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Solving two-dimensional ϕ^4 theory by discretized light-front quantization

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The recently proposed discretized light-front quantization (DLFQ) method is applied to ϕ^4 field theory in $1+1$ dimensions. We start with the normal-ordered Hamiltonian and perform calculations with and without finite-mass renormalization in order to elucidate its role. We find that finite-mass renormalization prevents the phase transition by restricting the theory to the weak-coupling region. Comparison with results obtained without mass renormalization demonstrates that both treatments can yield the same estimate of the critical coupling for which the mass gap vanishes. This DLFQ estimate of the critical coupling may be compared with other estimates. The invariant mass of various states is calculated as a function of bare coupling. In the weak-coupling region where we can easily extrapolate to the continuum limit we find evidence for scattering but there is no two-particle bound state in agreement with the well-known result established for constructive quantum field theory. In addition, we find no multiparticle bound states.

I. INTRODUCTION

Discretized light-front quantization (DLFQ) was proposed recently,[1,2] as a method to solve relativistic field-theory problems nonperturbatively. So far scalar two-dimensional Yukawa theory and both massless and massive two-dimensional QED (QED_2) have been studied in this scheme.[1,2] To understand the strengths and weaknesses of this technique it is desirable to apply DLFQ to a model field theory which has been studied previously by other means. Among the popular field theory models, two-dimensional ϕ^4 theory has received much attention in various approaches to solving strongly interacting field theories. For some of the recent work see Ref. 3. In particular, one can *quantitatively* test the various aspects of analytical and numerical techniques since many properties of $(\phi^4)_2$ have been established rigorously from the viewpoint of constructive quantum field theory.[4,5] In this work we investigate the $(\phi^4)_2$ theory with the DLFQ scheme.

The plan of this paper is as follows. In Sec. II the light-front quantization of self-coupled scalar-field models is reviewed. Discretization and the construction of the Hamiltonian and momentum operator are discussed in Sec. II. The distinct features of DLFQ that deserve detailed study are elaborated in Sec. III. Section IV contains our numerical results and comparisons with other methods. The summary and conclusions are presented in Sec. V.

II. REVIEW OF LIGHT-FRONT QUANTIZATION

Light-front quantization originated almost four decades ago from the work[6] of Dirac on the forms of relativistic dynamics. Formal foundations of the light-front quantization approach to quantum field theories were laid by Yan and collaborators.[7–10] In this section we first review certain results from Refs. 7 and 8 for the case of $(\phi^4)_2$ theory.

We start from the Lagrangian density

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi-m^2\phi^2)-\frac{\lambda}{4!}\phi^4 \ . \tag{2.1}$$

We choose λ greater than zero so that the Hamiltonian is bounded. Further, the mass parameter m^2 is chosen positive so that the vacuum state is the normal vacuum at least for small coupling.

As is well known[7] in light-front formulation the number of independent variables describing a dynamical system is reduced by half as compared with the conventional equal-time formulation. The equations of motion and the commutation relations between true dynamical variables are derived from Schwinger's action principle.[11]

In $1+1$ dimensions, the equation of motion is

$$\partial^+\partial^-\phi+m^2\phi+\frac{\lambda}{3!}\phi^3=0 \ . \tag{2.2}$$

Here

$$\partial^+=2\frac{\partial}{\partial x^-}$$

and

$$\partial^-=2\frac{\partial}{\partial x^+} \ ,$$

where

$$x^+=x^0+x^1$$

and

$$x^-=x^0-x^1 \ .$$

The metric tensor $g^{\mu\nu}$ is given by $g^{++}=g^{--}=0$, $g^{+-}=g^{-+}=2$. The commutation relation is given by

$$i[\phi(x^+,y^-),\phi(x^+,x^-)]\,|_{x^+}=-\tfrac{1}{4}\epsilon(y^- -x^-) \ , \tag{2.3}$$

where $\epsilon(x)$ is the antisymmetric step function:

$$\frac{\partial}{\partial x}\epsilon(x)=-2\delta(x)\ .$$

The commutation relation can be rewritten in the form

$$[\phi(x^+,x^-),\partial^+\phi(x^+,y^-)]\,|_{x^+}=i\delta(y^- - x^-)\ . \quad (2.4)$$

