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Colliding plane gravitational waves: A class of nondiagonal soliton solutions

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Using the inverse scattering method we construct a general class of nondiagonal solutions of the vacuum Einstein field equations describing the collision of two plane-fronted impulsive gravitational waves accompanied with gravitational shock waves with the same front. It is a two-parameter class of solutions: the first represents the angle between the directions of polarization of the two waves; the second determines the wave profile.

I. INTRODUCTION

It is well known that in the general theory of relativity one can construct plane-fronted gravitational waves[1] as regions of nonzero Riemann tensor, confined between two parallel planes, propagating at the speed of light in the direction of the normal to the planes. These waves transport a finite amount of energy per unit area and therefore if the distance between the two planes goes to zero, one gets an impulsive gravitational wave, namely, a wave with a δ-function profile in the curvature. Impulsive waves can be regarded as the most "elementary" gravitational waves. The existence of impulsive gravitational waves as exact solutions of Einstein's vacuum equations is a remarkable fact: on the contrary, no impulsive waves can exist in Maxwell's theory, because a δ-function profile in the electromagnetic energy will imply a square root of a δ function in the field variables, which is not permissible for physical descriptions. In addition, because of the nonlinearity of the Einstein equations, the interaction of gravitational waves exhibits properties and characteristics different from the interaction of electromagnetic waves. Therefore, exact solutions describing interaction of gravitational waves provide a scenario in which essential new features of the general theory of relativity can be revealed.

In 1971 Khan and Penrose[2] found an exact solution of the vacuum Einstein equations describing the collision of two impulsive gravitational plane waves with collinear polarization and they showed that the final result of the collision is the creation of a space-time singularity. Szekeres[3,4] showed that the occurrence of the singularity in no way depends on the amplitude of the incoming waves. The generalization of the Khan-Penrose solution for the case of waves with noncollinear polarization was accomplished by Nutku and Halil[5] in 1977.

The question whether the collision of gravitational waves necessarily produces a singularity was first answered by Bell and Szekeres[6] in 1974. They considered the collision of two pure electromagnetic waves with constant profile, and they showed that if the metric functions satisfy the O'Brian-Synge[7] conditions, then the electromagnetic waves must necessarily be accompanied by two impulsive gravitational waves, the interaction of which, unlike the case of pure gravitational impulsive waves, do not produce a curvature singularity.

Only recently a variety of new solutions describing the interaction of gravitational waves has been obtained, following the investigation done by Chandrasekhar[8] and one of us (V.F.) on the connections existing between the mathematical theory of black holes and the mathematical theory of colliding waves. The crucial point of this investigation is that in both cases the Einstein equations are reducible to the same Ernst equation. Thus it has been shown that the Nutku-Halil and the Kerr solutions have the same Ernst potential, and in this sense we say that they are "equivalent." This analogy has been generalized to the interaction of gravitational-electromagnetic plane waves and to the interaction of impulsive gravitational waves in a fluid background[9,10] and the solution "equivalent" to the Kerr-Newman metric has been found. Following this approach, a solution which is "equivalent" to the "distorted-black-hole" solution[11] has also been found.

All these solutions possess a common feature: the creation of a curvature singularity following the collision, due to the mutual focusing of the two waves, though the singularity can be considerably weakened by coupling with acoustic waves of sufficient amplitude, as shown in Ref. 10.

The analogy based on the Ernst equation has been developed further: it is well known that the Schwarzschild metric (as well as the Kerr metric) can be obtained by using the soliton method developed by Belinskii and Zakharov.[12,13] The soliton technique is applied to the equations which, in a different gauge, lead to the Ernst equation. The Schwarzschild metric can be obtained by using the Minkowski metric as a seed, and assuming that the space-time is stationary and axisymmetric. In a similar way, the Khan-Penrose solution (which is "equivalent" to the Schwarzschild solution) can be found by using as a seed the Kasner metric:

$$ds^2=t^{-2s_1s_2}(dt^2-dz^2)-t^{2s_1}(dx^1)^2-t^{2s_2}(dx^2)^2 \quad \text{with } s_1+s_2=1 \ , \qquad (1.1)$$

in the case $s_1=s_2=\frac{1}{2}$, and assuming that the space-time has two spacelike Killing vectors. Then the question whether the class of diagonal solutions derived in the case of arbitrary s_1 still represents the collision of gravitational waves, naturally arises. This question has been answered in a preceding paper[14] (this paper will be referred to hereafter as paper I). It has been shown that this class of solutions represents the collision of two linearly polarized impulsive gravitational waves of the same polarization, each supporting a gravitational shock wave. All members of this class develop a curvature singularity after the collision, with one exception: when $s_1=0$ the metric in the region of interaction is isometric to a part of the Schwarzschild interior $(1<r<2, M=1)$ and there is a *coordinate singularity,* corresponding to the Schwarzschild horizon, on the surface where the other solutions possess the curvature singularity.

Many other confirmations to the fact that a horizon can be created, as an alternative to the curvature singularity, by the collision of gravitational shock waves followed: Chandrasekhar and Xanthopoulos[15,16] have found a nondiagonal solution describing the collision of two impulsive + shock gravitational plane waves such that the interaction region is isometric to a part of the Kerr interior, and a class of two-parameter nondiagonal solutions describing the collision of electromagnetic + gravitational shock waves + impulsive gravitational waves which generalizes the Bell-Szekeres solution. All members of this class form a horizon in the region of interaction.

In this paper we discuss the solution already presented in a recent communication[17] which generalizes the results of paper I to the nondiagonal case. By using the inverse scattering method and the Kasner metric as a seed, we obtain a class of nondiagonal solutions of the vacuum Einstein equations describing the collision of two plane-fronted impulsive gravitational waves accompanied by gravitational shock waves with the same fronts, approaching each other from $-\infty$ and $+\infty$. The complete description of the space-time prior the instant of collision has been accomplished by making a proper extension of the metric across the null boundaries corresponding to the wave fronts at the instant of collision. The metric depends on two parameters: the first represents the angle between the directions of polarization of the two waves and the second determines the wave profile. When $s_1=s_2=\frac{1}{2}$ it reduces to the Nutku-Halil solution; when, in addition, $p=1$ it reduces to the Khan-Penrose solution. A subclass of the present solutions corresponding to integral values of the "wave profile parameter" has recently been found by Ernst, Diaz, and Hauser.[18] The solution presented in this paper is the most general solution for the collision of pure gravitational waves obtained so far. All members of the present class of solutions develop a curvature singularity after the instant of collision, except the case $s_1=0$, as in the diagonal case discussed in paper I.

The plan of the paper is the following. In Sec. II we shall give a brief review of the basic equations and of their soliton solution. In Sec. III we shall use this technique to obtain the solution corresponding to two real poles. In Sec. IV we shall extend the metric to the space-time preceding the instant of collision, in Sec. V we shall discuss the occurrence of the singularity on the surface $u^2+v^2=1$, and in Sec. VI we shall discuss the behavior of the complete solution.

