tau+h_0)$. The eigenstates of

$$h_0 = \frac{\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}{i}+gf\beta$$

are Dirac plane-wave states which we shall denote by $|k\rangle$ where k denotes all the quantum numbers (momentum, Dirac index, flavor, color,...) needed to specify the state:

$$h_0 = \frac{\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}{i}+gf\beta,\quad h_0|k\rangle = e_k|k\rangle,\quad \langle k|k\rangle = 1 , \quad (2.4)$$

and $e_k = \pm(k^2+g^2f^2)^{1/2}$.

With this notation and in the zero-temperature limit $\beta\to\infty$ we have

$$\frac{1}{\beta}\int d_4x\,\text{tr}\ln(\partial_\tau+h) = \int d_3x\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}\text{tr}\ln(h-\omega) . \quad (2.5)$$

With one integration by parts we obtain

$$\frac{1}{\beta}\int d_4x\,\text{tr}[\ln(-i\partial\!\!\!/+gfU)-\ln(-i\partial\!\!\!/+gf)]$$
$$= \int d_3x\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}\left[\frac{1}{h-\omega}-\frac{1}{h_0-\omega}\right]$$
$$= \sum_{e_\lambda<0}e_\lambda - \sum_{e_k<0}e_k . \quad (2.6)$$

The sums in (2.6) are over the negative-energy orbitals which lie to the left of the energy integration path. We could, at will, include some positive-energy orbitals (or exclude some negative-energy orbitals) in the sums (2.6) by adding a chemical potential: $h\Longrightarrow h-\mu$. We do not need to do so however in the study of the vacuum instability. The expression for the renormalized energy is thus

$$E = \sum_{e_\lambda<0}e_\lambda - \sum_{e_k<0}e_k + \frac{f^2}{16}\int d_3x\,\text{tr}(\partial U)(\partial U^\dagger)$$
$$-\frac{1}{\beta}\int d_4x\frac{g^2f^2}{4}\text{tr}\left[\frac{1}{-\partial^2+g^2f^2}\right]^2(\partial U)(\partial U^\dagger) . \quad (2.7)$$

The energies e_λ of the fermion orbits were evaluated numerically. The chiral field U was assumed to have a hedgehog shape:

$$U = e^{i\hat{\mathbf{r}}\cdot\boldsymbol{\tau}\theta(r)\gamma_5} \quad (2.8)$$

with an exponential profile

$$\theta(r) = n\pi e^{-r/R} , \quad (2.9)$$

the winding number n being an integer. Such a hedgehog shape is a common occurrence in soliton theory.[5] The choice of the exponential profile (2.9) is arbitrary except for the boundary conditions at $r=0$ and $r\Longrightarrow\infty$. An improvement in the shape of $\theta(r)$ can only make the instability, displayed below, worse.

Having fixed the shape of the chiral field U, we diagonalized the Dirac Hamiltonian (2.3) on a spherical basis of a free Dirac particle defined in Ref. 6. The momentum k of the basis states was *discretized* by introducing a boundary condition at a radius $r=D$ chosen to be larger than the soliton size R, typically $D>4R$. The results were checked to remain unchanged with a further increase of D. The basis was made *finite* by including only those basis states with momenta $k<\Lambda$. Basis states with all possible angular momenta were included compatible with the condition $k<\Lambda$. The maximum angular momentum was found to be of the order of the classical value $D\Lambda$, in fact a few percent lower. Further details are found in Ref. 6.

Special care was required to ensure the convergence of the energy as the cutoff momentum Λ was increased. For this purpose the counterterm, as expressed in (2.7), although formally correct, is not adequate for numerical evaluation and we rederive it now in a form which ensures numerical accuracy. The second line of (1.5), which represents the fermion contribution to the effective Lagrangian, can be expressed in terms of the Dirac Hamiltonians (2.3) and (2.4) thus

$$\mathcal{L}_F(U)=-\operatorname{tr}\ln(\partial_\tau+h)+\operatorname{tr}\ln(\partial_\tau+h_0)\ . \tag{2.10}$$