We construct the stress tensor $T^{\mu\nu}$ from the Lagrangian density $\mathcal{L}$ by

$$T^{\mu\nu}=\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial^\nu\phi-g^{\mu\nu}\mathcal{L}\ . \quad (2.5)$$

Now,

$$T^{++}=\partial^+\phi\partial^+\phi \quad (2.6)$$

and

$$T^{+-}=m^2\phi^2+\frac{2\lambda}{4!}\phi^4\ . \quad (2.7)$$

We note that both T^{++} and T^{+-} are positive definite. From the stress tensor $T^{\mu\nu}$ we construct the energy-momentum operator P^μ:

$$P^\mu=\tfrac{1}{2}\int dx^- T^{+\mu}\ . \quad (2.8)$$

The conversation of $T^{\mu\nu}(\partial_\mu T^{\mu\nu}=0)$ implies that both the operators P^+ and P^- are independent of the light-cone time x^+. We also note that P^+ and P^- commute with each other.

Let us introduce the free-field annihilation operator a (defined by $a\,|\,\text{vac}\rangle=0$) and the creation operator $a^\dagger$. In terms of these operators the free-field solution $\phi_0(x^+,x^-)$ can be written as

$$\phi_0(x^+,x^-)=\frac{1}{2\pi}\int\frac{dk^+}{2k^+}[a(k)e^{-ik\cdot x}+a^\dagger(k)e^{ik\cdot x}]\ . \quad (2.9)$$

The commutation relation between the fields imply the following commutation relation between a and $a^\dagger$:

$$[a(k^+),a^\dagger(k'^+)]=2\pi 2k^+\delta(k^+-k'^+)\ . \quad (2.10)$$

In the rest of this section we construct the light-front momentum and energy operator in the discretized version. In doing so we follow the conventions of Pauli and Brodsky.[1]

Discretization is introduced by the replacement

$$k^+\to k_n^+=\frac{2\pi}{L}n,\quad n=1,2,3,\ldots,\Lambda\ . \quad (2.11)$$

Since $k^+=k^0+k^1$, k^+ can be zero for a massive particle only when $k^1\to-\infty$. It is important to note that the above construction omits the zero-momentum states and neglects what is referred to as the zero-mode problem. As we discuss further below, this could be a significant issue. It is convenient to introduce the dimensionless variable

$$\xi=\frac{\pi x^-}{L}\ . \quad (2.12)$$

Then

$$\tfrac{1}{2}k^+x^-=n\xi\ . \quad (2.13)$$

In the discretized version, the free-field solution is given by

$$\phi_0(x^+,x^-)=\frac{1}{\sqrt{4\pi}}\sum_1^\Lambda\frac{1}{\sqrt{n}}(a_n e^{-ik_\mu^{(n)}x^\mu}+a_n^\dagger e^{ik_\mu^{(n)}x^\mu})\ . \quad (2.14)$$

Note that the factors $1/\sqrt{4\pi}$ and $1/\sqrt{n}$ are introduced so that a_n and $a_m^\dagger$ obey the commutation relation $[a_n,a_m^\dagger]=\delta_{n,m}$. The interacting field ϕ is chosen to coincide at $x^+=0$ with the free-field solution ϕ_0. We choose

$$\phi(x^-,0)=\phi_0(x^-,0)\ . \quad (2.15)$$

One also introduces operators K and H such that

$$P^+=\frac{2\pi}{L}K \quad (2.16)$$

and

$$P^-=\frac{L}{2\pi}H\ . \quad (2.17)$$

Thus K is the dimensionless momentum operator and H is the Hamiltonian operator with dimensions of mass squared. The invariant-mass operator $M^2=P^+P^-=KH$ is independent of L.