II. THE BASIC EQUATIONS AND THEIR SOLITON SOLUTIONS

Consider a space-time which admits globally a pair of commuting, spacelike Killing vectors, say $\partial/\partial x^1$ and $\partial/\partial x^2$. The metric of this space-time, which is said to be plane symmetric, can be written in the form

$$ds^2=\tilde{f}(dz^2-dt^2)+g_{ab}dx^a dx^b, \quad (a,b=1,2) \ , \qquad (2.1)$$

where $t=x^0$ and $z=x^3$, the metric functions depend only on t and z, and the gauge freedom has been used to put

$$\det(g)=t^2 \ . \qquad (2.2)$$

The Einstein equations in vacuum decomposes into two groups of equations:

$$(tg_{,t}g^{-1})_{,t}-(tg_{,z}g^{-1})_{,z}=0 \ ; \qquad (2.3)$$

$$(\ln\tilde{f})_{,t}=-t^{-1}+(4t)^{-1}tr(U^2+V^2) \ , \quad (\ln\tilde{f})_{,z}=(2t)^{-1}tr(U\cdot V) \ , \qquad (2.4)$$

where g denotes the matrix g_{ab}, and

$$U=tg_{,t}g^{-1}, \quad V=tg_{,z}g^{-1} \ . \qquad (2.5)$$

The integrability conditions for Eqs. (2.4) are automatically guaranteed if g satisfies Eq. (2.3). The system of Eqs. (2.2) and (2.3) can be solved by using the inverse scattering method. It has been shown by Belinskii and Zakharov that a couple of linear differential operators depending on a complex spectral parameter λ can be associated with Eq. (2.3) and that the solution of the original equations for the matrix g will be determined by the analytic structure of the eigenvalues λ_n in the λ plane. Once the number of poles has been fixed, a procedure for generating a new solution of Eqs. (2.2) and (2.3) from a known "seed" solution has been defined. The details of the procedure are extensively illustrated in Refs. 12, 13, and 19. We shall write here only the equations we have used to obtain our solution.

If $(g_0)_{ab}$ is the seed metric and $\lambda_k=\mu_k$ are the poles, where

$$\mu_k=\omega_k-z\pm[(\omega_k-z)^2-t^2]^{1/2} \quad (k=1,\ldots,n) \qquad (2.6)$$

and ω_k are arbitrary real or complex constant, the new solution is given by

$$g=t^{-n}\left(\prod_{k=1}^{n}\mu_k\right)g' \ , \qquad (2.7)$$

where

$$g'_{ab}=(g_0)_{ab}-\sum_{k,l}^{n}\frac{D_{kl}m_c^{(k)}(g_0)_{ca}m_d^{(l)}(g_0)_{db}}{\mu_k\mu_l} . \qquad (2.8)$$

$D_{kl}=(\Gamma_{kl})^{-1}$ and Γ_{kl} is an $n\times n$ matrix,

$$\Gamma_{kl}=\frac{m_c^{(k)}(g_0)_{cb}m_b^{(l)}}{\mu_k\mu_l-t^2} , \qquad (2.9)$$

and the vectors $m_a^{(k)}$ are constructed from the matrix $(\Phi_0)_{ab}$ associated with the assigned seed metric:

$$m_a^{(k)}=(m_0)_c^{(k)}[\Phi_0^{-1}(\mu_k,t,z)]_{ca} , \qquad (2.10)$$

$(m_0)_c^{(k)}$ being arbitrary real or complex constants depending on the nature of the poles. The matrix Φ_0 is obtained by solving the linear eigenvalue equations associated with the original equations (2.2) and (2.3) [see, for example, Ref. 19, Eqs. (2.7)]. Once Φ_0 has been found, the completion of the solution for the matrix g requires only algebraic operations. Finally, the solution for the metric coefficient $\tilde{f}$ is obtained explicitly by direct integration of Eqs. (2.4). The general result is

$$\tilde{f}=\tilde{f}_0 t^{-n^2/2}\frac{\left[\prod_{k=1}^{n}\mu_k\right]^{n+1}}{\left[\prod_{\substack{k,l=1\\k>l}}^{n}(\mu_k-\mu_l)^2\right]}\det\Gamma_{kl} , \qquad (2.11)$$

where n is the number of poles.

In the next section we shall study the two-soliton solution that can be obtained by using the Kasner metric (1.1) as seed. The matrix Φ_0 associated with this metric is

$$\Phi_0(\lambda,t,z)=\begin{bmatrix}(t^2+2z\lambda+\lambda^2)^{s_1} & 0\\ 0 & (t^2+2z\lambda+\lambda^2)^{s_2}\end{bmatrix} . \qquad (2.12)$$

III. THE NONDIAGONAL TWO-SOLITON SOLUTION

We shall evaluate the solution (2.7)–(2.11) in the case of two real poles ($n=2$):

$$\begin{aligned}\mu_1&=\omega_1-z-[(\omega_1-z)^2-t^2]^{1/2} ,\\ \mu_2&=\omega_2-z+[(\omega_2-z)^2-t^2]^{1/2} ,\\ \omega_1&=-\omega_2=1 .\end{aligned} \qquad (3.1)$$

This choice of the constants ω_k defines the solution in the region outside the two light cones:

$$(1-z)^2-t^2=0 \text{ and } (-1-z)^2-t^2=0 .$$

From Eqs. (3.1) it is apparent that

$$2\omega_k\mu_k=\mu_k{}^2+2z\mu_k+t^2$$

and therefore, from Eq. (2.12) we get

$$\Phi_0^{-1}(\mu_k,t,z)=\begin{bmatrix}\left[\frac{1}{2\omega_k\mu_k}\right]^{s_1} & 0\\ 0 & \left[\frac{1}{2\omega_k\mu_k}\right]^{s_2}\end{bmatrix} . \qquad (3.2)$$

From Eq. (2.10) it follows that

$$m_1^{(k)}=\left[\frac{1}{2\omega_k\mu_k}\right]^{s_1}(m_0)_1^k \text{ and } m_2^{(k)}=\left[\frac{1}{2\omega_k\mu_k}\right]^{s_2}(m_0)_2^k , \qquad (3.3)$$

and from Eq. (2.9)

$$\Gamma_{kl}=\frac{1}{\mu_k\mu_l-t^2}\left[t^{2s_1}\left[\frac{1}{2\omega_k\mu_k}\right]^{s_1}\left[\frac{1}{2\omega_l\mu_l}\right]^{s_1}(m_0)_1^k(m_0)_1^l+t^{2s_2}\left[\frac{1}{2\omega_k\mu_k}\right]^{s_2}\left[\frac{1}{2\omega_l\mu_l}\right]^{s_2}\times(m_0)_2^k(m_0)_2^l\right] . \qquad (3.4)$$

If we introduce the quantities

$$\begin{aligned}&a=\frac{1}{2\mu_1\omega_1},\quad b=\frac{1}{2\mu_2\omega_2} ,\\ &A=\mu_1{}^2-t^2,\quad B=\mu_2{}^2-t^2,\quad C=\mu_1\mu_2-t^2 ,\end{aligned} \qquad (3.5)$$

the determinant of the matrix Γ_{kl} is

$$\det\Gamma=\frac{t^2}{4ABC^2}\{(\mu_1-\mu_2)^2[t^{4s_1}(ab)^{2s_1}(p-1)^2+t^{4s_2}(ab)^{2s_2}(p+1)^2]+q^2[(a^{2s_2}b^{2s_1}+a^{2s_1}b^{2s_2})C^2+2abAB]\} . \qquad (3.6)$$

