Local and global gravitomagnetic effects in Kerr spacetime

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The integral shift in orientation of a gyroscope in closed polar orbit in the Kerr spacetime is examined as an example of a global gravitomagnetic effect. The exact dependence of this effect on the mass and angular momentum parameters of the Kerr field is determined and the well-known weak-field slow-motion limit pertinent to forthcoming experiments is analyzed. The precession of the spin vector of a gyroscope stationed at a given point of the Kerr spacetime's symmetry axis is presented as a local counterpart of the above gravitomagnetic effect.

I. INTRODUCTION

According to the theory of general relativity (GR), the gravitational field of a rotating body contains magneticlike components which give rise to a set of characteristic effects coming under the heading of "frame dragging."

For example, the spin vector of a gyroscope placed at Earth's north pole is expected to rotate relative to the "fixed stars" at a rate which differs from the angular velocity of Earth itself relative to the same asymptotic frame. In this case, the test gyroscope is kept at a fixed point of the terrestrial field of gravity by forces acting on the gyroscope's center of mass. We are dealing, therefore, with a local gravitomagnetic effect.

An example of a global gravitomagnetic effect is obtained by considering a gyroscope carried by an artificial satellite in polar orbit around Earth. If the radius of the orbit is assumed to be constant, then the satellite returns to the same point of the polar axis periodically and one can compare the gyroscope's orientation at the end of one revolution with that at its beginning. Again, a difference is expected to be observed as a result of Earth's rotation.

Recent technological advances seem to have rendered both of the above effects measurable. Thus, various ground-based experiments have been proposed[1–3] in order to measure the local effect of the terrestrial field, while satellite gyroexperiments designed to measure the global one will soon be attempted.[4,5] As a result, theoretical interest in gravitomagnetic effects is and will be growing.

In this paper an exact analysis of the two examples described above is presented in the context of the Kerr field of gravity. Of course, this choice of a model spacetime and the exact treatment of the corresponding effects are not suggested by the direct demands placed on the part of the theory by experiments associated with the terrestrial field. In this case, the pertinent physical parameters are so small that a first-order approximation to both the spacetime geometry and the equations of motion and parallel transport is quite satisfactory. We believe, however, that the exact solution presented in this paper does not lack in theoretical interest. Not only because it applies in the case of a rotating black hole, but also because, as it is hoped to emerge from the following discussion, it sheds light on certain aspects of the corresponding approximate treatment.

The structure of the paper is as follows. In Sec. II the Kerr metric is expressed in terms of the systems of coordinates used in our analysis. Section III is devoted to a short presentation of the results regarding the local effect first obtained by Urani and Carlson.[6] The principal part of the paper consists of Sec. IV, where the global gravitomagnetic effect is analyzed. Lastly, in Sec. V, the local and global effects are examined in the limit cases where the source's angular momentum parameter is either vanishing or very small compared to the orbit's coordinate radius. This is done with the purpose of making clear the role played by the angular momentum in both the local and global gravitomagnetic effects.

II. THE KERR METRIC

The spacetime metric which represents Kerr's solution[7] of Einstein's field equations can be written in the form

$$ds^2=-(1-2Mr/\Sigma)dt^2-2(2Mr/\Sigma)a\sin^2\theta\, dt\, d\varphi +(\Sigma/\Delta)dr^2+\Sigma\, d\theta^2+(A/\Sigma)\sin^2\theta\, d\varphi^2 , \quad (2.1)$$

where

$$\Sigma\equiv r^2+a^2\cos^2\theta ,$$
$$\Delta\equiv r^2+a^2-2Mr , \quad (2.2)$$
$$A\equiv(r^2+a^2)^2-\Delta a^2\sin^2\theta ,$$

and M,a are real parameters. This was shown by Boyer and Lindquist[8] (BL) and the coordinate system (t,r,θ,φ) bears their name. In this system of coordinates the axial symmetry and time independence of the Kerr spacetime is made explicit, as the metric coefficients in (2.1) do not depend on the spacelike coordinate φ which takes values in the range $0\leq\varphi\leq 2\pi$, or on the timelike coordinate t, where $-\infty<t<\infty$.

The gravitational field represented by the spacetime of Kerr is assumed to correspond to an object of total mass M and angular momentum $J=Ma$. When $M^2\geq a^2$, this object is referred to as a rotating black hole. In such a case the coordinate t is timelike in the region $r>r_0(\theta)\equiv M+(M^2-a^2\cos^2\theta)^{1/2}$ but becomes null when the hypersurface $r_0(\theta)$, known as the static limit, is reached.

This is clear from (2.1), where the coefficient of dt^2 is negative as long as $r > r_0(\theta)$ and it vanishes when r becomes equal to $r_0(\theta)$. However, the particle orbits, considered in the following, lie outside the static limit. Thus, t will retain its timelike character independently of the values taken by the parameters M and a.

