Grassmann-valued processes for the Weyl and the Dirac equations

B. Gaveau
Universite P. et M. Curie, Mathematiques Tour 45-46, 4 Place Jussieu, 75252 Paris, France

L. S. Schulman
Department of Physics, Clarkson University, Potsdam, New York 13676
and Technion, Haifa, Israel
(Received 16 March 1987)

The time evolution of a massless particle satisfying the Weyl equation is described as a stochastic process on a space of Grassmann variables, in close formal analogy with the use of Brownian motion for Schrödinger evolution. The Grassmann process is then combined with a Poisson process previously used for the evolution of Dirac electrons. (In that process electrons propagate as massless left- or right-handed particles with random changes in direction occurring at an average rate given by the particle mass.) Electron motion is thus given as an expectation over the two processes and spatial position supersymmetrically acquires contributions from sums of products of Grassmann variables.

I. INTRODUCTION

Several publications have recently proposed definitions for a Feynman path integral for the Dirac equation, in particular, in the case of 1 space and 1 time dimension,[1–3] in momentum space for any number of dimensions,[4] and also a spinor chain formalism for 3 space and 1 time dimension.[5] Besides these articles there is an extensive literature on the subject taking many imaginative and interesting approaches (for example, Refs. 6 and 7). In this article we aim to extend the Feynman "checkerboard" approach,[8] in particular, along the lines initiated in Refs. 1 and 2. Our method will be to define a new kind of mathematical object which provides for the Dirac equation a natural generalization of the classical path integral.

Using Ref. 1, one can describe the Dirac electron in the following way. For a short time it proceeds as a massless particle at the speed of light, subject to the Weyl equation. Then it flips, reversing direction by 180° and now obeys the *other* (parity-reversed) Weyl equation. It continues to flip at random times with an average flip rate given by the particle's physical mass m. This random flipping is the Poisson process of Refs. 1, 2, and 5 and some physical speculations about it are given in earlier papers. The intervening massless propagation accords with the fact that the electron velocity operator only has eigenvalues of magnitude 1 (in units $c=1$).

In the present article we also express the intervening massless motion as a "sum over paths." From Ref. 1 there is already a path or a history of flips; that is, the overall development of the particle involves a sum over functions $N(t)$, each $N(t)$ being an integer-valued Poisson process giving the number of flips up to time t. Between flips the particle satisfies the Weyl equation and our goal is to express this evolution too as a sum over paths. For our purposes the latter concept is taken to mean that the particle is imagined to undergo some stochastic process and its propagator is obtained from a coherent sum over paths of that process. As will be seen below the way in which we achieve that goal is by allowing the process to take place on a space of Grassmann variables. A path is a Grassmann-variable-valued function $\theta(s)$ and the sum over paths employs Berezin-type integrals over the Grassmann variables. In fact we also rewrite the usual path integral in a slightly idiosyncratic form so as to bring out the way in which our Grassmann path sum parallels the standard formalism. Finally, in the last two sections we combine the massless propagator stochastic process with the Poisson process of Ref. 1 to obtain a process for the full Dirac equation.

Historically, associating a stochastic process with the Schrödinger equation has been one of the most fruitful developments in mathematical physics. Whether some of the advantages gained there will carry over to the formalism proposed in this paper remains to be seen. We also mention that although our approach is different from that of Jacobson,[5] the resemblance is sufficient to suggest that some of the interesting ideas he has on space-time structure apply to our Grassmann processes as well. That is, for Jacobson pairwise products of spinors are elementary spatial steps; for us they are pairwise products of Grassmann variables (consistent with general supersymmetric ideas). Moreover, for us as well as for Jacobson the flipping due to the mass and the massless propagation are conceptually distinct so that one would not expect to combine our two stochastic processes into a single one.