At this point we make a choice of the constants $(m_0)_a^{(k)}$:

$$\begin{aligned}&(m_0)_2^1(m_0)_2^2+(m_0)_1^1(m_0)_1^2=p,\quad (m_0)_2^1(m_0)_2^2-(m_0)_1^1(m_0)_1^2=1 ,\\ &(m_0)_2^1(m_0)_1^2-(m_0)_1^1(m_0)_2^2=q,\quad (m_0)_2^1(m_0)_1^2+(m_0)_1^1(m_0)_2^2=0 .\end{aligned} \qquad (3.7)$$

It follows that

$$(m_0)_2^1(m_0)_2^2=\tfrac{1}{2}(p+1),\quad (m_0)_1^1(m_0)_1^2=\tfrac{1}{2}(p-1),\quad (m_0)_2^1(m_0)_1^2=\tfrac{1}{2}q,\quad (m_0)_1^1(m_0)_2^2=-\tfrac{1}{2}q , \qquad (3.8)$$

and

$$q^2+p^2=1 . \tag{3.9}$$

The matrix D_{kl} can now be calculated and the final form of the metric components, written in terms of the poles μ_1 and μ_2, is

$$g_{11}=\frac{\mu_1\mu_2}{t^2}t^{2s_1}\left[1-\frac{t^{2s_1}}{ABC\,\det\Gamma}\left\{(p-1)^2(tab)^{2s_1}(a^2AC+2abAB+b^2BC)+q^2t^{2s_2}a^2b^2\left[\left[\frac{a}{b}\right]^{2s_1}AC-2AB+\left[\frac{b}{a}\right]^{2s_1}BC\right]\right\}\right], \tag{3.10}$$

$$g_{22}=\frac{\mu_1\mu_2}{t^2}t^{2s_2}\left[1-\frac{t^{2s_2}}{ABC\,\det\Gamma}\left\{(p+1)^2(tab)^{2s_2}(a^2AC+2abAB+b^2BC)+q^2t^{2s_1}a^2b^2\left[\left[\frac{a}{b}\right]^{2s_2}AC-2AB+\left[\frac{b}{a}\right]^{2s_2}BC\right]\right\}\right], \tag{3.11}$$

$$g_{12}=-\frac{\mu_1\mu_2}{t^2}\frac{qt^2}{ABC\,\det\Gamma}\{(p-1)t^{2s_1}[a^3b^{2s_1}AC-b^3a^{2s_1}BC+(a^2b^{2s_1+1}-b^2a^{2s_1+1})AB]+(p+1)t^{2s_2}[-a^3b^{2s_2}AC+b^3a^{2s_2}BC-(a^2b^{2s_2+1}-b^2a^{2s_2+1})AB]\} , \tag{3.12}$$

$$\tilde{f}=t^{-2s_1s_2-2}\frac{(\mu_1\mu_2)^3}{(\mu_1-\mu_2)^2}\det\Gamma . \tag{3.13}$$

With the change of coordinates

$$t=\sin\phi\sin\theta,\quad z=\cos\phi\cos\theta,\quad 0\le\phi\le\pi,\quad 0\le\theta\le\pi , \tag{3.14}$$

our solution takes the form

$$ds^2=f(d\phi^2-d\theta^2)+g_{ab}dx^adx^b \quad (a,b=1,2) , \tag{3.15}$$

where

$$f=\text{const}\times(\sin\phi\sin\theta)^{s_1^2+s_2^2-1}\left[4(1-\cos\phi)^2\left[p_1\left[\frac{1+\cos\phi}{1-\cos\phi}\right]^{2s_1}+p_2\left[\frac{1+\cos\phi}{1-\cos\phi}\right]^{2s_2}\right]+q^2\left\{2(\cos^2\phi-\cos^2\theta)+(1+\cos\theta)^2\left[\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{2s_1}+\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{2s_2}\right]\right\}\right], \tag{3.16}$$

$$\tilde{g}_{11}=-(\sin\phi\sin\theta)^{2s_1}\left[2(1+\cos\phi)^2\left[p_1\left[\frac{1+\cos\phi}{1-\cos\phi}\right]^{2s_1}+p_2\left[\frac{1-\cos\phi}{1+\cos\phi}\right]^{2s_1}\right]+\frac{q^2}{2}\frac{1+\cos\phi}{1-\cos\phi}\left\{-2(\cos^2\phi-\cos^2\theta)+\sin^2\theta\left[\left[\frac{1+\cos\theta}{1-\cos\theta}\right]^{2s_1}+\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{2s_1}\right]\right\}\right], \tag{3.17}$$

$$\tilde{g}_{22}=-(\sin\phi\sin\theta)^{2s_2}\left[2(1+\cos\phi)^2\left[p_1\left[\frac{1-\cos\phi}{1+\cos\phi}\right]^{2s_2}+p_2\left[\frac{1+\cos\phi}{1-\cos\phi}\right]^{2s_2}\right]+\frac{q^2}{2}\frac{1+\cos\phi}{1-\cos\phi}\left\{-2(\cos^2\phi-\cos^2\theta)+\sin^2\theta\left[\left[\frac{1+\cos\theta}{1-\cos\theta}\right]^{2s_2}+\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{2s_2}\right]\right\}\right], \tag{3.18}$$

$$\tilde{g}_{12}=-2q\sin^2\theta(1+\cos\phi)\left[\frac{(\sin\phi\sin\theta)^{2s_1}}{(1-\cos\phi)^{2s_1}}\sqrt{p_1}\left[\frac{\cos\phi-\cos\theta}{(1+\cos\theta)(1-\cos\theta)^{2s_1}}-\frac{\cos\phi+\cos\theta}{(1-\cos\theta)(1+\cos\theta)^{2s_1}}\right]-\frac{(\sin\phi\sin\theta)^{2s_2}}{(1-\cos\phi)^{2s_2}}\sqrt{p_2}\left[\frac{\cos\phi-\cos\theta}{(1+\cos\theta)(1-\cos\theta)^{2s_2}}-\frac{\cos\phi+\cos\theta}{(1-\cos\theta)(1+\cos\theta)^{2s_2}}\right]\right], \tag{3.19}$$

$$K=-\frac{1+\cos\phi}{8}\frac{f}{\text{const}\times(\sin\phi\sin\theta)^{s_1^2+s_2^2-1}}\ , \tag{3.20}$$

$$g_{ab}=-\frac{\tilde{g}_{ab}(1-\cos\phi)}{4K}\ , \tag{3.21}$$

and

$$p_1=\frac{(p-1)^2}{2^{4s_1}},\ \ p_2=\frac{(p+1)^2}{2^{4s_2}}\ . \tag{3.22}$$

Notice that the change of coordinates (3.14) restricts the solution to the region between the two light cones.

IV. THE EXTENSION OF THE SPACE-TIME PRIOR TO THE INSTANT OF COLLISION

The solution (3.15)–(3.22) is defined inside the circle $\cos^2\phi+\cos^2\theta=1$. When we put $s_1=s_2=\frac{1}{2}$ and $\cos\phi$ and $\cos\theta$ range from 0 to 1, our metric reduces to the metric describing the interaction region, respectively, of the Nutku-Halil solution, if $q\neq 0$, and of the Khan-Penrose solution, if $q=0$ [see Ref. 8, Eq. (80)]. According to the usual interpretation of the coordinates, $\cos\phi$ measures the time from the instant of collision, and $\cos\theta$ measures the distance normal to the planes (x^1,x^2) spanned by the two Killing vectors (in Chandrasekhar's notation $\eta=\cos\phi$ and $\mu=\cos\theta$).