Polar orbits are, by definition, curves which reach the symmetry axis of the spacetime under consideration. This axis consists of the points where $\sin\theta = 0$, and at such points the coordinate φ is undefined. Since the particle orbits examined in this paper are polar ones, it follows that the BL system of coordinates will not suffice for our purposes. We will also have to use the coordinate system $(\tilde{t},x,y,z)$ of Kerr and Schild[9] (KS) which is well defined on the symmetry axis of the Kerr spacetime. In the KS coordinates, the Kerr metric takes the form

$$ds^2 = -d\tilde{t}^2 + dx^2 + dy^2 + dz^2 + \frac{2Mr^3}{r^4 + a^2 z^2}\left[-d\tilde{t} + \frac{1}{r^2+a^2}[r(x\,dx + y\,dy) + a(x\,dy - y\,dx)] + \frac{z}{r}dz \right]^2 , \qquad (2.3)$$

where the function $r(x,y,z)$ is implicitly defined by the equation

$$\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1 \ . \qquad (2.4)$$

The relation between the BL and KS systems of coordinates is determined by the equations

$$\begin{aligned} d\tilde{t} &= dt - \frac{2Mr}{\Delta} dr \ , \\ d\psi &= d\varphi - \frac{2Mar}{(r^2+a^2)\Delta} dr \ , \\ x &= (r^2+a^2)^{1/2} \sin\theta \cos\psi \ , \\ y &= (r^2+a^2)^{1/2} \sin\theta \sin\psi \ , \\ z &= r\cos\theta \ , \end{aligned} \qquad (2.5)$$

and can be made to be a one-to-one mapping from (t,r,θ,φ) to $(\tilde{t},x,y,z)$ by a well-known process of restricting the overlapping coordinate patches.

It should be noted that on each $r =$ const hypersurface the coordinates $\tilde{t}$ and ψ differ only by an additive constant from t and φ, respectively. This follows from the first two of Eqs. (2.5) and it will be utilized in the following sections where particle orbits with constant r are examined.

III. GYROSCOPE ON THE SYMMETRY AXIS

According to Eq. (2.5), the symmetry axis of the Kerr field consists of points where

$$x = 0 = y, \quad |z| = r \ , \qquad (3.1)$$

in the KS system of coordinates. In then follows from Eqs. (2.3) and (2.5) that on the axis of symmetry the Kerr metric takes the form

$$\begin{aligned} ds^2 &\doteq -d\tilde{t}^2 + dx^2 + dy^2 + dz^2 + \frac{2Mr}{r^2+a^2}(-d\tilde{t} + dz)^2 \\ &\doteq -\left[1 - \frac{2Mr}{r^2+a^2}\right] dt^2 + dx^2 + dy^2 \\ &\quad + \left[1 - \frac{2Mr}{r^2+a^2}\right]^{-1} dz^2 \ , \end{aligned} \qquad (3.2)$$

where the dot above the equality sign denotes the fact that the given equation holds only when condition (3.1) is satisfied.

Let us now assume that a particle is forced to remain at the point $z = r$ of the symmetry axis. The world line of such a particle consists of events for which $(t,x,y,z) = (t(\tau_R),0,0,r)$, where τ_R denotes the particle's proper time. Thus, its four-velocity $\mathbf{u}_R$ is equal to

$$\mathbf{u}_R = \frac{dt}{d\tau_R}\partial_t = \left[1 - \frac{2Mr}{r^2+a^2}\right]^{-1/2} \partial_t \ . \qquad (3.3)$$

This follows from Eq. (3.2) and the fact that $\mathbf{u}_R \cdot \mathbf{u}_R = -1$.

The particle under consideration could represent a gyroscope carried by a rocket ship which fires its engines so as not to fall under the pull of the object that produces the Kerr field. Alternatively, it could represent a gyroscope used in a terrestrial experiment which is designed to measure the effect of the rotation of Earth on the spin of a gyroscope placed at the north pole. In the latter case it is obviously assumed that the Kerr metric gives a satisfactory approximation of the gravitational field of Earth.

In either of the above cases, one has to choose a frame of reference along the gyroscope's world line in which to measure the effects of the gravitational field. Such a frame can be chosen to be the set $(\partial_{\hat{x}}, \partial_{\hat{y}}, \partial_{\hat{z}})$, where

$$\begin{aligned} &\partial_{\hat{x}} \doteq \partial_x, \quad \partial_{\hat{y}} \doteq \partial_y \ , \\ &\partial_{\hat{z}} \doteq \left[1 - \frac{2Mr}{r^2+a^2}\right]^{1/2} \partial_z \ , \end{aligned} \qquad (3.4)$$

since, according to Eq. (3.2), this triad of spacelike vectors forms with $\mathbf{u}_R$ an orthonormal tetrad along the gyroscope's worldline.

The frame just defined can be identified with the walls of the rocket ship carrying the gyroscope. In that case, the rocket ship is not rotating relative to the distant stars, since the base vectors given by Eq. (3.4) point permanently in the direction of the coordinate axes (x,y,z) which are defined in terms of the asymptotic flat region of the Kerr field. It is not the same with the gyroscope's spin $\mathbf{S}_R$, however. One finds that[6,10] inside the rocket ship and, therefore, with respect to the asymptotic frame the vector $\mathbf{S}_R$ precesses according to the equation

$$\frac{d\mathbf{S}_R}{dt} = \mathbf{\Omega}_R \times \mathbf{S}_R \ , \tag{3.5}$$

where

$$\mathbf{\Omega}_R = \frac{2Mar}{(r^2+a^2)^2}\partial_{\hat{z}} \ . \tag{3.6}$$

The gyroscope precession described by Eqs. (3.5) and (3.6) is a local effect, since the gyroscope was assumed to have a fixed position in the Kerr field. Obviously, it is the gravitational analog of the precession of a magnetic dipole placed on the symmetry axis of an axially symmetric, time-independent magnetic field. For this reason, the precession of the inertial gyroscope described above is referred to as a gravitomagnetic effect.