In the discretized version, the momentum K and the Hamiltonian H are given by

$$K=\sum_1^\Lambda n a_n^\dagger a_n\ , \quad (2.18)$$

and

$$H=H_0+H_1+H_2\ , \quad (2.19)$$

where

$$H_0=\sum_n\frac{1}{n}a_n^\dagger a_n\left[m^2+\frac{\lambda}{4\pi}\frac{1}{2}\sum_k\frac{1}{k}\right]\ , \quad (2.20a)$$

$$H_1=\frac{1}{4}\frac{\lambda}{4\pi}\sum_{klmn}\frac{a_k^\dagger a_l^\dagger a_m a_n}{\sqrt{klmn}}\delta_{m+n,k+l}\ , \quad (2.20b)$$

$$H_2=\frac{1}{6}\frac{\lambda}{4\pi}\sum_{klmn}\left[\frac{a_k^\dagger a_l a_m a_n+a_n^\dagger a_m^\dagger a_l^\dagger a_k}{\sqrt{klmn}}\right]\delta_{k,m+n+l}\ . \quad (2.20c)$$

Since we are dealing with a quantum field theory one should expect divergences. For $(\phi^4)_2$ theory, the only divergent graph is the "tadpole" (one-loop self-energy) which is logarithmically divergent.[12] The logarithmically divergent additive term to m^2 in H_0 is the DLFQ manifestation of the tadpole contribution. We elaborate on this point in the Appendix. We can readily remove this divergence by considering the normal-ordered Hamiltonian[13] which we now adopt for our numerical work.

III. RELEVANT ISSUES

The DLFQ method proposed in Ref. 1 has four distinct features: (1) the quantization on the light-front surface instead of the usual equal-time formulation; (2) discretization in momentum space; (3) choice of a Fock-space basis with the normal (perturbative) vacuum as the lowest-energy state in the spectrum; (4) treatment of the mass parameter in the Langrangian as the *adjustable* bare mass and the lowest eigenvalue of the invariant-mass matrix as the *fixed* physical mass.

One may ask the question whether the equal light-cone time and the equal ordinary time formulations of the quantum field theories are equivalent to each other. This is a nontrivial issue since a spacelike surface can never be brought into a light-front surface by any finite Lorentz transformation.[8] For self-interacting scalar-field models the proof of the equivalence of the S matrices obtained in the two formulations to all orders in perturbation theory has been given by Chang and Yan.[8] It is still legitimate to ask whether the DLFQ method yields results consistent with that demonstration, whether there are multiparticle bound states, and how the method is implemented in the strong-coupling region. In the remainder of this paper we will show how the DLFQ yields results consistent with what is known from quantization in ordinary space-time. We also show evidence for a lack of multiparticle bound states. In addition we demonstrate that the method is currently limited to the weak-coupling region.

Discretization in momentum space of course breaks Lorentz invariance. The method proposed in Ref. 1 is to take advantage of the fact that K is a bounded operator which commutes with the Hamiltonian H. Further, since k^+ for each individual quanta can only be positive, there are only a finite number of basis states for a given finite K (provided we neglect the zero-mode problem). The exact spectra is only obtained as $K\to\infty$, the continuum limit. This is one of the central issues in the current effort. Here, we will concern ourselves with seeking the continuum limit by extrapolating to *large but finite* values of K.

Following Ref. 1 we have neglected the $k^+=0$ states which should, in principle, be included even at finite values of K. However, this opens the possibility of a zero-momentum condensate and would require an extension to the method proposed in Ref. 1. We defer this particular issue to a future effort.

The Hamiltonian is diagonalized in the subspace spanned by the basis states for a given K. Even though the box length L has been eliminated from the eigenvalues of the invariant-mass operator M^2, the eigenvalues and eigenvectors have dependence on K as we show in specific examples. It is of great interest to see just how the results in DLFQ method approach the continuum limit as a function of the coupling constant λ.

In the real world of $3+1$ dimensions renormalization is necessary for solving a field theory. How one incorporates renormalization without violating physical principles is a crucial issue. In order to avoid divergences in the $(1+1)$-dimensional model under present study, we need only the renormalization of the mass which can be accomplished by normal ordering with respect to the mass parameter appearing in the Lagrangian. Then we are left with the *finite*-mass renormalization which has its origin in many body interactions. In their treatment of the two-dimensional Yukawa model Pauli and Brodsky[1] followed the mass renormalization scheme introduced by Brooks and Frautschi[14] for that same model in ordinary space-time. In our case this implies that for a given value of λ and $m_{\text{phys}}{}^2$ one diagonalizes the Hamiltonian matrix for an initial guess for the bare mass m^2 and obtains the lowest eigenvalue e_1. Then one iterates to solve the nonlinear equation

$$e_1[m^2,\lambda]-m_{\text{phys}}{}^2=0 \qquad (3.1)$$

until convergence is achieved to within a required accuracy ($m_{\text{phys}}{}^2$ was chosen to be 1.0). By definition, this mass renormalization method preserves the mass gap and in the case of $(\phi^4)_2$ theory we now show that it restricts the solution to the weak-coupling region. Indeed, Brooks and Frautschi also showed that this mass renormalization scheme apparently avoided certain pathologies that otherwise occur at strong coupling in the two-dimensional Yukawa model.