Using the property $\operatorname{tr}\ln(\partial_\tau+h)=\operatorname{tr}\ln(-\partial_\tau+h)$ we write the action (2.10) thus

$$\mathcal{L}_F(U)=-\tfrac{1}{2}\operatorname{tr}\ln\left[1+\frac{1}{-\partial_\tau^2+h_0^2}V\right]\ , \tag{2.11}$$

where the interaction V is defined by the equation

$$\begin{aligned}&h^2=h_0^2+V\ ,\\&V=h^2-h_0^2=g^2f^2(UU^\dagger-1)+igf(\nabla U)\ .\end{aligned} \tag{2.12}$$

Although $UU^\dagger$ is equal to 1 formally, it is not strictly equal to 1 in the subspace spanned by the basis plane-wave states with $k<\Lambda$. Since h is diagonalized in this subspace it is essential to calculate $V=h^2-h_0^2$ as the difference between the squares of the matrices h and h_0 as evaluated in this subspace.[7] Otherwise the counterterms will not cancel the divergent parts of the orbital energy sums appearing in the energy (2.7). Let us denote by $|k\rangle$ the plane-wave basis states. No confusion should arise with the use of this symbol to denote also the eigenstates of h_0 in (2.4) since they could equally well be used. The fermion contribution to the energy (2.1) may be obtained from the effective Lagrangian (2.11):

$$\begin{aligned}E_F&=\frac{1}{\beta}\int d_4x\,\mathcal{L}_F(U)\\&=-\frac{1}{2\beta}\int d_4x\operatorname{tr}\left[\frac{1}{-\partial_\tau^2+h_0^2}V-\frac{1}{2}\left[\frac{1}{-\partial_\tau^2+h_0^2}V\right]^2+\cdots\right]\ .\end{aligned} \tag{2.13}$$

The divergent part, evaluated on the plane-wave basis $|k\rangle$, is

$$-\frac{1}{2}\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}\sum_k\left[\frac{\langle k|V|k\rangle}{k^2+g^2f^2-\omega^2}-\frac{1}{2}\frac{\langle k|V^2|k\rangle}{(k^2+g^2f^2-\omega^2)^2}\right]=-\frac{1}{4}\sum_k\frac{\langle k|V|k\rangle}{(k^2+g^2f^2)^{1/2}}+\frac{1}{16}\sum_k\frac{\langle k|V^2|k\rangle}{(k^2+g^2f^2)^{3/2}}\ . \tag{2.14}$$

These are the counterterms we subtract from the energy. The form of the energy, which was actually calculated is thus

$$\begin{aligned}E=&\sum_{e_\lambda<0}e_\lambda-\sum_{e_k<0}e_k+\frac{1}{4}\sum_k\frac{\langle k|V|k\rangle}{(k^2+g^2f^2)^{1/2}}\\&-\frac{1}{16}\sum_k\frac{\langle k|V^2|k\rangle}{(k^2+g^2f^2)^{3/2}}+\frac{f^2}{16}\int d_3x\operatorname{tr}(\partial U)(\partial U^\dagger)\ .\end{aligned} \tag{2.15}$$

This energy is identical in form to the energy (2.7). However, in the numerical calculations the Hamiltonian was diagonalized in the subset of basis states $|k\rangle$ with $k<\Lambda$ and the sums appearing in the counterterms were limited to the same subset. Finally V was considered as a matrix in this subset and was squared as such. The last term of (2.15), which is the kinetic energy of the chiral field, was evaluated in the form

$$\frac{f^2}{16}\int d_3x\operatorname{tr}(\partial U)(\partial U^\dagger)=\frac{f^2}{2}\int_0^\infty 4\pi r^2dr\left[\left[\frac{d\theta}{dr}\right]^2+\frac{2\sin^2\theta}{r^2}\right]\ , \tag{2.16}$$

which is derived from the hedgehog form (2.8).

III. RESULTS FOR THE HEDGEHOG FIELD

The results are more easily discussed if energies are expressed in units of gf and distances in units of $1/gf$. This eliminates the constant gf from the eigenvalue problem (2.3) which becomes, in these units,

$$h=\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}{i}+\beta U,\quad h|\lambda\rangle=\epsilon_\lambda|\lambda\rangle,\quad\langle\lambda|\lambda\rangle=1\ , \tag{3.1}$$

where $\nabla_i\equiv\partial/\partial(gfx_i)$ and where $\epsilon_\lambda\equiv e_\lambda/gf$. The eigenvalues of h_0 are then equal to $\epsilon_k=\pm(k^2+1)^{1/2}$. In these units, the energy (2.15) of the system is expressed thus