A global gravitomagnetic effect is described in the next section. Before we turn to it, however, let us note that, according to Eqs. (3.5) and (3.6), a frame of reference stationed at the point $z=r$ of the symmetry axis which is nonrotating relative to inertial guidance gyroscopes accompanying it is given by the triad $(\mathbf{L}_x, \mathbf{L}_y, \mathbf{L}_z)$, where

$$\begin{aligned} \mathbf{L}_x &= \cos(\Omega_R t)\partial_{\hat{x}} + \sin(\Omega_R t)\partial_{\hat{y}} \ , \\ \mathbf{L}_y &= -\sin(\Omega_R t)\partial_{\hat{x}} + \cos(\Omega_R t)\partial_{\hat{y}} \ , \\ \mathbf{L}_z &= \partial_{\hat{z}} \ . \end{aligned} \tag{3.7}$$

IV. GYROSCOPE IN POLAR ORBIT

Let us consider a gyroscope carried by an artificial satellite orbiting the source of the Kerr field of gravity. If we denote the gyroscope's proper time by τ, then its center of mass follows a timelike geodesic, $C(\tau)$, in the Kerr spacetime. The image of $C(\tau)$ in the BL system of coordinates is given by the functions $x^a(\tau)$, where $x^a = t, r, \theta, \varphi$ for $a = 0,1,2,3$, respectively. Thus the gyroscope's four-velocity $\mathbf{u}$ can be written as $\mathbf{u} = u^a \partial_a$, where $u^a = \dot{x}^a \equiv dx^a/d\tau$.

Carter[11] has shown that the equations satisfied by $x^a(\tau)$ admit the following first integrals:

$$\begin{aligned} \dot{t} &= (\Delta\Sigma)^{-1}(AE - 2Mar\Phi) \ , \\ \Sigma^2 \dot{r}^2 &= R(r) \equiv [(r^2+a^2)E - a\Phi]^2 - \Delta(\mu^2 r^2 + K) \ , \\ \Sigma^2 \dot{\theta}^2 &= K - \mu^2 a^2 \cos^2\theta - (aE\sin\theta - \Phi/\sin\theta)^2 \ , \\ \dot{\varphi} &= \Delta^{-1}[(2Mr/\Sigma)aE + (1 - 2Mr/\Sigma)\Phi/\sin^2\theta] \ . \end{aligned} \tag{4.1}$$

The constants μ, E, and Φ represent the rest mass, energy, and projection of angular momentum along the Kerr spacetime's symmetry axis, respectively, of the particle following $C(\tau)$. The fourth constant, K, appearing in Eqs. (4.1), corresponds to the fact that in the spacetime of Kerr there exists a skew-symmetric tensor field Y_{ab} satisfying the equation

$$Y_{ab;c} + Y_{ac;b} = 0 \ , \tag{4.2}$$

where the semicolon denotes covariant differentiation. Y_{ab} is known as a Killing-Yano tensor and Carter's fourth constant is equal to

$$K = Y^a Y_a, \quad Y^a \equiv Y^{ab} u_b \ . \tag{4.3}$$

In the following, it will be assumed that $\mu = 1$, since this implies no loss of generality.

Let us now demand that our satellite follows an orbit of constant r which crosses the symmetry axis of the Kerr field. According to the third of Eqs. (4.1), the necessary and sufficient condition for a timelike geodesic to cross the axis reads

$$\Phi = 0, \quad K > a^2 \ . \tag{4.4}$$

On the other hand, the orbit will remain on a hypersurface of constant r only if r is a double root of the equation $R(r) = 0$. These observations allow us to write Eqs. (4.1) in the form

$$\begin{aligned} \dot{t} &= AE/\Delta\Sigma \ , \\ \dot{\theta} &= [\epsilon(\dot{\theta})/\Sigma](K - a^2\cos^2\theta - a^2E^2\sin^2\theta)^{1/2} \ , \\ \dot{\varphi} &= 2MaEr/\Delta\Sigma \ , \end{aligned} \tag{4.5}$$

where $\epsilon(\dot{\theta})$ denotes the sign of $\dot{\theta}$ and E, K are now given by

$$\begin{aligned} E^2 &= r\Delta^2/(r^2+a^2)(r^3 - 3Mr^2 + a^2 r + Ma^2) \ , \\ K &= r(Mr^3 + a^2r^2 - 3Ma^2r + a^4) \\ &\quad \times (r^3 - 3Mr^2 + a^2r + Ma^2)^{-1} \ . \end{aligned} \tag{4.6}$$

Since E^2 must be positive, Eqs. (4.4) and (4.6) imply that spherical polar orbits exist for any r such that $r(r^3 - 3Mr^2 + a^2r + Ma^2) > 0$ and $r^2 > a^2$. If it is required that the orbit is stable then the range of its coordinate radius is restricted further by the condition $[d^2R(r)/dr^2] < 0$. As an illustration, consider the case of an extreme Kerr black hole where $a^2 = M^2$. Then spherical polar orbits exists for any r in the open interval $r_1 < r < \infty$, where $r_1 \approx 2.415M$, but stable ones can be found only for $r < r_2 \approx 5.275M$. In order to keep the argument physically credible, the timelike polar orbit $C(\tau)$ which appears in the following will be assumed to be stable.