IV. NUMERICAL PROCEDURE AND RESULTS

We denote a general state in the Fock-space basis as $|n_1{}^{m_1},n_2{}^{m_2},n_3{}^{m_3},\ldots\rangle$ in order to represent a state with m_1 quanta with n_1 units of momentum and so on. for a given K one has $K=n_1m_1+n_2m_2+\cdots$. Let us denote the square of the physical mass of the boson by $m_{\text{phys}}{}^2$. The finite-mass renormalization is implemented by insisting that for each value of K the lowest excitation (in other words, the mass gap with respect to the perturbative vacuum) has the invariant mass m_{phys}.

(a) $K=0$. Since we have neglected zero modes ($k^+=0$ states), the only basis state is the vacuum state $|\text{vac}\rangle$:

$$K|\text{vac}\rangle=0|\text{vac}\rangle\ . \qquad (4.1)$$

Hence

$$M^2|\text{vac}\rangle=0|\text{vac}\rangle\ . \qquad (4.2)$$

Thus $|\text{vac}\rangle$ is the only state with $M^2=0$.

(b) $K=1$. We have a single state $|1^1\rangle$ with

$$\langle 1^1|M^2|1^1\rangle=m^2=m_{\text{phys}}{}^2\ . \qquad (4.3)$$

Thus for $K=1$ finite-mass renormalization is not an issue.

(c) $K=2$. We have two states $|1\rangle=|2^1\rangle$ and $|2\rangle=|1^2\rangle$:

$$M_1{}^2=\langle 1|M^2|1\rangle=m^2=m_{\text{phys}}{}^2\ , \qquad (4.4)$$

$$M_2{}^2=\langle 2|M^2|2\rangle=4m^2+\frac{\lambda}{4\pi}=4m_{\text{phys}}{}^2+\frac{\lambda}{4\pi}\ , \qquad (4.5)$$

$$\langle 1|M^2|2\rangle=0\ . \qquad (4.6)$$

Again mass renormalization is not an issue. $|2\rangle$ is the state containing two comoving particles which are at rest with respect to each other. Since λ is greater than zero, M_2 is greater than $2m_{\text{phys}}$ in agreement with the well-known result that $(\phi^4)_2$ theory has no two-particle bound states.[5]

At this stage we have to specify how to approach the continuum limit. The situation at $K=2$ is quite instructive since we have a single two-particle state with each particle carrying momentum fraction $x=\frac{1}{2}$. (In general $x_i=k_i{}^+/K$). On the other hand, for any value of momentum one should have a continuum of two-particle states

TABLE I. The dimensionality N of the Hamiltonian matrix as a function of K.

K	N
2	2
4	5
6	11
8	22
10	42
12	77
14	135
16	231
18	385
20	627

such that the fractional momenta carried by the constituents vary *continuously* from 0 to 1 in each state. Obviously such a situation arises only if we take the limit $K \rightarrow \infty$. We concentrate primarily on this state which reappears at even values of K.

(d) $K \geq 4$. The dimensionality of the Hamiltonian matrix grows rapidly as K increases as illustrated in Table I. This table applies to an arbitrary scalar field theory in $1+1$ dimensions. In this application to $(\phi^4)_2$ theory these matrices are reducible to submatrices of approximately equal size for the even- and odd-particle cases.

We adopt the notation that states are identified by their $\lambda=0$ structure. For efficiency, we present the ratio of the mass of the lowest two-particle $|(K/2)^2\rangle$ to the mass ($\sqrt{e_1}=M_1$) of the lowest single-particle state $|K^1\rangle$. We choose $m=1.0$ for the calculations without mass renormalization. We summarize our results for this ratio in Fig. 1 as a function of K for different values of λ. These results alone indicate that the invariant mass of this state approaches $2m_{\rm phys}$ for large K in the weak-coupling region ($\lambda \leq 10.0$). Convergence becomes much slower as the coupling becomes stronger and we return to this issue below.