$$\begin{aligned}\frac{E}{gf}=&\sum_{\epsilon_\lambda<0}\epsilon_\lambda-\sum_{\epsilon_k<0}\epsilon_k+\frac{1}{4}\sum_k\frac{\langle k|V|k\rangle}{(k^2+1)^{1/2}}\\&-\frac{1}{16}\sum_k\frac{\langle k|V^2|k\rangle}{(k^2+1)^{3/2}}\\&+\frac{1}{2g^2}\int_0^\infty 4\pi y^2dy\left[\left[\frac{d\theta}{dy}\right]^2+\frac{2\sin^2\theta}{y^2}\right]\end{aligned} \tag{3.2}$$

with $y=gfr$. This way, the fermion contribution to the energy, which is the first line of expression (3.2), becomes independent of the constants g and f and the chiral field kinetic energy, the second line, is simply inversely proportional to g^2.

Figure 1 shows the spectrum of the fermion orbitals plotted as a function of the dimensionless soliton radius gfR in the case where the winding number of the chiral field is $n=1$. As the radius R becomes vanishingly small the chiral field becomes indistinguishable from its translationally invariant value $U=1$ and the orbitals become Dirac plane-wave orbitals with energies $\epsilon_k = \pm(k^2+1)^{1/2}$. This can be understood by realizing that the chiral field behaves like a potential whose volume integral vanishes when its range R goes to zero. The instability we shall display occurs at small values $gfR<1$ before any orbital crosses the zero-energy line. For these small radii, the orbitals which are filled in the $B=0$ sector are the negative-energy orbitals. This contrasts with the case of large radii where filling the negative-energy orbitals produces a $B=1$ state when the winding number is $n=1$ and, more generally, a $B=n$ state for any winding number n. This is further discussed in Ref. 6.

Figure 2 shows the fermion contribution to the energy, which is the first line of expression (3.2). If N_c is the number of colors, each fermion orbital contains N_c fermions so that the fermion contribution to the energy contains a factor N_c. In Fig. 2 we have plotted this energy

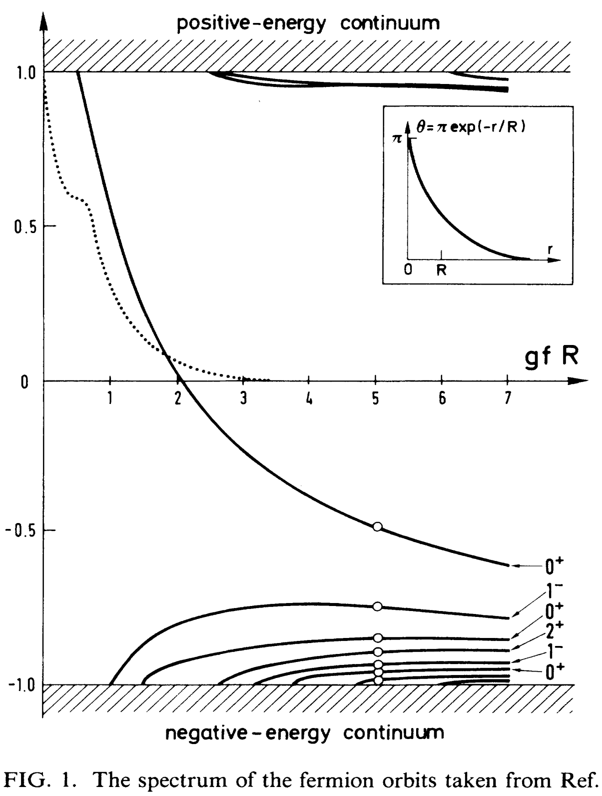

FIG. 1. The spectrum of the fermion orbits taken from Ref. 5 in the case where the winding number of the chiral field is $n=1$. The energies are in units of gf. The orbits are labeled by their parity and grand spin which is the sum of the angular momentum and isospin. The dotted curve is the energy of the $B=1$ soliton not discussed in this paper.