Combining the first and third of Eqs. (4.5) we find that

$$d\varphi/dt = 2Mar/A \ . \tag{4.7}$$

If we consider the family of timelike curves along which $r = \text{const}, \theta = \text{const}$ and φ, t vary according to Eq. (4.7) we obtain a congruence which is orthogonal to the $t = \text{const}$ hypersurfaces of Kerr's spacetime. With each member of this congruence we can associate an orthonormal tetrad $\{\mathbf{e}_{\hat{a}}\}$, where

$$\begin{aligned} \mathbf{e}_{\hat{t}} &\equiv (A/\Sigma\Delta)^{1/2}\partial_t + 2Mar/(A\Sigma\Delta)^{1/2}\partial_\varphi \ , \\ \mathbf{e}_{\hat{r}} &\equiv (\Delta/\Sigma)^{1/2}\partial_r \ , \\ \mathbf{e}_{\hat{\theta}} &\equiv (1/\Sigma)^{1/2}\partial_\theta \ , \\ \mathbf{e}_{\hat{\varphi}} &\equiv (\Sigma/A\sin^2\theta)^{1/2}\partial_\varphi \ . \end{aligned} \tag{4.8}$$

We then obtain a locally nonrotating frame (LNRF); i.e., a frame where Coriolis-type effects are absent. The construction of LNRF's is due to Bardeen[12] and some of their most interesting properties are discussed in Refs.

13–15 among others. For our purposes it suffices to note that Eqs. (4.5)–(4.8) imply that

$$\mathbf{u}=O\mathbf{e}_{\hat{t}}+P\mathbf{e}_{\hat{\theta}}\ , \tag{4.9}$$

where

$$O=(A/\Sigma\Delta)^{1/2}E,\quad P=\epsilon(\dot{\theta})(O^2-1)^{1/2}\ . \tag{4.10}$$

This means that our satellite moves in the θ direction of all the LNRF's it meets while in orbit around the source of the Kerr field. As a result, the orthonormal tetrad $\{\mathbf{e}_{(\hat{a})}\}$, where

$$\begin{aligned}&\mathbf{e}_{(\hat{0})}=\mathbf{u},\quad \mathbf{e}_{(\hat{1})}=\mathbf{e}_{\hat{r}}\ ,\\ &\mathbf{e}_{(\hat{2})}=P\mathbf{e}_{\hat{t}}+O\mathbf{e}_{\hat{\theta}},\quad \mathbf{e}_{(\hat{3})}=\mathbf{e}_{\hat{\varphi}}\ ,\end{aligned} \tag{4.11}$$

suggests itself as a natural choice for a frame comoving with the satellite.

However, the spatial legs $\mathbf{e}_{(\hat{i})}$, $i=1,2,3$, of the tetrad just defined are not parallel transported along $C(\tau)$. As a result, the gyroscope carried by the satellite will not retain a constant direction with respect to the vectors $\mathbf{e}_{(\hat{i})}$. Specifically, one can use the rotation coefficients $\Gamma_{\hat{a}\hat{b}\hat{c}}$ of the locally nonrotating base (4.8) given by Bardeen, Press, and Teukolsky[13] to show that

$$\dot{\mathbf{e}}_{(\hat{i})}\equiv e_{(\hat{i})}{}^{\hat{a}}{}_{;\hat{b}}u^{\hat{b}}\mathbf{e}_{\hat{a}}=-\mathbf{\Omega}_S\times\mathbf{e}_{(\hat{i})}\ , \tag{4.12}$$

where

$$\begin{aligned}&\Omega_S^{(\hat{1})}=\Gamma_{\hat{\theta}\hat{\varphi}\hat{t}}=2Ma^3r\Delta^{1/2}\sin^2\theta\cos\theta/A\Sigma^{3/2}\ ,\\ &\Omega_S^{(\hat{2})}=O\Gamma_{\hat{\varphi}\hat{r}\hat{t}}=OMa\sin\theta[(3r^2-a^2)(r^2+a^2)-(r^2-a^2)a^2\sin^2\theta]/A\Sigma^{3/2}\ ,\\ &\Omega_S^{(\hat{3})}=OP(\Gamma_{\hat{r}\hat{t}\hat{t}}+\Gamma_{\hat{r}\hat{\theta}\hat{\theta}})=OP(r^2+a^2)[M(3r^2-a^2)-(r^2+a^2)r]/A(\Delta\Sigma)^{1/2}\ .\end{aligned} \tag{4.13}$$

In order to obtain a frame which is parallel transported along $C(\tau)$ one can follow Marck's[16] construction which starts with Penrose's[17] observation that the vector Y^a defined by Eq. (4.3) is orthogonal to $\mathbf{u}$ and parallel transported along $C(\tau)$. The nonvanishing components of the Killing-Yano tensor Y_{ab} in the base $\{e_{\hat{a}}\}$ are given by

$$\begin{aligned}&Y_{\hat{t}\hat{r}}=-Y_{\hat{r}\hat{t}}=-(r^2+a^2)a\cos\theta/A^{1/2}\ ,\\ &Y_{\hat{t}\hat{\theta}}=-Y_{\hat{\theta}\hat{t}}=ar(\Delta/A)^{1/2}\sin\theta\ ,\\ &Y_{\hat{r}\hat{\varphi}}=-Y_{\hat{\varphi}\hat{r}}=-a^2(\Delta/A)^{1/2}\sin\theta\cos\theta\ ,\\ &Y_{\hat{\theta}\hat{\varphi}}=-Y_{\hat{\varphi}\hat{\theta}}=r(r^2+a^2)/A^{1/2}\ .\end{aligned} \tag{4.14}$$