In order to see if the requirements for a consistent description of colliding gravitational waves are satisfied by our solution, it is convenient to introduce a system of null coordinates u and v, which are related to ϕ and θ by the equations

$$\begin{aligned}\cos\phi&=u\sqrt{1-v^2}+v\sqrt{1-u^2}\ ,\\ \cos\theta&=u\sqrt{1-v^2}-v\sqrt{1-u^2}\ .\end{aligned} \tag{4.1}$$

The resulting metric is

$$ds^2=\frac{4f(u,v)du\,dv}{[(1-u^2)(1-v^2)]^{1/2}}+g_{ab}(u,v)dx^a dx^b\ . \tag{4.2}$$

The region of the (u,v) plane where the collision takes place is

$$0\leq u\leq 1\ \ \text{and}\ \ 0\leq v\leq 1\ . \tag{4.3}$$

We shall divide the space-time into four regions:

region I $u<0,\ \ v<0$,

region II $u<0,\ \ v>0$,

region III $u>0,\ \ v<0$,

region IV $0\leq u<1,\ \ 0\leq v<1$.

As usual, the extension of the metric to regions I, II, and III is accomplished by the Penrose-Khan substitution:

$$u\rightarrow uH(u)\ \ \text{and}\ \ v\rightarrow vH(v)\ ,$$

where $H(u)$ and $H(v)$ are the Heaviside functions. This algorithm for the extension ensures that the resulting metric is C^0 and that the curvature scalars behave consistently with the assumed character of the colliding fronts: for example, if in regions II and III we have gravitational shock waves, we expect that the corresponding metric is Petrov type N, and only the Weyl scalars Ψ_4 (in region II) or Ψ_0 (in region III) are different from zero. Thus, on the null boundary $u=0$ and $0<v<1$ which separates one of the incoming waves from the interaction region, Ψ_4 must be continuous, while Ψ_0 an Ψ_2 have in general a H-function discontinuity. In addition, if we assume that across the null boundaries the metric functions satisfy the O'Brian-Synge conditions, permitting some normal derivatives of the metric to be discontinuous across them, gravitational impulse waves, exhibiting a δ-function profile in the curvature, must necessarily accompany the shock waves. As a consequence, on the null boundary $u=0$ and $0<v<1$, the Weyl scalar Ψ_0 has δ-function discontinuity. (For a detailed discussion of the behavior of the metric functions and of the Weyl scalars on the null boundaries see paper I.)

The extended metric in region II is obtained by the substitution

$$\cos\phi=v\ \ \text{and}\ \ \cos\theta=-v$$

in Eqs. (3.15)–(3.20). The resulting extended metric is

$$f=\text{const}\times[\alpha_1(1+v)^{2s_1^2}(1-v)^{2(s_1-1)^2}+\alpha_2(1-v)^{2s_1^2}(1+v)^{2(s_1-1)^2}]\ , \tag{4.4}$$

$$g_{11}=-(1+v)^{4s_1}\left[\frac{\alpha_1}{\alpha_1(1-v)^{1-2s_1}(1+v)^{2s_1-1}+\alpha_2(1-v)^{2s_1-1}(1+v)^{1-2s_1}}+\frac{\alpha_2}{\alpha_1(1-v)^{1-6s_1}(1+v)^{-2s_1-1}+\alpha_2(1-v)^{-2s_1-1}(1+v)^{1-6s_1}}\right]\ , \tag{4.5}$$

$$g_{22}=-(1+v)^{4s_2}\left[\frac{\alpha_2}{\alpha_1(1-v)^{1-2s_1}(1+v)^{2s_1-1}+\alpha_2(1-v)^{2s_1-1}(1+v)^{1-2s_1}}+\frac{\alpha_1}{\alpha_1(1-v)^{-3+2s_1}(1+v)^{6s_1-5}+\alpha_2(1-v)^{6s_1-5}(1+v)^{-3+2s_1}}\right]\ , \tag{4.6}$$

$$g_{12}=-\frac{4qv(\sqrt{p_1}-\sqrt{p_2})}{\alpha_1(1-v)^{1-2s_1}(1+v)^{2s_1-1}+\alpha_2(1-v)^{2s_1-1}(1+v)^{1-2s_1}}\ , \tag{4.7}$$

where

$$\alpha_1=2p_1+\frac{q^2}{2} \quad \text{and} \quad \alpha_2=2p_2+\frac{q^2}{2} . \tag{4.8}$$

Because our general solution is symmetrical in u and v, the extension to region III is obtained by interchanging v with u in expressions (4.4)–(4.7). The only difference is that g_{12} changes sign, but this is equivalent to a change of the coordinate x^1 to $-x^1$, or x^2 to $-x^2$ in region III.

It is convenient at this point to write the metric in region II in an alternative form:

$$ds^2=4e^{g(v)}du\,dv -(1-v^2)\left[\chi(v)(dx^2)^2+\frac{1}{\chi(v)}[dx^1-q_2(v)dx^2]^2\right] , \tag{4.9}$$

where the functions $g(v)$, $\chi(v)$, and $q_2(v)$ are

$$\begin{aligned} g(v)&=\ln\left[\frac{f(v)}{\sqrt{1-v^2}}\right] ,\\ \chi&=-\frac{1-v^2}{g_{11}(v)} ,\\ q_2&=-\frac{g_{12}}{g_{11}} . \end{aligned} \tag{4.10}$$

In order to be acceptable as representing a single plane wave, the metric (4.4)–(4.7) must satisfy the following requirements: it must not be degenerate, it must be free of curvature singularities, and it must be a solution of the Einstein vacuum equations. In Ref. 15 all these conditions have been summarized into a single equation involving only the metric functions and its first derivatives:

$$\frac{1}{4}e^{-g}\left[-\frac{2}{(1-v^2)^2}+\frac{2v}{1-v^2}g'+\frac{1}{2\chi^2}[(\chi')^2+(q_2')^2]\right] =0 , \tag{4.11}$$

where the prime indicates differentiation with respect to v.

This equation is satisfied by our metric when extended into regions II, as well as the corresponding equation in terms of u is satisfied by the extended metric in region III.

We conclude that our class of solutions represents the collision of gravitational shock waves of different wave profile and of different polarization: the angle between the two directions of polarization is defined by the parameter p, as in the Nutku-Halil solution, and the wave profile is defined by the parameter s_1. Each shock wave is accompanied by an impulsive wave with the same front and with the same polarization.

V. THE OCCURRENCE OF A SINGULARITY ON THE SURFACE $u^2+v^2=1$

We shall analyze in this section the nature of the singularity on the surface $u^2+v^2=1$. The complexity of the metric coefficients makes the analysis of the Weyl scalars very difficult. Thus, we concentrate our attention on the Weyl scalar Ψ_2, which is different from zero in the region of interaction, and which has two important properties: (1) it involves only first derivatives of the metric functions, (2) it is the only Weyl scalar which is invariant under a simultaneous rescaling of the tetrad vectors $\mathbf{l}$ and $\mathbf{n}$. It is therefore plausible to think that if Ψ_2 is singular, the metric will be singular and vice versa.