II. FEYNMAN-KAC RECAST

We begin by giving a new proof of the classical Feynman-Kac formula for the heat equation, stressing aspects that will be useful later. We start from the Trotter formula giving the heat propagator of

$$K=\frac{1}{2}\frac{d^2}{dx^2}-V ,$$

$$e^{tK}=\exp\left[t\left[\frac{1}{2}\frac{d^2}{dx^2}-V\right]\right]$$
$$=\lim_{n\to\infty}\left[\exp\left[\frac{-t}{n}V\right]\exp\left[\frac{1}{2}\frac{t}{n}\frac{d^2}{dx^2}\right]\right]^n ,$$

where the product is time ordered. The Feynman-Kac formula gives a way to separate the kinetic-energy part and the potential-energy part in the Trotter formula. For any $1\le k\le n$, let G_k be normalized independent Gaussian variables; we then have

$$\exp\left[\frac{1}{2}\frac{t}{n}\frac{d^2}{dx^2}\right]=\left\langle\exp\left[\left[\frac{t}{n}\right]^{1/2}G_k\frac{d}{dx}\right]\right\rangle . \quad (1)$$

Thus by introducing the random variable G_k we have managed to take the "square root" of d^2/dx^2. It follows that

$$\left[\exp\left[\frac{-t}{n}V\right]\exp\left[\frac{1}{2}\frac{t}{n}\frac{d^2}{dx^2}\right]\right]^n$$
$$=\left\langle\prod_{k=1}^{n}\exp\left[\frac{-t}{n}V\right]\exp\left[\left[\frac{t}{n}\right]^{1/2}G_k\frac{d}{dx}\right]\right\rangle . \quad (2)$$

However, within the averaging (angular) brackets it is easy to commute $\exp(-tV/n)$ and $\exp[(t/n)^{1/2}G_k d/dx]$ since the latter is simply a translation operator; for example,

$$\exp\left[\left[\frac{t}{n}\right]^{1/2}G_3\frac{d}{dx}\right]\exp\left[\frac{-t}{n}V\right]\exp\left[\left[\frac{t}{n}\right]^{1/2}G_2\frac{d}{dx}\right]\exp\left[-\frac{t}{n}V\right]\exp\left[\left[\frac{t}{n}\right]^{1/2}G_1\frac{d}{dx}\right]$$
$$=\exp\left[\left[\frac{t}{n}\right]^{1/2}G_3\frac{d}{dx}\right]\exp\left\{-\frac{t}{n}\left[V(x)+V\left[x+\left[\frac{t}{n}\right]^{1/2}G_2\right]\right]\right\}\exp\left[\left[\frac{t}{n}\right]^{1/2}(G_1+G_2)\frac{d}{dx}\right]$$
$$=\exp\left\{-\frac{t}{n}\left[V\left[x+\left[\frac{t}{n}\right]^{1/2}G_3\right]+V\left[x+\left[\frac{t}{n}\right]^{1/2}G_2+\left[\frac{t}{n}\right]^{1/2}G_3\right]\right]\right\}\exp\left[\left[\frac{t}{n}\right]^{1/2}(G_1+G_2+G_3)\frac{d}{dx}\right]$$

and finally

$$\exp(tK)=\lim_{n\to\infty}\left[\exp\left[-\frac{t}{n}V\right]\exp\left[\frac{1}{2}\frac{t}{n}\frac{d^2}{dx^2}\right]\right]^n$$
$$=\lim_{n\to\infty}\left\langle\exp\left[\frac{-t}{n}\sum_{k=2}^{n}V\left[x+\sum_{j=k}^{n}\left[\frac{t}{n}\right]^{1/2}G_j\right]\right]\exp\left[\left[\frac{t}{n}\right]^{1/2}\sum_{k=1}^{n}G_k\frac{d}{dx}\right]\right\rangle \quad (3)$$

which is exactly the discretized version of the Feynman-Kac formula. The term

$$\sum_{j=k}^{n}\left[\frac{t}{n}\right]^{1/2}G_j$$

is identified with the discretized version of the Brownian path at time $s=kt/n$. [In Appendix A we relate (3) to the form of the functional integral more commonly used in the physics literature.]