It then follows that the vector

$$\boldsymbol{\lambda}_{(\hat{3})}\equiv-\mathbf{Y}/K^{1/2}=-O(r^2+a^2)a\cos\theta(KA)^{-1/2}\mathbf{e}_{(\hat{r})}+ar\sin\theta(\Delta/KA)^{1/2}\mathbf{e}_{(\hat{\theta})}+P(r^2+a^2)r(KA)^{-1/2}\mathbf{e}_{(\hat{\varphi})} \tag{4.15}$$

is a unit timelike vector normal to $\mathbf{u}$ and parallel transported along $C(\tau)$. It is not hard to verify, on the other hand, that the unit vectors

$$\boldsymbol{\lambda}'_{(\hat{1})}\equiv O(r/\beta)(r^2+a^2)(KA)^{-1/2}\mathbf{e}_{(\hat{r})}+\beta a^2\sin\theta\cos\theta(\Delta/KA)^{1/2}\mathbf{e}_{(\hat{\theta})}+P\beta a\cos\theta(r^2+a^2)(KA)^{-1/2}\mathbf{e}_{(\hat{\varphi})} \tag{4.16}$$

and

$$\boldsymbol{\lambda}'_{(\hat{2})}\equiv\beta P(r^2+a^2)[\Sigma/A(K+r^2)]^{1/2}\mathbf{e}_{(\hat{\theta})}-\beta a\sin\theta[\Sigma\Delta/A(K+r^2)]^{1/2}\mathbf{e}_{(\hat{\varphi})}\ , \tag{4.17}$$

where

$$\beta^2=(K+r^2)/(K-a^2\cos^2\theta) \tag{4.18}$$

are orthogonal to $\boldsymbol{\lambda}'_{(\hat{3})}$. A much longer calculation which makes use of Eq. (4.15) shows that

$$\dot{\boldsymbol{\lambda}}'_{(\hat{2})}=-\dot{\Psi}\boldsymbol{\lambda}'_{(\hat{1})},\quad \dot{\boldsymbol{\lambda}}'_{(\hat{1})}=\dot{\Psi}\boldsymbol{\lambda}'_{(\hat{2})}\ , \tag{4.19}$$

where

$$\dot{\Psi}=EK^{1/2}(K-a^2)/(r^2+K)(K-a^2\cos^2\theta)\ . \tag{4.20}$$

It then follows from Eq. (4.19) that the vectors

$$\boldsymbol{\lambda}_{(\hat{1})}\equiv\cos\Psi(\tau)\boldsymbol{\lambda}'_{(\hat{1})}-\sin\Psi(\tau)\boldsymbol{\lambda}'_{(\hat{2})} \tag{4.21}$$

and

$$\boldsymbol{\lambda}_{(\hat{2})}\equiv\sin\Psi(\tau)\boldsymbol{\lambda}'_{(\hat{1})}+\cos\Psi(\tau)\boldsymbol{\lambda}'_{(\hat{2})} \tag{4.22}$$

are parallel transported along $C(\tau)$. Together with $\boldsymbol{\lambda}_{(\hat{3})}$ defined by Eq. (4.15) they form an inertial frame whose origin is identified with the center of mass of the gyroscope carried by our satellite.

So far the polar geodesic $C(\tau)$ and the comoving frames $\{\mathbf{e}_{(\hat{i})}\}$ and $\{\boldsymbol{\lambda}_{(\hat{i})}\}$ have been specified only locally, i.e., in terms of the coordinates r and θ that correspond to the satellite's position at a given instant. Thus we are not in a position to determine the shift in the gyroscope's orientation each time the carrier satellite completes a revolution around the source of the Kerr field. This requires a complete integration of the equations of motion and parallel transport such that the geodesic under consideration and the frames defined along it return to the same point in space from which they emanate.

Let us, therefore, assume that our satellite is launched into a near miss orbit by the rocket ship of the previous section which is stationed at the point $z=r$ of the symmetry axis. The satellite's initial direction is identified with the x axis of the KS coordinate system and the $\varphi=0$ direction on the $r=$const hypersurface in the BL coordinates. By continuity, this choice implies the identification

$$(\mathbf{e}_{\hat{t}},\mathbf{e}_{\hat{r}},\mathbf{e}_{\hat{\theta}},\mathbf{e}_{\hat{\varphi}})\,|_{\tau=0}=(\partial_{\hat{t}},\partial_{\hat{z}},\partial_{\hat{x}},\partial_{\hat{y}}) \,, \tag{4.23}$$

where $\tau=0$ is the value of the proper-time parameter at the initial point-event $C(0)$ of $C(\tau)$. Noticing that Eqs. (2.2) and (4.10) imply that

$$O_{\theta=0}=[(K+r^2)/(a^2+r^2)]^{1/2} \tag{4.24}$$

and taking into account Eq. (3.2), we conclude that the initial conditions

$$\begin{aligned}&(t,x,y,z)_{\tau=0}=(0,0,0,r) \,,\\ &(\dot{t},\dot{x},\dot{y},\dot{z})_{\tau=0}=([(K+r^2)/\Delta]^{1/2},[(K-a^2)/(a^2+r^2)]^{1/2},0,0) \,,\end{aligned} \tag{4.25}$$

where K is determined by r via Eq. (4.6), guarantee that the satellite launched by the rocket ship will follow the polar orbit described in terms of the BL coordinates earlier in this section.

Integrating the equations of motion (4.5) one finds that[18–20] the satellite completes one revolution in coordinate time T_t, where

$$\begin{aligned}T_t=&4E(r/\Delta Q^{1/2})[r^3+a^2(2M+r)]K(k)\\ &+4EQ^{1/2}(1-E^2)^{-1}[K(k)-E(k)] \,.\end{aligned} \tag{4.26}$$

In this equation

$$Q\equiv K-a^2E^2,\quad k^2\equiv a^2(1-E^2)/Q \,, \tag{4.27}$$

and $K(k)$ and $E(k)$ denote the complete elliptic integrals of the first and second kind,[21] respectively. In terms of the parameters M, a, and r which characterize our problem the argument of the elliptic integrals is given by the expression

$$k^2=\frac{(a/r)^2(r^4-Mr^3+2a^2r^2+a^4)}{(r^4+2a^2r^2-4Ma^2r+a^4)} \tag{4.28}$$

which is obtained by combining Eqs. (4.6) and (4.27).