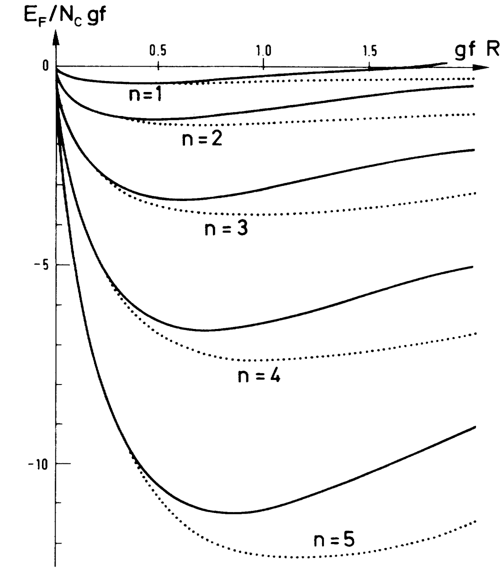

FIG. 2. The fermion one-loop contribution to the energy, for various values of the winding number n, as a function of the dimensionless size parameter gfR. The exponential profile (2.9) was used. The energy is expressed in units of gfN_c where N_c is the number of colors. The solid line is the exact calculation. The dashed line is the second-order approximation discussed in Sec. IV.

divided by $N_c gf$. We see that for small radii the fermion contribution to the energy is negative and that the negative value increases almost quadratically with the winding number.

To the fermion energy we need to add the kinetic energy of the chiral field, the second line of (3.2). It is a linear function of R:

$$\frac{1}{2g^2}\int_0^\infty 4\pi y^2 dy\left[\left[\frac{d\theta}{dy}\right]^2+\frac{2\sin^2\theta}{y^2}\right]=\frac{a}{2g^2}gfR\ . \qquad (3.3)$$

For the exponential profile (2.9) the constant a is equal to 61.64, 163.19, 323.36, 544.47, and 827.46 for winding numbers n, respectively, equal to 1, 2, 3, 4, and 5. The kinetic energy of the chiral field is a straight line with a slope equal to $a/2g^2$, as shown in Fig. 4. In Fig. 3 we plot the total energy (3.2) in a typical case with $g=4$ and $N_c=3$. We see that even when the kinetic energy of the chiral field is included, the energy becomes more negative as the winding number increases.

When the fermion contribution to the energy is negative, it is always possible to find a coupling constant g strong enough to make the total (fermion + meson) energy negative. Recall that the total energy (2.15) or (3.2) is measured relative to the energy of the translationally invariant system $U=1$. The renormalization condition which defines the counterterm is chosen such that f (which is the decay constant in the translationally invariant vacuum) keeps a constant value. Under these conditions a negative energy means that the system with a localized hedgehog field has a lower energy than the

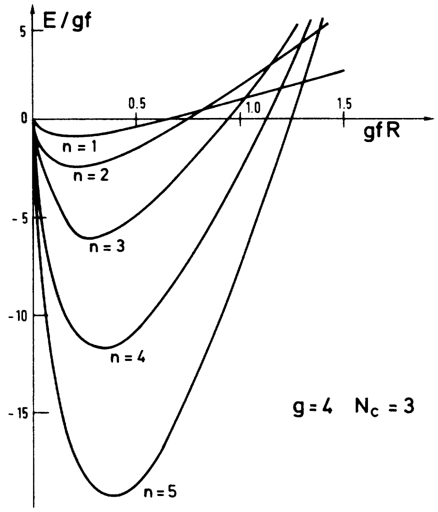

FIG. 3. General behavior of the fermion and chiral field contributions of the energy as a function of the dimensionless size parameter gfR. The energies are expressed in units of gf.

translationally invariant system for which $U=1$. Since the energy density is localized in a region of space of size roughly equal to R, one could arrange an infinite array of nonoverlapping hedgehogs such that the energy per unit volume would be lower energy than that of the translationally invariant system.

As the coupling constant g decreases, the minimum energy occurs for decreasing values of R. In order to determine whether the hedgehog field has lower energy than the constant field for any value of the coupling constant, we need to know how the fermion contribution to the energy behaves at small values of R. Fortunately there is an expansion valid for small values of R. This expansion is presented in the next section, where we will show that the

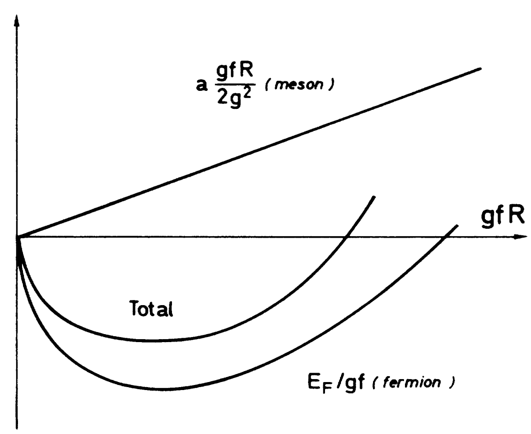

FIG. 4. Total (fermion + chiral field) energy in units of gf plotted against the dimensionless size parameter gfR. The case depicted is for a coupling constant $g=4$ and for $N_c=3$ colors.

instability occurs at all values of the coupling constant and independently of the (hedgehog) shape of the chiral field.