III. GRASSMANN VARIABLES

Following Refs. 9 and 10 we shall first introduce three Grassmann variables θ_α, $\alpha=1,2,3$, with the relation $\theta_\alpha\theta_\beta+\theta_\beta\theta_\alpha=0$. We also define the Berezin integral of polynomials in θ_α by

$$\int 1\,d\theta_\alpha=0 ,$$
$$\int\theta_\beta d\theta_\alpha=\delta_{\alpha\beta} . \quad (4)$$

Call $d\boldsymbol{\theta}=d\theta_1 d\theta_2 d\theta_3$ and let $\boldsymbol{\theta}$ be the vector of components θ_α. Let $\sigma_1,\sigma_2,\sigma_3$ be the three Pauli matrices which anticommute with each other and have square unity. We have $\sigma_1\sigma_2\sigma_3=iI$ and by expanding the exponential we can check that

$$\int\exp(i\boldsymbol{\theta}\cdot\boldsymbol{\sigma})\exp(\boldsymbol{\theta}M\boldsymbol{\theta})d\boldsymbol{\theta}=I+i\sum\epsilon_{\alpha\beta\gamma}\sigma_\alpha m_{\beta\gamma} , \quad (5)$$

where $M=(m_{\beta\gamma})$ is an antisymmetric 3×3 matrix. Let us take $m_{\beta\gamma}=\frac{1}{2}\sum\eta_\alpha\epsilon_{\alpha\beta\gamma}$ ($\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol). Then Eq. (5) becomes

$$\frac{\exp(i\boldsymbol{\eta}\cdot\boldsymbol{\sigma})}{\cosh\lambda}=I+i(\boldsymbol{\sigma}\cdot\boldsymbol{\eta})=\int\exp(i\boldsymbol{\theta}\cdot\boldsymbol{\sigma})\exp(\boldsymbol{\theta}M\boldsymbol{\theta})d\boldsymbol{\theta} \quad (6)$$

with $\tanh\lambda=|\boldsymbol{\eta}|$. Equation (6) is a fundamental identity analogous to (1). In (1) we take a square root by introducing an auxiliary integral. In (6) we separate factors $\boldsymbol{\sigma}$ and $\boldsymbol{\eta}$ through another sort of integral.

IV. GRASSMANN STOCHASTIC PROCESSES

For any time s let us introduce Grassmann variables $\theta_\alpha(s)$, $\alpha=1,2,3$, satisfying the relations

$$\theta_\alpha(s)\theta_\beta(s')+\theta_\beta(s')\theta_\alpha(s)=0 \text{ for any } \alpha,\beta,s,s' . \quad (7)$$

This family gives a Grassmann-valued path; for any s we can also introduce the Berezin integral with respect to $d\boldsymbol{\theta}(s)$ as above and define a path integral of a function $F(\boldsymbol{\theta}(\cdot))$ by

$$\int\mathcal{D}\boldsymbol{\theta}(\cdot)F(\boldsymbol{\theta}(\cdot))=\int\prod_s d\boldsymbol{\theta}(s)F(\boldsymbol{\theta}(\cdot)) . \quad (8)$$

Let us now write the Weyl equation

$$i\frac{\partial\psi}{\partial t}=H\psi\equiv\sum_{\alpha=1}^{3}\sigma_\alpha\left[i\frac{\partial}{\partial x_\alpha}+eA_\alpha\right]\psi\ . \tag{9}$$

The propagator is given by the Trotter formula with time ordering understood:

$$\exp[t\sigma\cdot(\partial-ie\,\mathbf{A})]=\prod_{s\le t}[I+\sigma\cdot(\partial-ie\,\mathbf{A})ds]$$
$$=\lim_{n\to\infty}\left[I+\sigma\cdot(\partial-ie\,\mathbf{A})\frac{t}{n}\right]^n\ . \tag{10}$$