The satellite's proper time period, on the other hand, turns out to be equal to T, where

$$T=4(r^2/Q^{1/2})K(k)+4Q^{1/2}(1-E^2)^{-1}[K(k)-E(k)] \,. \tag{4.29}$$

Similarly one finds that the satellite's angle of longitude increases by $\delta\varphi$, where

$$\delta\varphi=8MaE(r/\Delta Q^{1/2})K(k) \tag{4.30}$$

during a complete oscillation in latitude. This implies that at $\tau=T$ our satellite crosses the symmetry axis moving in the $(\cos\delta\varphi\partial_{\hat{x}}+\sin\delta\varphi\partial_{\hat{y}})$ direction, and by the same argument that led us to Eq. (4.23) we conclude that

$$(\mathbf{e}_{\hat{t}},\mathbf{e}_{\hat{r}},\mathbf{e}_{\hat{\theta}},\mathbf{e}_{\hat{\varphi}})_{\tau=T}=(\partial_{\hat{t}},\partial_{\hat{z}},\,\cos\delta\varphi\partial_{\hat{x}}+\sin\delta\varphi\partial_{\hat{y}},\,-\sin\delta\varphi\partial_{\hat{x}}+\cos\delta\varphi\partial_{\hat{y}}) \,. \tag{4.31}$$

Thus, at the end of the first revolution the satellite frame $\{\mathbf{e}_{(\hat{a})}\}$ is found to have rotated by $\delta\varphi$ about the polar axis. Of course, the smooth evolution of $\{\mathbf{e}_{(\hat{a})}\}$ along $C(\tau)$, which projects on a smooth curve on the $r=$const hypersurface, and the spherical topology of the latter, imply that the base vectors $\mathbf{e}_{(\hat{r})}$ and $\mathbf{e}_{(\hat{\theta})}$ have undergone a rotation of 2π in the time interval T.

We are now in a position to determine the change in the spin vector $\mathbf{S}_s$ of a gyroscope carried by our satellite in the period of one revolution. Since the vector $\mathbf{S}_s$ remains constant in the inertial frame $\{\boldsymbol{\lambda}_{(\hat{i})}(\tau)\}$ all that is required is to calculate the difference between $\{\boldsymbol{\lambda}_{(\hat{i})}(\tau)\}$ and $\{\boldsymbol{\lambda}_{(\hat{i})}(0)\}$.

To this effect, let us assume that $\Psi(0)=0$. It then follows from Eqs. (4.15)–(4.22) that the frame $\{\boldsymbol{\lambda}_{(\hat{i})}(0)\}$ is given by

$$\begin{aligned}&\{\boldsymbol{\lambda}_{(\hat{1})}(0)\}=\cos Z\,\mathbf{e}_{(\hat{r})}(0)+\sin Z\,\mathbf{e}_{\hat{\varphi}}(0) \,,\\ &\{\boldsymbol{\lambda}_{(\hat{2})}(0)\}=\mathbf{e}_{\hat{\theta}}(0) \,,\\ &\{\boldsymbol{\lambda}_{(\hat{3})}(0)\}=-\sin Z\,\mathbf{e}_{\hat{r}}(0)+\cos Z\,\mathbf{e}_{(\hat{\varphi})}(0) \,,\end{aligned} \tag{4.32}$$

where the angle Z is such that

$$\tan Z=(a/r)[(K+r^2)/(K-a^2)]^{1/2} \,. \tag{4.33}$$

From the same equations it also follows that Eqs. (4.32) and (4.33) express the relation between the vectors $(\boldsymbol{\lambda}'_{(\hat{1})}(T),\boldsymbol{\lambda}'_{(\hat{2})}(T),\boldsymbol{\lambda}_{(\hat{3})}(T))$ and the frame $\{\mathbf{e}_{(\hat{i})}(T)\}$. Then, according to Eqs. (4.21) and (4.22) it suffices to calculate $\Psi(T)$ in order to obtain the inertial frame

$\{\boldsymbol{\lambda}_{(\hat{\imath})}(T)\}$ in terms of $\{\mathbf{e}_{(\hat{\imath})}(T)\}$.

The angle $\Psi(T)$ is obtained by integration of Eq. (4.20). With the help of the second of Eqs. (4.5) we find that[22]

$$\Psi(T)=4Er^2[(K-a^2)/(KQ)^{1/2}(K+r^2)]K(k) -4E[(K-a^2)/(KQ)^{1/2}][K(k)-\Pi(n,k)] , \tag{4.34}$$

where

$$n\equiv a^2/K \tag{4.35}$$

and $\Pi(n,k)$ denotes the complete elliptic integral of the third kind.[21]

On the basis of the above results, we can describe the total shift of the gyroscope's spin vector $\mathbf{S}_s$ in the following terms. At the end of each revolution about the source of the Kerr field of gravity, the satellite returns to the point $z=r$ of the symmetry axis with its comoving frame $\{\mathbf{e}_{(\hat{a})}\}$ rotated by $\delta\varphi$, as given by Eq. (4.30) about the symmetry axis itself and in the same sense in which the source is rotating. Relative to the satellite frame, the vector $\mathbf{S}_s$ is found rotated by $\Psi(T)$, as given by Eq. (4.35), about an axis lying in the $\mathbf{e}_{(\hat{r})}$-$\mathbf{e}_{(\hat{\varphi})}$ plane and making an angle Z, as given by Eq. (4.33), with the negative $\mathbf{e}_{\hat{\varphi}}$ axis.