The results begin to depend significantly on whether mass renormalization is adopted when λ exceeds about 10. In the $(\phi^4)_2$ model the only dimensionless parameter is λ/m^2 and the differences in these results for increasing λ can be easily understood. In the case of $\lambda=24$ the dashed curve represents $\lambda/m^2=24$ for all K while the solid curve represents λ/m^2 decreasing with K to the point where it is approximately 18 at $K=16$.

For the moment, let us concentrate on results in the weak-coupling regime which are less dependent on the normalization issue. Does the fact that the invariant mass of the state $|(K/2)^2\rangle$ approaches $2m_{\rm phys}$ in the continuum limit imply that $(\phi^4)_2$ theory is a free-field theory? To answer this question, we study the Fock-space decomposition of this state for $\lambda=2.4$ as a function of K. Let us denote the square of the coefficient of the state $|(K/2)^2\rangle$ by C_0 and the sum of the squares of the coefficients of all two-particle state components of this state by C_1. In Table II we present C_0 and C_1 as a function of K. The fact that C_0 differs from unity with increasing K indicates the presence of scattering in the continuum limit. C_1 remains close to unity indicating that the dominant mixing of the two-particle state $|(K/2)^2\rangle$ is with other two-particle states (all of which have higher invariant masses at $\lambda=0$). It has been known previously[15] that the renormalized coupling for ϕ^4 is nonvanishing $1+1$ dimensions, indicating the nontrivial nature of the theory. We have therefore obtained results consistent with that conclusion.

TABLE II. C_0, the square of the coefficient in the Fock-space expansion of the state $|(K/2)^2\rangle$ and C_1, the sum of the squares of the coefficients of all two-particle state components of this state as function of K at $\lambda=2.4$.

K	C_0	C_1
2	1.0000	1.0000
4	0.9894	1.0000
6	0.9762	0.9996
8	0.9627	0.9993
10	0.9496	0.9992
12	0.9375	0.9991
14	0.9263	0.9991

So far our attention has been focused on the state which has an invariant mass of $2m_{\rm phys}=2m$ at $\lambda=0$. The matrix diagonalization gives the invariant mass and Fock-space composition of many multiparticle states. We noted before that Table I gives the number of states as a function of K. The lowest excitation is the single-particle state whose mass is fixed at $m_{\rm phys.}=m=1.0$. Then at $K=4$, there are 2 two-particle states, 1 three-particle state, and 1 four-particle state. All these states reappear at K values which are integer multiples of 4. Let us arbitrarily select the case with mass renormalization for the moment. In Fig. 2 we present the invariant mass of these four states as function of K for $\lambda=2.4$ As before we denote a state by its Fock-space structure at $\lambda=0$ and the mass at $\lambda=0$ is shown as a horizontal reference line. The results for these three- and four-particle states indicate the lack of bound states in the continuum limit since the convergence is similar to that of the two-particle state.

We now return to the question of the behavior of the

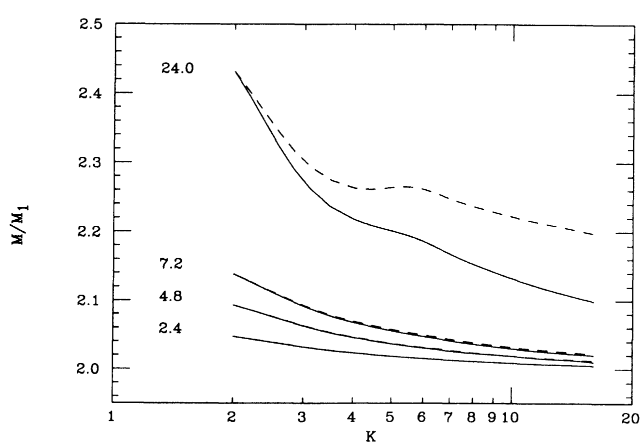

FIG. 1. Invariant mass of the two-particle state $|(K/2)^2\rangle$ as a function of K for different values of λ. Solid lines: with mass renormalization; dashed lines: without mass renormalization. Smooth lines are drawn through results obtained at even values of K.