Using (6), each factor in the second member of (10) can be written as

$$\left[I+\sigma\cdot(\partial-ie\,\mathbf{A})\frac{t}{n}\right]_k=\int d\theta\left[\frac{kt}{n}\right]\exp\left[i\sigma\cdot\theta\left[\frac{kt}{n}\right]\right]\exp\left[\theta\left[\frac{kt}{n}\right]M_k\theta\left[\frac{kt}{n}\right]\right]\ , \tag{11}$$

where the index $1\le k\le n$ denotes the kth term in the chronological product (10) and where

$$(M_k)_{\alpha\beta}\equiv(\nabla_k)_{\alpha\beta}+(\mathcal{A}_k)_{\alpha\beta}\ ,$$
$$(\nabla_k)_{\alpha\beta}\equiv\frac{1}{2}\sum_\gamma\epsilon_{\gamma\alpha\beta}\left[-i\frac{\partial}{\partial x_\gamma}\right]\frac{t}{n}\ , \tag{12}$$
$$(\mathcal{A}_k)_{\alpha\beta}\equiv\frac{1}{2}\sum_\gamma\epsilon_{\gamma\alpha\beta}(-eA_\gamma)\frac{t}{n}\ ,$$

and in the limit $n\to+\infty$, we obtain

$$e^{tK}=\lim_{n\to\infty}\left[I+\sigma\cdot(\partial-ieA)\frac{t}{n}\right]^n$$
$$=\lim_{n\to\infty}\int\prod_{k=1}^{n}d\theta\left[\frac{kt}{n}\right]\prod_{k=1}^{n}\left\{\exp i\sigma\cdot\theta\left[\frac{kt}{n}\right]\exp\left[\theta\left[\frac{kt}{n}\right](\mathcal{A}_k+\nabla_k)\theta\left[\frac{kt}{n}\right]\right]\right\}\ . \tag{13}$$

In this way, the analogy between Eqs. (1) and (6) has been carried forward to an analogy between (2) and (13).

V. WEYL PROPAGATOR AS EXPECTATION OF GRASSMANN STOCHASTIC PROCESS

Now, we disentangle the second member of (13) in the same way that we did the Feynman-Kac formula. First we remark that any operator-valued function which is even in the θ's commutes with any function of the θ's containing no spatial operator and using the Trotter formula to separate $\mathcal{A}$ and ∇ we can rewrite the integrand in (13) as

$$\prod_{k=1}^{n}\exp\left[i\sigma\cdot\theta\left[\frac{kt}{n}\right]\right]\prod_{k=1}^{n}\exp\left[\theta\left[\frac{kt}{n}\right]\mathcal{A}_k\theta\left[\frac{kt}{n}\right]\right]\exp\left[\theta\left[\frac{kt}{n}\right]\nabla_k\theta\left[\frac{kt}{n}\right]\right] \tag{14}$$

where the first product is time ordered with earlier times on the right. The commutation relations for the σ_α can be used to show that

$$\prod_{k=1}^{n}\exp\left[i\sigma\theta\left[\frac{kt}{n}\right]\right]=\exp\left[i\sigma\sum_{k=1}^{n}\theta\left[\frac{kt}{n}\right]\right]$$
$$\times\exp\left[-\sum_{j>k}\theta\left[\frac{jt}{n}\right]\theta\left[\frac{kt}{n}\right]\right] \tag{15}$$

but we shall continue to write the left-hand side of (15) as a time-ordered product. Moreover, we have

$$\exp\left[\theta\left[\frac{kt}{n}\right]\mathcal{A}_k\theta\left[\frac{kt}{n}\right]\right]=\exp\left[-e\,\mathbf{A}_k\cdot\delta\xi\left[\frac{kt}{n}\right]\right]\ , \tag{16}$$

where

$$\delta\xi_\gamma\left[\frac{kt}{n}\right]=\frac{t}{n}\sum_{\alpha<\beta}\epsilon_{\gamma\alpha\beta}\theta_\alpha\left[\frac{kt}{n}\right]\theta_\beta\left[\frac{kt}{n}\right] \tag{17}$$

and also

$$\exp\left[\theta\left[\frac{kt}{n}\right]\nabla_k\theta\left[\frac{kt}{n}\right]\right]=\exp\left[-i\partial_k\cdot\delta\xi\left[\frac{kt}{n}\right]\right]\ . \tag{18}$$