To compute Ψ_2 we started from its expression given in Ref. 9 in terms of the Ernst potential:

$$\Psi_2=-\frac{1}{8f}(\cot^2\phi-\cot^2\theta)+\frac{(E_{,\phi}-E_{,\theta})(E^*_{,\phi}+E^*_{,\theta})}{2f(1-|E|^2)^2} . \tag{5.1}$$

In terms of the metric components the Ernst potential is

$$E=\frac{-\sin\phi\sin\theta-g_{11}-ig_{12}}{-\sin\phi\sin\theta+g_{11}-ig_{12}} . \tag{5.2}$$

When g_{11} and g_{12} are written explicitly, E is a very complicated expression of the coordinates, and its derivatives are very difficult to handle, even with the help of a computer. It has been easier to write the expression (5.1) directly in terms of the derivatives of g_{11} and g_{12} which are not too difficult to evaluate. The final result is

$$\Psi_2=\frac{\cos^2\theta-\cos^2\phi}{8f\sin^2\phi\sin^2\theta}+\frac{O_-O_+}{8fA^2\tilde{g}_{11}^{\,2}} , \tag{5.3}$$

where

$$\begin{aligned} A&=\frac{4\sin\phi\sin\theta}{1-\cos\phi}K ,\\ O_\mp&=\tilde{g}_{11}(A_{,\phi}\mp A_{,\theta}\mp i\tilde{g}_{12,\phi}+i\tilde{g}_{12,\theta}) -(A\mp i\tilde{g}_{12})(\tilde{g}_{11,\phi}\mp\tilde{g}_{11,\theta}) . \end{aligned} \tag{5.4}$$

Remembering that, according to the definitions (4.1), $u^2+v^2\rightarrow 1$ implies that $\phi\rightarrow 0$ and therefore $\sin\phi\rightarrow 0$, it is easy to check that the first term appearing in Ψ_2 is always singular when $\phi\rightarrow 0$. In order to analyze the second term of Ψ_2, we notice that all functions appearing in Eqs. (5.3) and (5.4) can be expressed in terms of polynomials (or ratio of polynomials) in $\sin\phi$. For example, the expression (3.17) for $\tilde{g}_{11}$ can be written as

$$\tilde{g}_{11}=-\sin^{2s_1}\theta\left[8\left[p_1 2^{4s_1}\sin^{-2s_1}\phi+\frac{p_2}{2^{4s_1}}\sin^{6s_1}\phi\right] +2q^2\sin^{2(s_1-1)}\phi f_a(\theta)\right] , \tag{5.5}$$

where

$$f_a(\theta)=-2(\cos^2\phi-\cos^2\theta) +\sin^2\theta\left[\left[\frac{1+\cos\theta}{1-\cos\theta}\right]^{2s_1}+\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{2s_1}\right] . \tag{5.6}$$

If we write all the metric functions and their derivatives in a similar way, at the end we see that the second term of Ψ_2 can be written as

$$\frac{O_-O_+}{8fA^2\tilde{g}_{11}{}^2}=\frac{1}{x^{2s_1{}^2}}\,\frac{a_1x^{\alpha_1(s_1)}+a_2x^{\alpha_2(s_1)}+\cdots+i(b_1x^{\beta_1(s_1)}+b_2x^{\beta_2(s_1)}+\cdots)}{c_1x^{\gamma_1(s_1)}+c_2x^{\gamma_2(s_1)}+\cdots}\ , \tag{5.7}$$

where $x=\sin\phi$ and $\alpha_i(s_1)$,$\beta_i(s_1)$ and $\gamma_i(s_1)$ are polynomials of *first* degree in s_1.

The asymptotic behavior of this term has been analyzed by using a graphic method: we have plotted all the exponents appearing in the real part and in the imaginary part of the numerator (we call the corresponding graphics, respectively, H_1 and H_2, see Fig. 1) and all the exponents appearing in the denominator (graphic H_3, Fig. 1) versus s_1. From these graphics we know, for a given value of s_1, which is the maximum positive exponent and the minimum negative exponent both in the numerator and in the denominator. For example, when $s_1<-0.5$ and $x\to 0$ the maximum and the minimum terms in the real part are, from the graphics H_1 and H_3 in Fig. 1:

$$\mathrm{Re}\left[\frac{O_-O_+}{8fA^2\tilde{g}_{11}{}^2}\right]\to\frac{1}{x^{2s_1{}^2}}\,\frac{a_1x^{8-12s_1}+a_2x^{20s_1-14}}{c_1x^{10-18s_1}+c_2x^{22s_1-2}}\ . \tag{5.8}$$

The first term, both in the numerator and in the denominator, goes to zero, because $8-12s_1$ and $10-18s_1$ are positive for $s_1<-0.5$. Therefore the asymptotic behavior is dictated by the minimum negative exponents

$$\mathrm{Re}\left[\frac{O_-O_+}{8fA^2\tilde{g}_{11}{}^2}\right]\to\frac{\mathrm{coef}(\theta)}{x^{2s_1{}^2}}\,\frac{x^{20s_1-14}}{x^{22s_1-2}}\to\frac{\mathrm{coef}(\theta)}{x^{2s_1{}^2+2s_1+12}}\ . \tag{5.9}$$

If the exponent is positive in the range of interest there will be a singularity, unless the coefficient in front is zero. From Eq. (5.9) we see that for $s_1<-0.5$ the exponent is positive and therefore the real part of the second term of Ψ_2 is singular. This analysis has been extended to the other intervals of s_1 both for the real and the imaginary part, and we also checked carefully the coefficients of the powers of x. The result is that Ψ_2 is always singular on $u^2+v^2=1$, except when $s_1=0$. In this case the metric is

$$f=\frac{\mathrm{const}\times(3p-5)^2}{4}[(1+p_k\cos\phi)^2+q_k{}^2\sin^2\phi]\ , \tag{5.10}$$

$$\tilde{g}_{11}=-\frac{(3p-5)^2}{8}(1+\cos\phi)^2\ , \tag{5.11}$$

$$\tilde{g}_{22}=-\frac{(3p-5)^2}{8}\,\frac{1+\cos\phi}{1-\cos\phi}\{[(1+p_k\cos\phi)^2+q_k{}^2\sin^2\phi]^2\sin^2\theta+4q_k{}^2\sin^2\phi\cos^2\theta\}\ , \tag{5.12}$$