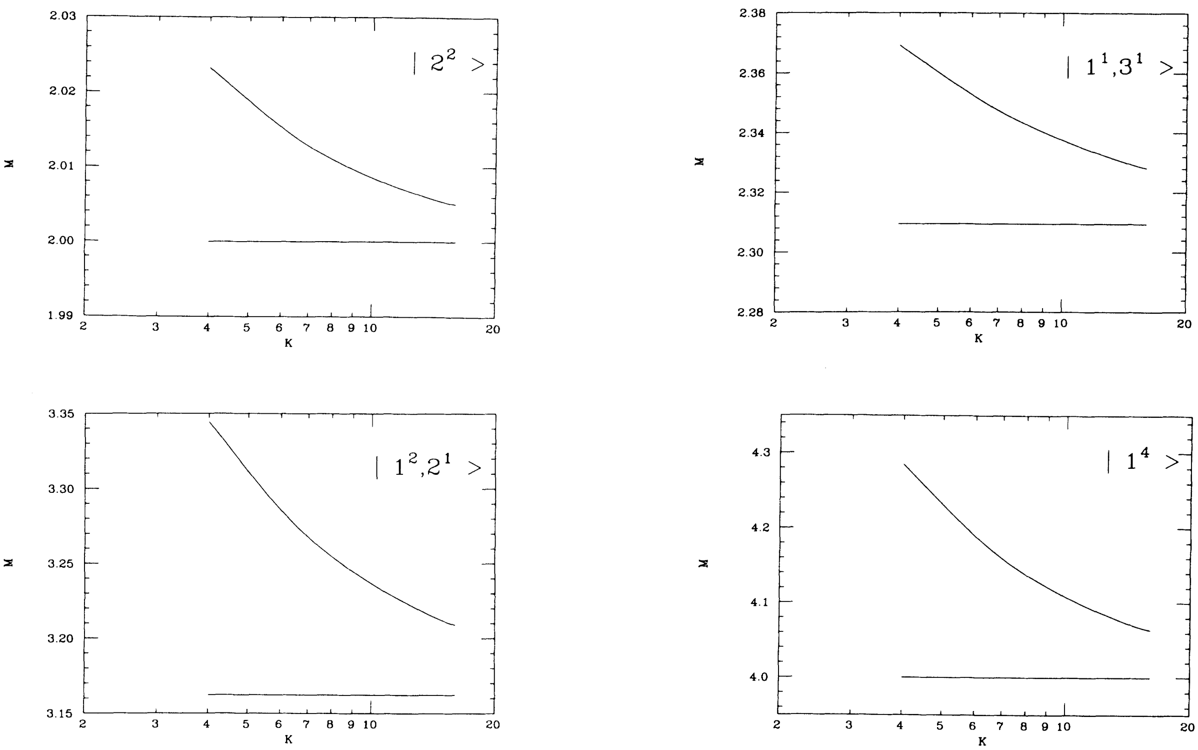

FIG. 2. Invariant mass of all the states at $K=4$ (except lowest one which is fixed by $m_{\text{phys}}{}^2=1.0$) as function of K for $\lambda=2.4$. The horizontal reference line shows the mass at $\lambda=0$. Smooth lines are drawn through results obtained at values of K which are multiples of 4.

results with increased coupling. Without mass renormalization, the DLFQ results yield a single-particle invariant mass squared e_1 which decreases with increasing λ and eventually becomes negative for all values of $K \geq 4$. The calculations which incorporate the finite-mass renormalization define the value of the mass gap and thus avoid this vanishing mass gap. We now ask whether we have really gained anything by the finite-mass gap constraint. The answer is no and is demonstrated in the following way by considering the results at $K=16$ as a typical example. Here the mass gap is found to vanish at the critical coupling $\lambda_c=43.9$. The values of λ/m^2 with mass renormalization are plotted in Fig. 3 as a function of λ and are clearly seen to approach λ_c as $\lambda \rightarrow \infty$. Thus, it is impossible to go to strong coupling (to exceed λ_c) by simply adopting mass renormalization in the DLFQ method. The above conclusion should not be so surprising since the $(\phi^4)_2$ theory depends only on one dimensionless parameter λ/m^2.

The actual value of λ_c changes with K and our value at $K=16$ should be compared with Chang's Hartree result[12] of 54.3 when expressed in our conventions for the coupling constant. For comparison, another numerical method recently introduced[16] obtains $22.8 \leq \lambda_c \leq 51.6$. We do not dwell on the significance of our result for λ_c since we have restricted our discussions to a single phase of the theory. If, however, we had obtained λ_c in the continuum limit and found it be larger than Chang's result then we would have identified a serious problem with the DLFQ method.