All the $\delta\xi$ are even functions of the $\theta(kt/n)$. By the supersymmetric rules of Berezin[11] (or by the definition of supermanifold) they define spatial variables. More precisely, if η is function of an even number of Grassmann variables and $f(\mathbf{x})$ is an indefinitely differentiable function, we have, by definition,

$$[\exp(\eta\cdot\partial)f](\mathbf{x})=f(\mathbf{x}+\eta)\ , \tag{19}$$

where the second member is defined by Taylor's formula around $\mathbf{x}$. If we now define

$$\xi\left[\frac{kt}{n}\right]=\sum_{j=1}^{k}\delta\xi\left[\frac{jt}{n}\right], \tag{20}$$

we obtain

$$\prod_{k=1}^{n}\exp\left[\theta\left[\frac{kt}{n}\right]\mathcal{A}_k\theta\left[\frac{kt}{n}\right]\right]\exp\left[\theta\left[\frac{kt}{n}\right]\nabla_k\theta\left[\frac{kt}{n}\right]\right]=\exp\left[-e\sum_{k=1}^{n}\mathbf{A}\left[\mathbf{x}-i\xi\left[\frac{(k-1)t}{n}\right]\right]\cdot\delta\xi\left[\frac{kt}{n}\right]\right]\exp[-i\partial\cdot\xi(t)] \tag{21}$$

and in the limit $n\to+\infty$, the propagator for the Weyl equation can be written as

$$\exp[t\sigma\cdot(\partial-ieA)]=\int\mathcal{D}\theta(\cdot)\exp\left[i\sigma\cdot\int_0^t\theta(s)\right]\exp\left[-e\int_0^t\mathbf{A}(\mathbf{x}-i\xi(s))\delta\xi(s)\right]\exp[-i\partial\cdot\xi(t)], \tag{22}$$

(where the first exponential is understood to be a time-ordered product.

VI. ANALOGY WITH OTHER FORMS

If we compare this formula with the one given, for example, in Ref. 12 for the imaginary time propagator of the Pauli equation, we see that $-i\xi(s)$ is the Grassmann analog of Brownian motion. The last term in the integrand of (22) is the usual translation operator by $-i\xi(t)$. The middle exponential is the analog of the nonanticipating Itô integral of the electromagnetic field along the Grassmann Brownian path; the first exponential is the spin contribution which has been separated as an odd function of the θ's (the last two terms are even functions of the θ's). We also note that the integral $\int_0^t\theta(s)$ is taken without ds; in some sense $\theta(s)$ is the Grassmann analog of white noise.

VII. THE DIRAC EQUATION: POISSON PROCESS

We now turn to the Dirac equation; an electron is represented by a four-component spinor

$$\psi=\begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix},$$

where ψ_+ and ψ_- are two-component spinors. We employ the Weyl representation of the Dirac equation, so that ψ satisfies

$$i\frac{\partial}{\partial t}\psi=\begin{bmatrix}\sigma\cdot(i\partial+e\,\mathbf{A}) & 0\\ 0 & -\sigma\cdot(i\partial+e\,\mathbf{A})\end{bmatrix}\psi+\begin{bmatrix}0 & m\\ m & 0\end{bmatrix}\psi,$$

where m is the mass of the particle (we have taken $\hbar=1$, $c=1$). Multiply the equation by $-i$ and define $\Phi=e^{-mt}\psi$ to obtain

$$\frac{\partial\Phi}{\partial t}=\begin{bmatrix}\sigma\cdot(\partial-ie\,\mathbf{A}) & 0\\ 0 & -\sigma\cdot(\partial-ie\,\mathbf{A})\end{bmatrix}\Phi+\begin{bmatrix}-m & -im\\ -im & -m\end{bmatrix}\Phi. \tag{23}$$