$$\tilde{g}_{12}=q(3p-5)(1+\cos\phi)^2\cos\theta\ , \tag{5.13}$$

$$K=-\frac{(3p-5)^2(1+\cos\phi)}{32}[(1+p_k\cos\phi)^2+q_k{}^2\sin^2\phi]\ ,$$

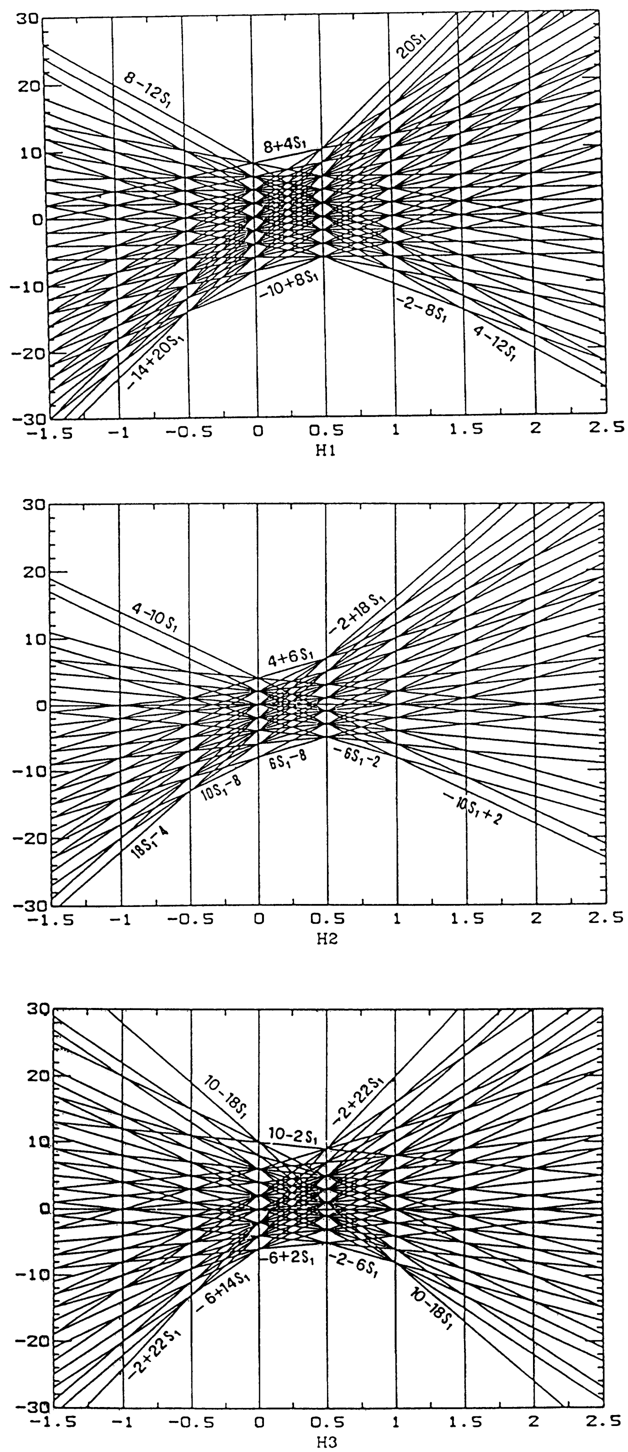

FIG. 1. The exponents of x in Eq. (5.7) are plotted versus s_1.

where

$$p_k=\frac{3-5p}{3p-5},\quad q_k=\frac{4q}{3p-5}\ .$$

When $q\neq 0$ the solution (5.10)–(5.14) in the interaction region is isometric to a part of the Taub-Newman-Unti-Tamburino (NUT) solution (see, e.g., Ref. 20) *in the nonstationary region inside the horizon.* The explicit coordinate transformation is

$$\cos\phi=\frac{t'-m}{\sigma},\quad x^2=\sigma\varphi\ ,$$

and the constants appearing in the Taub-NUT solutions m,l,σ are related to our parameters by the relations

$$p\sigma=m,\quad q\sigma=-l\ ,$$

$$\sigma=(m^2+l^2)^{1/2},\quad \text{const}=\sigma^2\ .$$

When $q=0$ the interaction region is isometric to a part of the Schwarzschild solution *inside* the horizon (see paper I). It is therefore clear that when $s_1=0$ the solution in the interaction region is *Petrov type D.* In addition, in both cases ($q\neq 0$ and $q=0$) the isometry maps the surface $u^2+v^2=1$ onto the horizon of the Taub-NUT and Schwarzschild solutions; therefore we conclude that when $s_1=0$ the collision of pure gravitational plane wave does not end in a singularity, but develops a horizon on the surface $u^2+v^2=1$ and the metric can be extended to the future. These "degenerate" solutions ($s_1=0$ or $s_1=1$) are of particular interest and will be analyzed in a separate paper.

VI. THE BEHAVIOR OF THE COMPLETE SOLUTION AS FUNCTION OF s_1

It remains to analyze the behavior of the complete solution (metric in the interaction region + extension) when s_1 assumes different values. The analysis is quite complicated, but it can be simplified by considering the diagonal case ($q=0$) as a reference, and by making graphics of the metric functions, to give a visual picture of the situation.

First of all let us write the metric in the *diagonal* case both in region IV and in region II.

Region IV:

$$ds^2=f(d\phi^2-d\theta^2)+g_{11}(dx^1)^2+g_{22}(dx^2)^2\ , \tag{6.1}$$

$$f=\sin\theta^{-2s_1s_2}(1+\cos\phi)^{2(1+s_2)}(1-\cos\phi)^{s_1(1+s_1)}\ , \tag{6.2}$$

$$g_{11}=-\sin\theta^{2s_1}(1+\cos\phi)^{s_1-1}(1-\cos\phi)^{1+s_1}\ , \tag{6.3}$$

$$g_{22}=-\sin\theta^{2s_2}(1+\cos\phi)^{2-s_1}(1-\cos\phi)^{-s_1}\ . \tag{6.4}$$

Region II:

$$ds^2=\frac{f(v)du\,dv}{(1-v^2)^{1/2}}+g_{11}(dx^1)^2+g_{22}(dx^2)^2\ , \tag{6.5}$$

$$f(v)=(1+v)^{2s_2^2}(1-v)^{2s_1^2}\ , \tag{6.6}$$

$$g_{11}(v)=-(1+v)^{2s_1-1}(1-v)^{2s_1+1}\ , \tag{6.7}$$

$$g_{22}(v)=-(1+v)^{2s_2+1}(1-v)^{1-2s_1}\ . \tag{6.8}$$

In this region the only nonvanishing Weyl scalar is

$$\Psi_4=\frac{1}{2}\frac{s_1-s_2}{(1+v)^{2s_1^2+1}(1-v)^{2s_1^2+1}}[3+3(s_1-s_2)v-4s_1s_2v^2]\ . \tag{6.9}$$

The metric in region III is obtained from Eqs. (6.6)–(6.9) by substitution $v\to u$. From Eq. (6.1) it is clear that the exponent of $(1-\cos\phi)$ is positive for $s_1<-1$ and $s_1>0$, and it is negative for $-1<s_1<0$. Therefore, when $\phi\to 0$, or $u^2+v^2\to 1$, f has two possible types of behaviors: (a) it goes to zero if $s_1<-1$ or $s_1>0$ and (b) it diverges if $-1<s_1<0$.

In region II we have the "critical" surface $v\to 1$, where, if $s_1\neq 0$, f goes to zero. Notice that all the metric functions (6.6)–(6.8) tend to 1 when $v\to 0$, where region II matches with the flat space-time preceding the collision (region I). The behavior of g_{11} and g_{22} in regions IV and II follows.