V. SUMMARY AND CONCLUSIONS

DLFQ, a recently proposed method to solve field theories is applied to $(\phi^4)_2$ theory to understand the strengths and weaknesses of this scheme. Quantization on the light front leads to a spectrum which is in agreement with properties established within constructive quantum field theory for the $(\phi^4)_2$ scalar field model.[4,5] The physi-

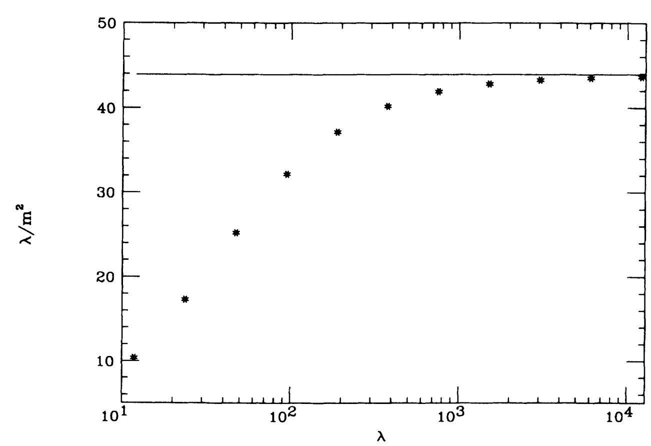

FIG. 3. The intrinsic dimensionless coupling λ/m^2 as a function of λ in the scheme with $m_{\text{phys}}{}^2=1.0$. The bare coupling at which the mass gap vanishes in the calculation with fixed mass parameters in the Lagrangian ($m=1.0$) is shown as a horizontal line. Both results correspond to $K=16$.

cal spectrum emerges only after we take the limit of the dimensionless momentum operator $K \to \infty$. In this effort we have obtained results with modest values of K which are sufficient to sense the continuum limit for weak coupling. We obtained evidence for a lack of multiparticle bound states.

Calculations with a fixed mass parameter in the Lagrangian produce a vanishing mass gap as coupling strength increases. We believe this to be a manifestation in the DLFQ method of the nontrivial vacuum structure of the theory. A positive viewpoint of this breakdown is as follows: had the DLFQ method not revealed *some* pathology using the perturbative vacuum with strong coupling we would have a major concern about the overall validity of the method. A finite-mass renormalization amounts to preserving the mass gap with respect to the normal vacuum and preserves the phase structure. The strength of the dimensionless coupling λ/m^2 always remains *below* the critical coupling (at which the mass gap vanishes) thus restricting the theory to the weak-coupling region. The slow convergence of results with increased coupling may also be a direct consequence of the underlying nontrivial vacuum structure. Lastly, we emphasize the fact that we have neglected the important question of the zero-mode problem altogether in the present study. For a boson field theory the presence of zero-momentum condensates may be essential for a study of its vacuum structure.

In conclusion we would summarize our own efforts in conjunction with the results of Refs. 1 and 2 in the following way. While certain limitations have been identified, the DLFQ method has continued to yield results in test applications which are in agreement with results from other methods and it has produced a number of interesting predictions.

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APPENDIX

In this appendix we outline the calculation of the self-energy graph for ϕ^4 theory at the one-loop level. In $1+1$ dimensions we have

$$-i\Sigma(p^2) = -i\frac{\lambda}{2}\int \frac{d^2k}{(2\pi)^2}\frac{i}{k^2-m^2+i\epsilon} . \tag{A1}$$

$\Sigma(p^2)$ is independent of p^2. In space-time variables, $d^2k = dk^0 dk^1$. After the Wick rotation, i.e., $k^0 \to ik^4$, we have the well-known result

$$\Sigma = \frac{\lambda}{4\pi}\ln\frac{\Lambda}{m}$$

or

$$m_{\text{phys}}{}^2 = m^2 + \frac{\lambda}{4\pi}\ln\frac{\Lambda}{m} .$$

Here m_{phys} is the physical mass and Λ is the high-momentum cutoff.