We assume that $\mathbf{A}$ does not depend on time t. Let us now recall the method for solving (23) presented in Ref. 1. We introduce a Poisson process with flipping rate m. Call $N(t)$ the value of the process at time t [$N(t)$ is an integer-valued process starting at time $t=0$ from 0]. Let $T_1<T_2<\cdots$ be the times of the jumps of $N(t)$ so that $T_1,T_2-T_1,\ldots,T_k-T_{k-1},\ldots$ are independently distributed stochastic times with exponential law

$$\text{Prob}(\tau\geq t)=e^{-mt}. \tag{24}$$

Finally we define the random variable

$$\epsilon(t)=(-1)^{N(t)}\epsilon_0, \tag{25}$$

where ϵ_0 is $+1$ or -1. Then the value of $\Phi_\pm(\mathbf{x},t)$ is given by the formula

$$\Phi_{\epsilon_0}(\mathbf{x},t)=E\left[(-i)^{N(t)}\exp\left[\int_0^t\epsilon(s)\sigma\cdot(\partial-ie\,\mathbf{A})ds\right]\times\Phi_{\epsilon(t)}(\cdot,t=0)\mid\epsilon(0)=\epsilon_0\right], \tag{26}$$

where the notation

$$\exp\left[\int_0^t\epsilon(s)\sigma\cdot(\partial-ie\,\mathbf{A})ds\right]\Phi_{\epsilon(t)}(\cdot,t=0)$$

means that the 2×2 matrix-valued operator

$$\exp\left[\int_0^t\epsilon(s)\sigma\cdot(\partial-ie\,\mathbf{A})ds\right]$$

acts on the two-component spinor $\Phi_{\epsilon(t)}(\cdot,t=0)$ (taken at time $t=0$) and the symbol E denotes the expectation over the realizations of the Poisson process. The factor $(-i)^{N(t)}$ takes into account the fact that at each jump of $N(s)$ we have to multiply by the phase factor $-i$; this is due to the term $-im$ in the matrix $\left(\begin{smallmatrix}-m & -im\\ -im & -m\end{smallmatrix}\right)$ appearing in (23). This fact was absent in Ref. 1 because we were writing the formula in imaginary time, so that $-im$ was m. Formula (26) reduces the computation of the propagator of the Dirac equation to that of the Weyl equation.

Remark. When $\mathbf{A}$ depends on t explicitly, Eq. (26) is no longer true as it stands. We have to interpret

$$\exp\left[\int_0^t\epsilon(s)\sigma\cdot(\partial-ie\,\mathbf{A}(\cdot,s))ds\right]$$

as a chronological product.

VIII. THE DIRAC EQUATION: DOUBLE STOCHASTIC PROCESS

We combine formula (26) with the previous formalism of Grassmann-valued paths. Let us fix a sample of the Poisson process. Then

$$\exp\left[\int_0^t \epsilon(s)\sigma\cdot(\partial - ie\,\mathbf{A})ds\right] = \exp\{(-1)^n\epsilon_0(t-T_n)[\sigma\cdot(\partial-ie\,\mathbf{A})]\}\exp[(-1)^{n-1}\epsilon_0(T_n-T_{n-1})\sigma\cdot(\partial-ie\,\mathbf{A})]\times\cdots$$
$$\times\exp\{-\epsilon_0(T_2-T_1)[\sigma\cdot(\partial-ie\,\mathbf{A})]\}\exp\{\epsilon_0T_1[\sigma\cdot(\partial-ie\,\mathbf{A})]\}\ , \tag{27}$$

where $n=N(t)$ is the number of jumps up to time t of the Poisson process and $T_1<T_2<\cdots$ are the times of the jumps.

Now we introduce the Grassmann variables: as before, for each time t, we define $\theta(t)$ satisfying the relations

$$\theta_\alpha(s)\theta_{\alpha'}(s')+\theta_{\alpha'}(s')\theta_\alpha(s)=0 \tag{28}$$

for any $\alpha,\alpha'=1,2,3$, and $s,s'\geq 0$. In (27), we rewrite each block using the Grassmann path integral (22): more precisely we write