Region IV $u^2+v^2\to 1$:

$$g_{11}\to 0,\quad s_1>-1,\quad g_{22}\to 0,\quad s_1<0\ ,$$

$$g_{11}\to\infty,\quad s_1<-1,\quad g_{22}\to\infty,\quad s_1>0\ .$$

Region II, $v\to 1$:

$$g_{11}\to 0,\quad s_1>-\tfrac{1}{2},\quad g_{22}\to 0,\quad s_1<\tfrac{1}{2}\ ,$$

$$g_{11}\to 0,\quad s_1<-\tfrac{1}{2},\quad g_{22}\to 0,\quad s_1>\tfrac{1}{2}\ .$$

In any event the Weyl scalars are singular both on $u^2+v^2\to 1$ and on $v\to 1$ if $s_1\neq 0$. A similar situation arises when we consider the behavior of the metric functions in the nondiagonal case $q\neq 0$. The analysis is only more cumbersome, because there are some more intervals of s_1 to take into account, but the result is qualitatively the same: depending on the value of s_1, some of the metric functions (or all of them) diverge, and some

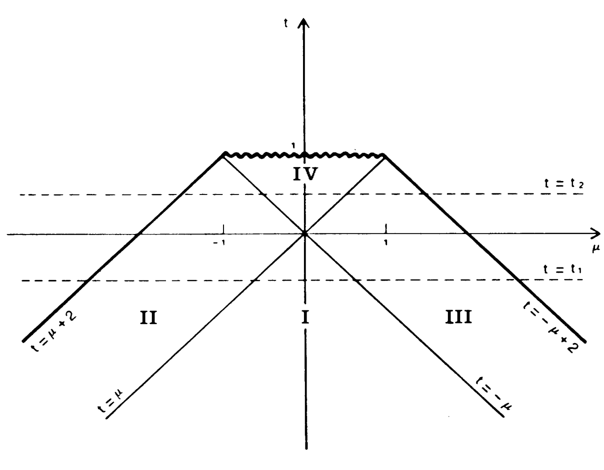

FIG. 2. Regions I, II, III, and IV expressed in terms of μ, representing the direction of propagation of the waves, and t representing the time.

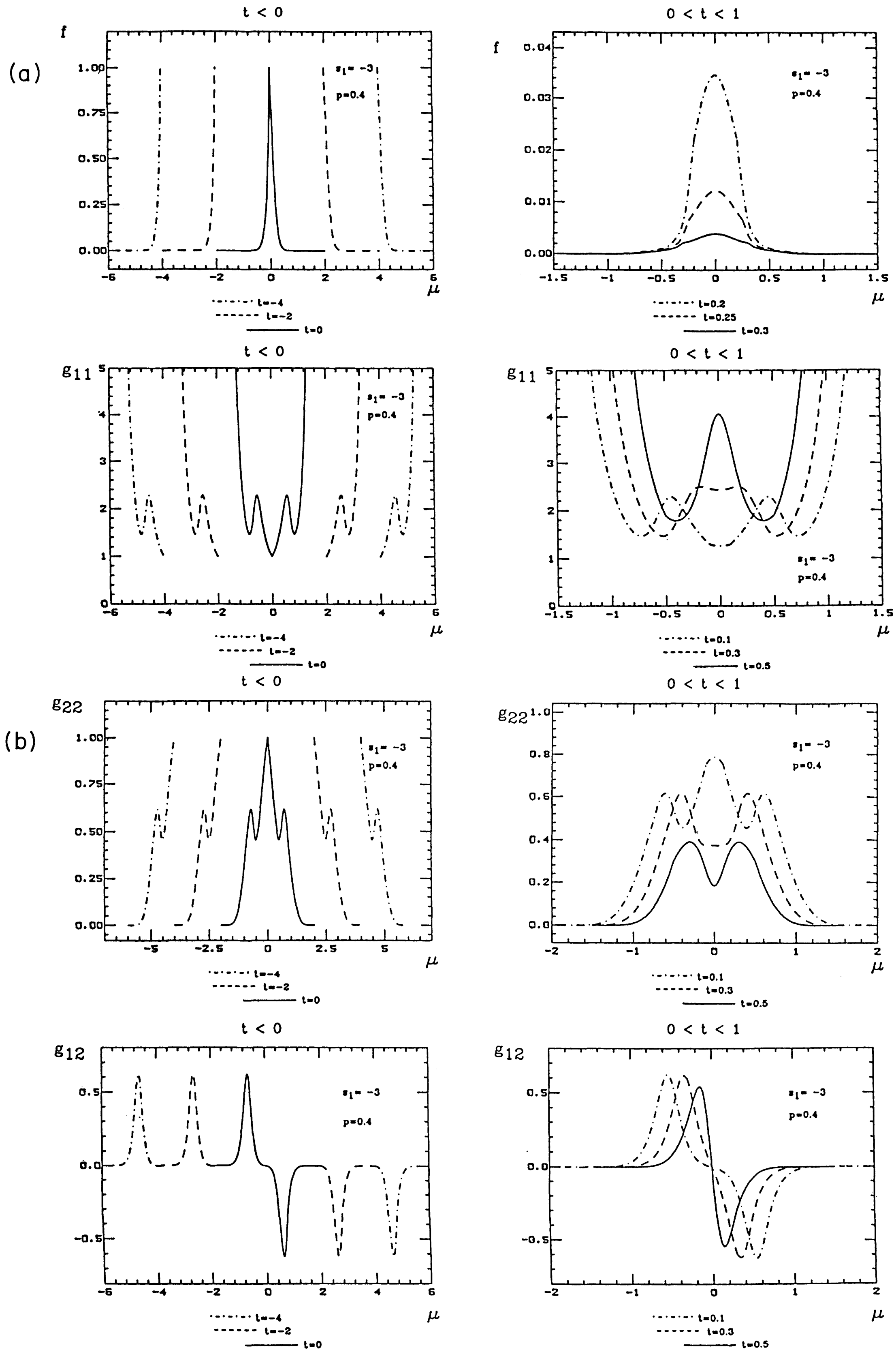

FIG. 3. (a) The metric functions f and g_{11} are plotted for $s_1=-3$ and $p=0.4$, versus the propagation direction, before and after the collision. (b) The same for g_{22} and g_{12}.