IV. SMALL-SIZE EXPANSION

In this section we derive an expansion which converges when the chiral field is localized in a small region of space of size $R<1/gf$. The chiral field does not necessarily have a hedgehog shape. We use this expansion to show analytically that the energy is lowered, relative to the translationally invariant case, when the chiral field is localized in a sufficiently small region of space.

The expansion of the logarithm in the fermion action (1.7) yields an expansion which converges for chiral fields which are localized in a small region of space. Subtracting the counterterm (1.8) the fermion one-loop contribution to the energy becomes

$$\begin{aligned} E_F &= \frac{1}{\beta}\int d_4x\,\mathcal{L}_F(U) \\ &= -\frac{1}{2\beta}\int d_4x\,\mathrm{tr}[\ln(1+GV)+\tfrac{1}{2}G^2V^2] \\ &= \frac{1}{\beta}\int d_4x\,\mathrm{tr}[\tfrac{1}{4}GVGV-\tfrac{1}{4}G^2V^2+\tfrac{1}{8}(GV)^4 \\ &\qquad +\tfrac{1}{12}(GV)^6+\cdots] \end{aligned} \tag{4.1}$$

with the notation

$$G\equiv\frac{1}{-\partial^2+g^2f^2},\quad V\equiv igf(\partial\!\!\!/ U)\ . \tag{4.2}$$

Because V is odd in the γ matrices, only even powers contribute to the expansion.

Consider first the lowest- (second-)order contribution:

$$E_F^{(2)}=\frac{1}{8\beta}\int d_4x\,\mathrm{tr}[G,V]^2<0\ . \tag{4.3}$$

This second-order contribution, which turns out to be the dominant contribution for chiral fields localized in a small region of space, is negative. Indeed G and V are Hermitian, their commutator $[G,V]$ is therefore pure imaginary and its square negative. This explains why, for example, the curves in Fig. 2 all start out negative at small values of R.

We evaluate the expression (4.3) using a normalized plane-wave basis $|k\rangle$ defined thus

$$\langle x\,|\,k\rangle=\frac{1}{\sqrt{\beta\Omega}}e^{ik_\mu x_\mu},\quad G\,|\,k\rangle=\frac{1}{k^2+g^2f^2}\,|\,k\rangle\ . \tag{4.4}$$

In our Euclidean metric $k_\mu=x_\mu=k_0\tau+\mathbf{k}\cdot\mathbf{r}$. The second-order energy becomes

$$\begin{aligned} E_F^{(2)} &= \frac{\mathrm{tr}}{4\beta}\sum_{k_1k_2}\langle k_1\,|\,V\,|\,k_2\times k_2\,|\,V\,|\,k_1\rangle \\ &\quad\times\left[\frac{1}{(k_1{}^2+g^2f^2)(k_2{}^2+g^2f^2)}\right. \\ &\qquad\left.-\frac{1}{(k_1{}^2+g^2f^2)^2}\right]\ . \end{aligned} \tag{4.5}$$

We consider a time-independent chiral field U which we scale by setting

$$U=W(\mathbf{r}/R)\ . \tag{4.6}$$

The scaling of the matrix element $\langle k_1 \mid V \mid k_2 \rangle$ can then be expressed as

$$\langle \mathbf{k}_1 \mid V \mid \mathbf{k}_2 \rangle = \delta_{k_{01},k_{02}} igf \langle \mathbf{k}_1 \mid (\nabla U) \mid \mathbf{k}_2 \rangle = \delta_{k_{01},k_{02}} igf \frac{1}{\Omega} R^2 L(R\mathbf{q})\ , \tag{4.7}$$

where $L(R\mathbf{q})$ is the Fourier transform of (∇U):

$$L(R\mathbf{q}) = \int d_3x\, e^{iR\mathbf{q}\cdot\mathbf{x}} (\nabla U)\ . \tag{4.8}$$