In light-cone variables we have

$$\Sigma = \frac{i\lambda}{4}\int \frac{dk^+dk^-}{(2\pi)^2}\frac{1}{k^+k^- - m^2 + i\epsilon} , \tag{A2}$$

i.e.,

$$\Sigma = \frac{i\lambda}{4}\int \frac{dk^+}{2\pi k^+}\int \frac{dk^-}{2\pi}\frac{1}{k^- - (m^2+i\epsilon)/k^+} . \tag{A3}$$

If $k^+=0$ is neglected, $\Sigma = 0$ which directly leads to a contradiction.[10] This contradiction underscores our claim in the text that it is important to address the zero-mode problem. One way to proceed is to use the integral representation

$$\frac{1}{k^+k^- - m^2 + i\epsilon} = -i\int_0^\infty d\alpha\, e^{i\alpha(k^+k^- - m^2 + i\epsilon)} . \tag{A4}$$

Then

$$\Sigma = \frac{\lambda}{4}\frac{1}{2\pi}\int_0^\Lambda \frac{d\alpha}{\alpha}e^{i\alpha(-m^2+i\epsilon)} . \tag{A5}$$

Next introduce the covariant regularization

$$\int_0^\infty \frac{d\alpha}{\alpha}e^{i\alpha(-m^2+i\epsilon)} \to \int_0^\infty \frac{d\alpha}{\alpha}\left(e^{i\alpha(-m^2+i\epsilon)} - e^{i\alpha(-\Lambda^2+i\epsilon)}\right) , \tag{A6}$$

where we have introduced the high-momentum cutoff Λ. Using

$$\int_0^\infty \frac{d\alpha}{\alpha}\left(e^{i\alpha(a+i\epsilon)} - e^{i\alpha(b+i\epsilon)}\right) = \ln\frac{a}{b} , \tag{A7}$$

we have

$$\Sigma(p^2) = \frac{\lambda}{4\pi}\ln\frac{\Lambda}{m} \tag{A8}$$

which agrees with the previous result. Coming back to the additive logarithmically divergent term in H_0, it can be written as

$$\frac{1}{2}\sum_k \frac{1}{k} \to \frac{1}{2}\int_{m^2}^{\Lambda^2+m^2}\frac{dk}{k} = \ln\frac{\Lambda}{m} . \tag{A9}$$

[1]H. C. Pauli and S. J. Brodsky, Phys. Rev. D **32**, 1993 (1985); **32**, 2001 (1985).

[2]T. Eller, H. C. Pauli, and S. J. Brodsky, Phys. Rev. D **35**, 1493 (1987).

[3]T. Barnes and G. J. Daniell, Phys. Rev. D **28**, 2045 (1983); C. S. Hsue, K. Kummel, and P. Ueberholz, *ibid*. **32**, 1435 (1985); J. Abad, J. G. Esteve, and A. F. Pacheco, *ibid*. **32**, 2729 (1985); P. M. Stevenson and I. Roditi, *ibid*. **33**, 2305 (1985).

[4]B. Simon, *The* $P(\phi)_2$ *Euclidean (Quantum) Field Theory* (Princeton University Press, Princeton, New Jersey, 1974).

[5]J. Glimm and A. Jaffe, *Quantum Physics: A. Functional Integral Point of View* (Springer-Verlag, New York, 1981).

[6]P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[7]S.-J. Chang, R. G. Root, and T.-M. Yan, Phys. Rev. D **7**, 1133 (1973).

[8]S.-J. Chang and T.-M. Yan, Phys. Rev. D **7**, 1147 (1973).

[9]T.-M. Yan, Phys. Rev. D **7**, 1760 (1973).

[10]T.-M. Yan, Phys. Rev. D **7**, 1780 (1973).

[11]J. Schwinger, Phys. Rev. **82**, 914 (1951); **91**, 713 (1953).

[12]S.-J. Chang, Phys. Rev. D **13**, 2778 (1976).

[13]S. Coleman, Phys. Rev. D **11**, 2088 (1975).

[14]E. D. Brooks and S. C. Frautschi, Z. Phys. C **23**, 263 (1984).

[15]F. Cooper, B. Freedman, and D. Preston, Nucl. Phys. **B210**, 210 (1982).

[16]M. Funke, U. Kaulfuss, and H. Kummel, Phys. Rev. D **35**, 621 (1987).