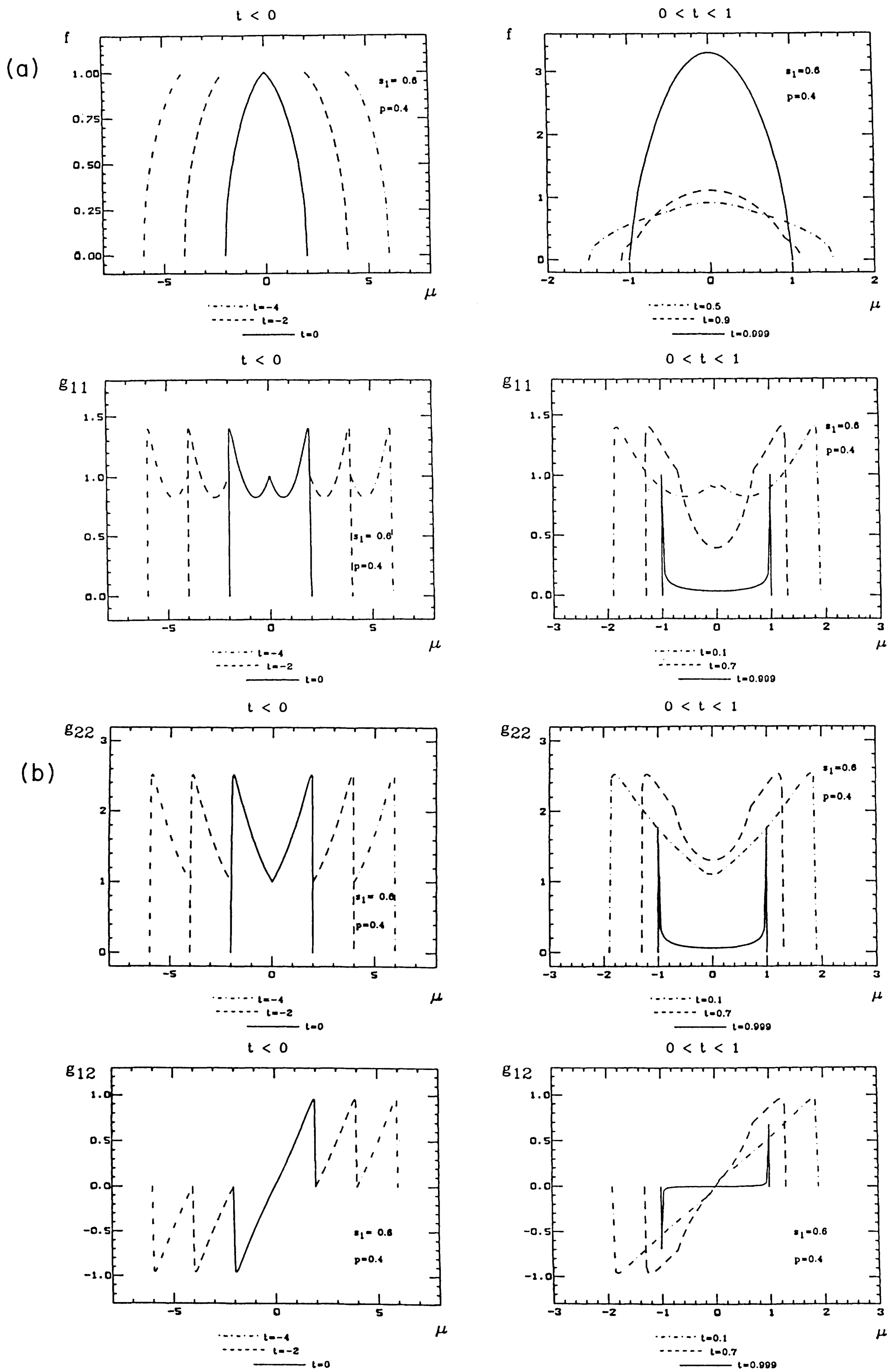

FIG. 4. (a) The metric functions f and g_{11} are plotted, as in Fig. 3, for $s_1=0.6$ and 0.4. (b) The same for g_{22} and g_{12}.

(or all of them) tend to zero, but there is always a curvature singularity on the surface $s^2+v^2=1$, except for $s_1=0$ as shown in Sec. V. In the extended regions, when $v \to 1$ (or $u \to 1$) the situation is similar, and the existence of a curvature singularity on that surface is a consequence of the fact that the metric is singular on $u=0$ and $v=1$ (or on $v=0$ and $u=1$), as stated by a theorem proved by Matzner and Tipler in Ref. 21.

Before discussing the graphics of the metric functions, it is better to identify regions I, II, III, and IV, in terms of $\mu=\cos\theta$, representing the direction of propagation of the two waves, and $t=\cos\phi$, representing the time. The new time coordinate does not coincide with the coordinate t used in Secs. II and III. From Eqs. (4.1) it follows that, in the region of interaction,

$$u=\cos\frac{\phi+\theta}{2} \quad \text{and} \quad v=\sin\frac{\theta-\phi}{2} \tag{6.10}$$

and therefore

$$\begin{aligned} 2u^2 &= 1+\cos\phi\cos\theta-\sin\phi\sin\theta \\ &= 1+t\mu-[(1-t^2)(1-\mu^2)]^{1/2} , \\ 2v^2 &= 1-\cos\phi\cos\theta-\sin\phi\sin\theta \\ &= 1-t\mu-[(1-t^2)(1-\mu^2)]^{1/2} . \end{aligned} \tag{6.11}$$

Now, in region IV $0 \le u \le 1$ and $0 \le v \le 1$; it follows that region IV corresponds to the region

$$-t \le \mu \le t, \quad 0 \le t \le 1 ,$$

in the (μ,t) plane. The instant of collision is $t=0$, the instant when the singularity is created is $t=1$.

If $t<0$ the two waves are traveling one against the other in the μ direction. They are single plane waves propagating in flat space-time; therefore, according to the definitions (4.1) and to the algorithm for the extension, the null coordinate u and v are, respectively,

$$u=\frac{t+\mu}{2} \quad \text{and} \quad v=\frac{t-\mu}{2} ;$$

one wave will be a function of v only (in region II), the other a function of u only (in region III). In region II $0 \le v \le 1$, therefore it corresponds to the region

$$t-2 \le \mu \le t, \quad t<0 ,$$

in the (μ,t) plane. In region III $0 \le u \le 1$, and the corresponding region in the (u,v) plane is

$$-t \le \mu \le -t+2, \quad t<0 .$$

Thus the situation which is illustrated in Fig. 2, is the following.

(a) $t=t_1<0$. The two waves are, respectively, confined in the two regions: $t-2 \le \mu \le t$ (progressive wave) and $-t \le \mu \le -t+2$ (regressive wave). The space-time is flat for $t \le \mu \le -t$.

(b) $t=t_2>0$. For $t-2 \le \mu \le -t$ there is the still incoming first wave. For $-t \le \mu \le t$ the two waves interact. For $t \le \mu \le -t+2$ there is the still incoming second wave.

As the time tends to 1, the region of interaction expands and it is maximum for $t=1$ when the singularity is created on $-1 \le \mu \le 1$.

In Figs. 3(a) and 3(b) we illustrate the behavior of the metric functions f, g_{11}, g_{22}, and g_{12} for $s_1=-3$ and $p=0.4$, both for $t \le 0$ and for $0<t<1$. Each function is plotted for several values of time, and it is shown how the two waves approach each other for $t<0$, they collide for $t=0$, and they mix for $0<t<1$. For this value of s_1, f, g_{22}, and g_{12} tend to zero as time tends to 1, while g_{11} diverges.

Before the collision $(t<0)$ f, g_{11}, and g_{22} tend to 1, while g_{12} tends to zero on $\mu=t$ (corresponding to $v=0$ and $u<0$) and on $\mu=-t$ (corresponding to $u=0$ and $v<0$). In fact on these surfaces the metric joins continuously to the Minkowski metric. On $\mu=t-2$ ($v=1$ and $u<0$) and on $\mu=-t+2$ ($u=1$ and $v<0$), f, g_{22}, and g_{12} tend to zero, while g_{11} diverges. Remember that on these surfaces there is a curvature singularity, as well as on the surface $u^2+v^2=1$, independently from the behavior of the metric functions.

Finally in Figs. 4(a) and 4(b) f, g_{11}, g_{22}, and g_{12} are plotted for $s_1=0.6$ and $p=0.4$ In this case only f diverges when $t \to 1$ (while it does not diverge in the diagonal case $p=1$), and the remaining metric functions go to zero.

VII. CONCLUDING REMARKS

We have obtained in this paper a two-parameter class of nondiagonal solutions of the vacuum Einstein equations, which describes the collision of plane gravitational shock waves, each supporting an impulsive wave with the same front. The solution reduces to the Nutku-Halil solution if $s_1=s_2=\frac{1}{2}$, and to the Khan-Penrose solution when $s_1=s_2=\frac{1}{2}$ and $p=1$.

It has been shown that the collision of pure gravitational waves does not necessarily produce a curvature singularity; in fact, in some cases a horizon can be created *when the two waves have parallel or nonparallel polarization.* It is remarkable that when the horizon forms, the space-time in the region of interaction is *Petrov type D.* These particular solutions will be analyzed in a subsequent paper.

All the calculations presented in this paper have been checked by using the symbolic manipulation program *SMP* available on VAX 8600.

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