V. THE SCHWARZSCHILD AND LENSE-THIRRING LIMIT CASES

Let us now consider two limit cases of the exact results obtained in the last two sections. The first corresponds to the vanishing of the parameter a which reduces the metric of Kerr to that of Schwarzschild. The second case assumes that (a/r) is very small. This allows one to consider the line element

$$ds^2=-(1-2M/r)dt^2+(1-2M/r)^{-1}dr^2 +r^2(d\theta^2+r^2\sin^2\theta\, d\varphi^2)-4M(a/r)\sin^2\theta\, dt\, d\varphi \tag{5.1}$$

as a first approximation to the one given by Eq. (2.1). Since the metric (5.1) represents the Lense-Thirring[23] solution of Einstein's equations, we can say that the second case to be examined in the following corresponds to the Lense-Thirring region of the Kerr spacetime.

A. $a=0$

Equation (3.6) shows that no local effect on a gyroscope's spin is found when the angular momentum of the source vanishes. The nonvanishing of the mass parameter, however, is sufficient to produce a global effect. Equations (4.28) and (4.35) show that the parameters k and n vanish and, taking into account the fact that

$$K(0)=E(0)=\Pi(0,0)=\pi/2 , \tag{5.2}$$

we find that the rest of the equations of Sec. III give

$$\begin{aligned} &E=(r-2M)/[r(r-3M)]^{1/2} ,\\ &K=Q=Mr^2/(r-3M) ,\\ &O^2=(r-2M)/(r-3M),\quad P^2=M/(r-3M) ,\\ &\Omega_S^{(\hat{3})}=-\dot{\Psi}=-(M/r^3)^{1/2} ,\\ &\delta\varphi=\tan Z=0 ,\\ &\Psi=2\pi(1-3M/r)^{1/2} . \end{aligned} \tag{5.3}$$

The last three of these equations, together with the subset of Eqs. (4.15)–(4.23) which define the frames of reference used in our analysis, lead to the following well-known result.[24] The orbit of the satellite is confined in the x-z plane and the spin vector $\mathbf{S}_s$ of the gyroscope precesses with angular velocity $-(M/r^3)^{1/2}\partial_{\hat{y}}$ relative to the carrier satellite. Thus, in one revolution the vector $\mathbf{S}_s$ rotates by the angle Ψ given by the last of Eqs. (5.3). During the same time, however, the satellite frame rotates by 2π relative to the fixed stars, since at each instant its axes correspond to the r-θ-φ directions. As a result the gyroscope's spin rotates by $2\pi-\delta\Psi$ relative to the fixed stars which amounts to $3\pi M/r$ when $(M/r)\ll 1$. For a satellite in near-Earth orbit $(M/r)\sim 7\times 10^{-10}$ in units where $c=G=1$ and the gyroscope precesses at a rate of ~ 8 arcsec/yr.

B. $0<(a/r)\ll 1$

It is of interest to obtain approximate expressions, valid to first order in (a/r), for the exact results presented in the last two sections. First, because in experiments designed to measure the terrestrial gravitomagnetic field (a/r) is a very small quantity indeed, amounting to $\sim 5\times 10^{-7}$ when r is set equal to Earth's radius. Second, because such an approximation leads to a very clear intuitive picture of the exact results obtained above.

Starting with Eq. (3.6) for the local effect on the symmetry axis, we find that

$$\Omega_R\approx\frac{2Ma}{r^3}\partial_z . \tag{5.4}$$

In the case of the satellite in polar orbit, on the other hand, most of the parameters of the problem are of second and higher order in (a/r). This holds for E,Q and the Lorentz-boost factors O and P for example. From Eqs. (4.6), (4.28), and (4.35), on the other hand, we find that

$$\begin{aligned} &K\approx Mr^2(r-3M)^{-1}[1+(a/M)(a/r)] ,\\ &k^2\approx(a/r)^2 ,\\ &n\approx(a/M)(a/r)(1-3M/r) . \end{aligned} \tag{5.5}$$

Since for small k and n

$$\begin{aligned} &K(k)\approx(\pi/2)[1+(k/2)^2] ,\\ &E(k)\approx(\pi/2)[1-(k/2)^2] ,\\ &\Pi(n,k)\approx(\pi/2)[1+(n/2)+(k/2)^2] , \end{aligned} \tag{5.6}$$

the last two of Eqs. (5.5) imply that to first order in (a/r)

$$K(k)\approx E(k)\approx \pi/2$$

and (5.7)

$$\Pi(n,k)\approx(\pi/2)[1-(a/M)(a/r)] \; .$$

Using the expressions for E and Q given by Eqs. (5.3), (5.5), and (5.7) we find that Eqs. (4.30), (4.33), and (4.34) take the form

$$\delta\varphi\approx 4\pi a(M/r^3)^{1/2} \; , \tag{5.8}$$

$$\tan Z\approx(a/r)[(r-2M)/M]^{1/2} \; , \tag{5.9}$$

and

$$\Psi(T)\approx 2\pi(1-3M/r)^{1/2} \; , \tag{5.10}$$

respectively.