Setting $q=k_2-k_1$ and using (4.7) the second-order energy becomes

$$E_F^{(2)} = (gf)^2 R^4 \frac{\mathrm{tr}}{(2\pi)^3} \int d_3q \mid L(R\mathbf{q}) \mid^2 F(q)\ , \tag{4.9}$$

where

$$F(q) = \frac{1}{4\beta\Omega} \sum_k \left[\frac{1}{(k^2+g^2f^2)[(k+q)^2+g^2f^2]} - \frac{1}{(k^2+g^2f^2)^2} \right] . \tag{4.10}$$

The function $F(q)$ can be evaluated by the well-known method of writing

$$F(q) = \frac{1}{4\beta\Omega} \sum_k \int_0^1 dx \left[\frac{1}{[k^2+g^2f^2+q^2x(1-x)]^2} - \frac{1}{(k^2+g^2f^2)^2} \right] \tag{4.11}$$

and by using the identity

$$\frac{1}{\beta\Omega} \sum_k f(k^2) = \frac{1}{(2\pi)^4} \pi^2 \int k^2 d(k^2) f(k^2)\ . \tag{4.12}$$

We find

$$F(q) = \frac{1}{64\pi^2} \left[2 - \frac{2gf}{q} \left[1 + \left[\frac{q}{2gf} \right]^2 \right]^{1/2} \ln \frac{\left[1 + \left[\frac{q}{2gf} \right]^2 \right]^{1/2} + \frac{q}{2gf}}{\left[1 + \left[\frac{q}{2gf} \right]^2 \right]^{1/2} - \frac{q}{2gf}} \right] . \tag{4.13}$$

Let us set $\mathbf{y}=R\mathbf{q}$ in (4.9). We then obtain the second-order fermion energy in the form

$$E_F^{(2)} = (gf)^2 R \frac{\mathrm{tr}}{(2\pi)^3} \int d_3k \mid L(\mathbf{y}) \mid^2 F(y/R)\ . \tag{4.14}$$

This expression has also been obtained by Zuk[8] and Tudor.[9] The limit $R \to 0$ of the function $F(k/R)$ can be obtained from (4.13):

$$F(y) \underset{y/R\to\infty}{\sim} \frac{1}{64\pi^2} \left[2 - 2\ln \frac{y^2}{2g^2R^2f^2} \right] . \tag{4.15}$$

Substituting (4.15) into (4.14) we obtain a limiting form for the second-order energy at small R:

$$E_F^{(2)}(R) \underset{R\to 0}{\sim} \alpha gfR + \delta gfR \ln gfR\ , \tag{4.16}$$

where the coefficients α and δ are

$$\alpha = gf \frac{\mathrm{tr}}{(2\pi)^3} \int d_3y \mid L(\mathbf{y}) \mid^2 \frac{1}{64\pi^2} \left[2 - 2\ln \frac{y^2}{2} \right] ,$$
$$\delta = gf \frac{\mathrm{tr}}{(2\pi)^3} \int d_3y \mid L(\mathbf{y}) \mid^2 \frac{4}{64\pi^2} > 0\ . \tag{4.17}$$

Since δ is positive, the energy (4.16) will have a (negative) infinite slope at the origin $R=0$:

$$\frac{d}{dR} E_F^{(2)} \underset{R\to 0}{\sim} \alpha gf + \delta gf (\ln gfR + 1) \underset{R\to 0}{\to} -\infty\ . \tag{4.18}$$

A dimensional argument shows that the kinetic energy of the chiral field is a positive linear function of the size R as shown in Fig. 4. Therefore, no matter how small the coupling constant g is, the total (fermion + chiral field) energy will become negative at sufficiently small values of R. This result was also noted by Soni.[3] The dashed curves in Fig. 2 show the second-order energy calculated with the chiral field of hedgehog shape (2.8) and exponential profile (2.9). The second-order energy (4.3) is seen to approximate the one-loop fermion energy sufficiently well to display the minimum.