$$\exp[\epsilon_k(T_k-T_{k-1})\sigma\cdot(\partial-ie\,\mathbf{A})] = \int_{[T_{k-1},T_k]}\mathcal{D}\theta(\cdot)\exp\left[i\sigma\cdot\int_{T_{k-1}}^{T_k}\theta(s)\right]\exp\left[-e\int_{T_{k-1}}^{T_k}\epsilon_k\,\mathbf{A}(\mathbf{x}-i\epsilon_k\xi(s))\cdot\delta\xi(s)\right]$$
$$\times\exp\{-i\epsilon_k\partial\cdot[\xi(T_k)-\xi(T_{k-1})]\}\ , \tag{29}$$

where $\epsilon_k=(-1)^{k-1}\epsilon_0$ and $\xi(s)$ is constructed as in (17) and (20). The symbol

$$\int_{[T_{k-1},T_k]}\mathcal{D}\theta(\cdot)\cdots$$

denotes the Grassmann path integral over the Grassmann path θ between time T_{k-1} and time T_k.

We now combine all the factors in (27) rewritten as in (29). Using the fact that an even function of Grassmann variables commutes with anything and commuting the Hamiltonian operators as in (21) and (19), we obtain

$$\exp\left[\int_0^t\epsilon(s)\sigma\cdot(\partial-ie\,\mathbf{A})ds\right]=\int\mathcal{D}\theta(\cdot)\exp\left[i\sigma\int_0^t\theta(s)\right]\exp\left[-e\int_0^t\mathbf{A}(\mathbf{x}-i\zeta(s))\delta\zeta(s)\right]\exp\left[-i\partial\cdot\int_0^t\delta\zeta(s)\right]\ , \tag{30}$$

where as usual the factor involving $\sigma\cdot\theta$ is a time-ordered product and we have denoted

$$\delta\zeta(s)=\epsilon(s^+)\delta\xi(s),\quad \zeta(s)=\int_0^s d\zeta(s)\ . \tag{31}$$

$\delta\xi(s)$ is given by (17) and $\epsilon(s^+)=\lim_{s\to s^+}\epsilon(s)$ is the value of $\epsilon(s)$ immediately after s.

Remark. We can expand (31) as

$$\zeta(t)=\sum_{k=1}^n\epsilon_k(\xi(T_k)-\xi(T_{k-1}))+\epsilon_{n+1}(\xi(t)-\xi(T_n)) \tag{32}$$

with the convention $T_0=0$, $\zeta(0)=0$, $\epsilon_k=(-1)^{k+1}\epsilon_0$, and $0<T_1<T_2<\cdots<T_n<t$ are the n jump times before t.

We can combine (30) and (26) to obtain

$$\Phi_{\epsilon_0}(\mathbf{x},t)=E\left[\int\mathcal{D}\theta(-i)^{N(t)}\exp\left[i\sigma\cdot\int_0^t\theta(s)\right]\exp\left[-e\int_0^t\mathbf{A}(\mathbf{x}-i\zeta(s))\cdot\delta\zeta(s)\right]\Phi_{\epsilon(t)}(\mathbf{x}-i\zeta(t))\,|\,\epsilon(0)=\epsilon_0\right]\ ,$$

where we have used the fact that

$$\left[\exp\left[-i\partial\int_0^t\delta\zeta(s)\right]f\right](\mathbf{x})=f(\mathbf{x}-i\zeta(t))\ .$$

Equation (33) gives the time evolution of the Dirac state vector as an expectation over both the Poisson and the Grassmann processes.

Remark. The propagators (22) and (33) differ in all the following ways: superficially, significantly, and cryptically. Superficially because in one case [Eq. (22)] we write the operator and in the other [Eq. (33)] we write the solution of the initial-value problem. The significant differences are the factor $(-i)^{N(t)}$ in (33), the use of four rather than two components and finally the cryptic difference. The latter is the appearance of a factor $\epsilon(\cdot)$ in the definition of $\zeta(\cdot)$, while for $\xi(\cdot)$ the concept of flipping [hence $\epsilon(\cdot)$] is absent. We emphasize this point because the expression for $\zeta(\cdot)$ in terms of Grassmann variables and flips [Eq. (31)] is essential to the space-spin relation implicit in our formalism. Similar ideas have been emphasized by Jacobson.[5]

ACKNOWLEDGMENT

This work was supported in part by NSF Grant No. PHY85 18806.