From this point on let it be assumed that (M/r) is also very small, being equal to $\sim 7\times 10^{-10}$ in the case of a satellite in near-Earth orbit. This assumption allows us to adopt a Euclidean point of view with regard to space relations and to interpret Eqs. (5.8)–(5.10) as follows. At the end of the first revolution the satellite frame has turned into $(\partial_{\hat{x}'},\partial_{\hat{y}'},\partial_{\hat{z}'})=(\cos\delta\varphi\partial_{\hat{x}}+\sin\delta\varphi\partial_{\hat{y}},-\sin\delta\varphi\partial_{\hat{x}}+\cos\delta\varphi\partial_{\hat{y}},\partial_{\hat{z}})$, where $\delta\varphi$ is now given by Eq. (5.8). According to Eq. (5.10) and the remarks at the end of the last section, on the other hand, the vector $\mathbf{S}_s$ has rotated by $3\pi(M/r)$ in the direction $\mathbf{N}=\cos Z\partial_{\hat{y}'}-\sin Z\partial_{\hat{z}'}$ relative to the satellite frame. Equation (5.9) allows us to write $\cos Z\approx 1$, $\sin Z\approx Z\approx(a/r)(r/M)^{1/2}$. Therefore, the overall change of $\mathbf{S}_s$ relative to the "fixed stars," i.e., the frame $(\partial_{\hat{x}},\partial_{\hat{y}},\partial_{\hat{z}})$ can be described as a rotation of $[\delta\varphi-3\pi(r/M)\sin Z]=\pi a(M/r^3)^{1/2}$ about $\partial_{\hat{z}}$ and a rotation of $3\pi(M/r)\cos Z=3\pi(M/r)$ about $\partial_{\hat{y}'}$.

The same picture emerges from a different point of view. Specifically, the assumption that (a/r) and (M/r) are very small allows us to write Eq. (4.13) in the form

$$\begin{aligned}&\Omega_S^{(\hat{r})}\approx 0 \; ,\\ &\Omega_S^{(\hat{\theta})}\approx 3(Ma/r^3)\sin\theta \; ,\\ &\Omega_S^{(\hat{\varphi})}\approx -(M/r^3)^{1/2} \; ,\end{aligned} \tag{5.11}$$

where now the components of $\mathbf{\Omega}_S$ refer to the usual spherical polar coordinates of three-dimensional Euclidean space. The vector $\mathbf{\Omega}_s$ represents the rate of rotation of the gyroscope spin relative to the satellite frame which is now identified with the base $(\partial_{\hat{r}},\partial_{\hat{\theta}},\partial_{\hat{\varphi}})$ at each point of the orbit. Since the satellite frame rotates with angular velocity $\mathbf{w}=\dot{\varphi}\partial_{\hat{z}}+\dot{\theta}\partial_{\hat{\varphi}}$ relative to the Cartesian base $(\partial_{\hat{x}},\partial_{\hat{y}},\partial_{\hat{z}})$, we conclude that the spin vector $\mathbf{S}_s$ precesses relative to the "fixed stars" with angular velocity $\mathbf{\Omega}$ where

$$\begin{aligned}\mathbf{\Omega}=\mathbf{\Omega}_s+\mathbf{w}=&3(Ma/r^3)\sin\theta\partial_{\hat{\theta}}\\ &-(M/r^3)^{1/2}\partial_{\hat{\varphi}}+\dot{\varphi}\partial_{\hat{z}}+\dot{\theta}\partial_{\hat{\varphi}} \; .\end{aligned} \tag{5.12}$$

It now follows from Eq. (4.5) that

$$\dot{\theta}\approx(M/r^3)^{1/2}(1+3M/2r), \quad \dot{\varphi}\approx 2Ma/r^3 \; . \tag{5.13}$$

Thus, Eq. (5.12) can be written as

$$\begin{aligned}\mathbf{\Omega}=&(Ma/r^3)\sin\theta\partial_{\hat{\theta}}+(2Ma/r^3)\cos\theta\partial_{\hat{r}}\\ &+\tfrac{3}{2}(M/r)(M/r^3)^{1/2}\partial_{\hat{\varphi}} \; .\end{aligned} \tag{5.14}$$

Equivalently,

$$\mathbf{\Omega}=(1/r^3)[3(\mathbf{J}\cdot\partial_{\hat{r}})\partial_{\hat{r}}-\mathbf{J}]+\tfrac{3}{2}\mathbf{v}\times\nabla(M/r) \; , \tag{5.15}$$

where

$$\mathbf{J}\equiv Ma\,\partial_{\hat{z}}, \quad \mathbf{v}\equiv(M/r)^{1/2}\partial_{\hat{\theta}} \; . \tag{5.16}$$

According to Eq. (5.13), $\dot{\theta}>>\dot{\varphi}$. Therefore, the vector $\mathbf{v}$ defined in Eq. (5.16) can be identified with the satellite's velocity and the orbit can be considered to be planar during each revolution. Then, the first term on the right-hand side of Eq. (5.15) will represent the rate of rotation of the component of $\mathbf{S}_s$ normal to the plane of the orbit, while the second will stand for the angular velocity of the projection of $\mathbf{S}_s$ in the orbit plane relative to the "fixed stars."

Equation (5.15) gives the well-known formula for calculating the rate of spin precession in the weak-field and slow-motion limit.[25,26] Averaged over one period, the first term gives $\pi a(M/r^3)^{1/2}$, while from the second we obtain $3\pi(M/r)$ for the integrated rotation of the gyroscope spin components normal to and in the orbit plane, respectively. These values are the same with the ones obtained earlier in this subsection and, for a near-earth orbit, they amount to $\sim 4.2\times 10^{-11}$ and 6.5×10^{-9} per revolution, or ~ 0.05 and 8 arcsec/yr, respectively.

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