Each extra factor GV in the expansion (4.1) contributes an extra factor gfR. Indeed (4.7) shows that V contains a factor $(gfR)^2$. Each factor GV involves an extra q integration which contributes a factor $1/R$ when the substitution $\mathbf{y}=R\mathbf{q}$ is made as in (4.14). Therefore the expansion (4.1) converges for small values of gfR. The argument is independent of the shape of the chiral field, which need not have a hedgehog shape as in the numerical example discussed in Sec. III. All that is required is that the chiral field be localized in space. In the linear σ model one could calculate the contribution to the energy of the quantum fluctuations of the chiral field (boson one-loop contributions). In the dimensionless units defined in Sec. III, the boson quantum fluctuations would be proportional to the coupling constant g. Since the fermions cause an instability for any value of g it is possible, although not proved, that the boson one-loop contribution will not upset the instability. The present calculation does not show, however, how other degrees of freedom such as, for example, vector mesons, will change the behavior of the system at small values of R. If however the energy of the quark-loop calculation would increase when another field, interacting with the fermions, is added, the energy could then be lowered again by simply setting the classical value of this field to zero.

It is difficult to assess all the physical consequences of the instability we have found. It may be viewed as displaying a limit to the domain of validity of effective theories, such as the σ model, to short-distance phenomena. It would be important to study whether such instabilities also plague theories involving Higgs fields, such as the Weinberg-Salam model.

Finally it may be useful to compare our approach with that of Dyakonov and Petrov[10] who view the problem differently. Their Lagrangian consists of the (nonrenormalized) fermion Lagrangian (2.11) alone. They do not include a term representing the kinetic energy of the chiral field. The latter is assumed to be generated dynamically in the fermion one-loop approximation. The Lagrangian (2.11) used by Dyakonov and Petrov can be written in the form

$$\mathcal{L}_F(U) = -\tfrac{1}{2}\mathrm{tr}\ln(1+GV) - \tfrac{1}{4}\mathrm{tr}G^2V^2 + \tfrac{1}{4}\mathrm{tr}G^2V^2 \qquad (4.19)$$

using the notation (4.2).

We have purposely added and subtracted our counterterm in order to display the contribution of the Lagrangian to the kinetic energy of the chiral field. Indeed the counterterm has the form of this kinetic energy since it is equal to

$$\tfrac{1}{4}\mathrm{tr}G^2V^2 = \frac{f^2}{16}(\partial_\mu U)(\partial^\mu U^\dagger) a(g,\Lambda/gf) \;, \qquad (4.20)$$

where $a(g,\Lambda/gf)$ is the logarithmically divergent function:

$$a(g,\Lambda/gf) = \frac{4g^2N_c}{(2\pi)^4}\int^\Lambda \frac{d_4k}{(k^2+g^2f^2)^2} = \frac{4g^2N_c}{(2\pi)^4}\pi^2\left[\ln\left[\frac{\Lambda^2}{g^2f^2}+1\right] - \frac{\Lambda^2}{\Lambda^2+g^2f^2}\right] . \qquad (4.21)$$

If the momentum cutoff Λ is chosen such that $a(g,\Lambda/gf)=1$ the kinetic energy of the chiral field is generated entirely by the fermion loop. Typical values obtained from (4.21) are $\Lambda/gf=2$ for $g=2$ and $\Lambda/gf=1$ for $g=8$. In the range $3 \lesssim g \lesssim 9$ we have in fact $a(g,\Lambda/gf)=1$ for $\Lambda \approx 8f$ independently of g. When the fermion-loop momentum is cut off in this way no instability of the vacuum seems to occur.

Our work is most easily compared to that of Dyakonov and Petrov if we write the Lagrangian in the form

$$\mathcal{L}(U) = -\tfrac{1}{2}\mathrm{tr}\ln(1+GV) - \tfrac{1}{4}\mathrm{tr}G^2V^2 + \frac{f^2}{16}(\partial_\mu U)(\partial^\mu U^\dagger)\;[a(g,\Lambda/gf)+b] \;. \qquad (4.22)$$

Dyakonov and Petrov set $b=0$ (no bare kinetic energy term for the chiral field in the Lagrangian) and they cut off the Fermi-loop momentum at $\Lambda \approx 8f$. In our approach we do not assume that the chiral field kinetic energy is generated by the fermion loop alone and we choose b such that $a+b=1$ thereby maintaining the chiral field kinetic energy at its desired value as Λ extends to infinity. When $\Lambda \gtrsim 8f$ the constant b becomes negative. It is this difference between the Lagrangians that explains the difference between our results and theirs.

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