APPENDIX A

We here relate Eq. (3) to the form of the functional integral more commonly found in the physics literature. The sum of the G's is a Brownian motion, so we define

$$b_k=-\left[\frac{t}{n}\right]^{1/2}\sum_{j=1}^k G_j,\quad k=1,\ldots,n\ . \tag{A1}$$

Thus

$$\langle(\text{“}db\text{”})^2\rangle=\left\langle\frac{t}{n}G^2\right\rangle=\frac{t}{n}=\text{“}dt\text{”}\ .$$

Equation (3) can now be written

$$e^{tK}=\left\langle \exp\left[-\frac{t}{n}\sum_{j=1}^{n-1}V(x-b_n+b_j)\right]\exp\left[-b_n\frac{d}{dx}\right]\right\rangle . \tag{A2}$$

To see that this is a propagator apply it to a function $\psi(\cdot)\equiv\psi(\cdot,0)$ (the initial condition). The second exponential in (A2) is simply a translation operator and we get

$$\psi(x,t)=(e^{tK}\psi)(x)=\left\langle \exp\left[-\frac{t}{n}\sum_{j=1}^{n-1}V(x-b_n+b_j)\right]\psi(x-b_n)\right\rangle \tag{A3}$$

which can be rewritten

$$\psi(x,t)=\int dy\left\langle \exp\left[-\frac{t}{n}\sum_{j=1}^{n-1}V(y+b_j)\right]\delta(x-y-b_n)\right\rangle\psi(y) . \tag{A4}$$

The usual propagator is the kernel of this integral and denoting it by $G(x,t;y)$ we have

$$G(x,t;y)=\left\langle \exp\left[\frac{-t}{n}\sum_{j=1}^{n-1}V(y+b_j)\right]\delta(x-y-b_n)\right\rangle . \tag{A5}$$

Let $\xi(s)$ be a Brownian motion path beginning at the point y [$\xi(0)=y$ and $\xi(jt/n)$ corresponds to $y+b_j$]. Let E_t be the expectation over all such Brownian motion paths (Wiener measure). Then (A5) becomes

$$G(x,t;y)=E_t\left[\exp\left[-\int_0^t ds\, V(\xi(s))\right]\delta(x-\xi(t))\right] . \tag{A6}$$

Finally, one can use the unnormalized conditional expectation E_{xt} (see, for example, Ref. 13, Chap. 9) to complete the identification

$$G(x,t;y)=E_{xt}\left[\exp\left[-\int_0^t ds\, V[\xi(s)]\right]\right] , \tag{A7}$$

since E_{xt} and E_t are related by

$$E_{xt}(Q)=E_t[Q\delta(x-\xi(t))] .$$

As presented in Ref. 13, Eq. (A7) is the usual Feynman-Kac formula and analytically continues to the path integral.

[1]B. Gaveau, T. Jacobson, M. Kac, and L. S. Schulman, Phys. Rev. Lett. **53**, 419 (1984).

[2]T. Jacobson and L. S. Schulman, J. Phys. A **17**, 375 (1984).

[3]I. Ichinose, J. Math. Phys. **25**, 1810 (1984).

[4]B. Gaveau, J. Funct. Anal. **58**, 310 (1984).

[5]T. Jacobson, J. Phys. A **17**, 2433 (1984).

[6]A. O. Barut and N. Zanghi, Phys. Rev. Lett. **52**, 2009 (1984); A. O. Barut and I. H. Duru, *ibid.* **53**, 2355 (1984).

[7]F. Ravndal, Phys. Rev. D **21**, 2823 (1980).

[8]R. P. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw Hill, New York, 1965).

[9]J. Schwinger, *Quantum Kinematics and Dynamics* (Benjamin, New York, 1970).

[10]F. Berezin, *The Method of Second Quantization* (Academic, New York, 1966).

[11]F. Berezin, *Analysis and Algebra with Anticommuting Variables* (Nauka, Moscow, 1982).

[12]B. Gaveau and J. Vauthier, J. Funct. Anal. **44**, 388 (1984).

[13]L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).