Positivity of total energy in general relativity

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A simple argument for energy positivity is given.

I. INTRODUCTION

The positivity of the total energy is a well-established subject due to Schön and Yau[1] and Witten.[2] Recently one of us gave a simple proof of energy positivity which is based on special 2 + 1 foliations of the three-dimensional spacelike Cauchy surface Σ (Ref. 3). This proof gives the energy in terms of manifestly positive expressions, each of them depending on canonical variables describing degrees of freedom of the gravitational field. In the present paper we give an important improvement and generalization of the above argument. The idea of the proof is based on the observation that the energy plays a double role in general relativity. It is a "gravitational charge" defined by a surface integral (similar to the one defining the electric charge in electrodynamics). On the other hand, the energy is a Hamiltonian. It is given by a three-dimensional volume integral. This way we obtain the energy (surface integral on the left-hand side) as a sum of manifestly positive expressions (volume integral on the right-hand side). The equivalence of these two expressions is given by the constraint equation.

We assume that the Cauchy surface Σ has a topology of R^3. We will consider 2 + 1 foliations of Σ. Two topologically different situations will be considered: $R^3=R^2\times R^1$ and $R^3-\{x_0\}=S^2\times R_+$, the first being referred to as "topologically flat foliation" and the second as "nested spheres" or "radial foliations." For the sake of calculational convenience let the coordinates x^k $(k=1,2,3)$ on Σ be adapted to the foliation in such a way that two-dimensional leaves of foliation correspond to $x^3=\text{const}$. [$x^A(A=1,2)$ are coordinates on each two-dimensional leaf separately.] The foliation together with parametrization of leaves is thus given by a real function τ on Σ, where $\tau(x^1,x^2,x^3)=x^3$. In the topologically flat case x^3 is an "asymptotically Cartesian" coordinate and in the case of nested spheres x^3 is a radial coordinate. The surface Σ is equipped with the Riemannian (positive-definite) metric g_{ij}. Let M^i be a unit vector field on Σ orthogonal to the leaves of the foliation. There are two such fields and our construction does not depend on the choice of one of them. Define the "acceleration" field

$$a^i=M^i{}_{|j}M^j , \tag{1}$$

where $M^i{}_{|j}$ denotes the covariant derivative of M^i with respect to the metric connection on Σ. Moreover let k_{AB} be a second fundamental form (external curvature) of two-dimensional leaves of our foliation as embedded in Σ (i.e., minus one-half of the Lie derivative of the induced two-dimensional metric g_{AB} with respect to the vector field M^i). Let $k=\tilde{g}^{AB}k_{AB}$ be the two-dimensional trace of this object. ($\tilde{g}^{AB}$ is a two-dimensional inverse of g_{AB} on each leaf separately.)

There is a Gauss-Codazzi formula relating the three-dimensional scalar curvature $\mathcal{R}$ at each point x in Σ with the internal and external geometry of the two-dimensional leaf passing through x:

$$(\det g)^{1/2}\mathcal{R}=(\det g)^{1/2}(R+k^2-k_{AB}k^{AB}) +2\partial_i[(\det g)^{1/2}(M^ik+a^i)] , \tag{2}$$

where R is the two-dimensional Riemann scalar curvature of the leaf.[4] We have

$$(\det g)^{1/2}=\lambda\,|\nabla\tau|^{-1} , \tag{3}$$

where $|\nabla\tau|=(g^{33})^{1/2}$ and $\lambda=(\det g_{AB})^{1/2}$ is a two-dimensional volume element on each leaf. One can easily prove that

$$a^i=\tfrac{1}{2}(g^{ij}-M^iM^j)\partial_j\phi , \tag{4}$$

where $\phi=\ln g^{33}$. Multiplying both sides of (2) by $|\nabla\tau|$ we obtain the result

$$2\partial_i(\lambda M^ik+\lambda a^i)+\lambda R=\lambda\mathcal{R}+\lambda(k_{AB}k^{AB}-\tfrac{1}{2}k^2) +\tfrac{1}{2}\lambda\tilde{g}^{AB}(\partial_A\phi)(\partial_B\phi) +\lambda kM^i\partial_i\phi-\tfrac{1}{2}\lambda k^2 . \tag{5}$$

II. TOPOLOGICALLY FLAT FOLIATIONS

For the sake of simplicity of calculations we assume that our metric g_{ij} is asymptotically flat in the following strong sense:

$$g_{ij}=f\gamma_{ij} , \tag{6}$$

where f behaves like $(1+2Er^{-1})$ for $r\rightarrow\infty$:

$$\gamma_{ij}=\delta_{ij}+O(1/r^{1+\epsilon}),\quad \partial_k\gamma_{ij}=O(1/r^{2+\epsilon}),\quad \epsilon>0 . \tag{7}$$

The left-hand side of formula (5) is the total divergence due to the Gauss-Bonnet theorem. We integrate it over Σ.

Theorem. Under the above assumptions on the behavior of g_{ij} the integral of the left-hand side of (5) over Σ is equal to $16\pi E$. The theorem is true in the general asymptotically flat case[5] but our strong condition makes the proof easier (see the Appendix).

We will show in a future work that the total integral of the right-hand side of (5) is positive when calculated for Cauchy data satisfying the constraint equation

$$(\det g)^{1/2}\mathcal{R}=(\det g)^{-1/2}(P^{ij}P_{ij}-\tfrac{1}{2}P^2)+16\pi\mathcal{E}_{\rm mat}\ , \quad (8)$$

where P^{ij} is the Arnowitt-Deser-Misner (ADM) momentum,[6] $P=g_{ij}P^{ij}$, and $\mathcal{E}_{\rm mat}=(\det g)^{1/2}T_{\mu\nu}n^{\mu}n^{\nu}$ is the energy density of the matter fields interacting with the gravitational field. Here n^{μ} is a unit timelike vector orthogonal to Σ.

Inserting (8) into (5) we obtain on the right-hand side the following manifestly positive terms:

$$(\lambda)^{-1}g^{33}P^{ij}P_{ij}=A_1\geq 0\ , \quad (9)$$

$$16\pi(g^{33})^{1/2}\mathcal{E}_{\rm mat}=A_2\geq 0\ , \quad (10)$$

$$\lambda(k_{AB}k^{AB}-\tfrac{1}{2}k^2)=A_3\geq 0\ , \quad (11)$$

$$\tfrac{1}{2}\lambda\tilde{g}^{AB}(\partial_A\phi)(\partial_B\phi)=A_4\geq 0\ . \quad (12)$$

Moreover, we have the manifestly negative term

$$B_1=-\tfrac{1}{2}(\lambda)^{-1}g^{33}P^2 \quad (13)$$

and another term

$$B_2=\lambda k M^i\partial_i\phi-\tfrac{1}{2}\lambda k^2\ . \quad (14)$$

There is however a gauge freedom consisting in (asymptotically trivial) changes of Σ and of coordinates on Σ. The terms A_m and B_n are not gauge invariants, but the total integral over Σ is invariant due to the fact that the integral of the left-hand side is a surface integral at infinity. To prove that this integral is positive it is sufficient to choose a gauge which leaves only manifestly positive terms. The situation is like in linear algebra where the proof of positivity of a quadratic form consists in choosing coordinates (i.e., a "gauge") which diagonalizes the form. The form is positive if and only if there is a "gauge" which leaves only manifestly positive terms. Using this analogy we could call our method a "diagonalization" of the total energy. To eliminate the term B_1 we use the maximality condition for $\Sigma(P=0)$. To diagonalize the term B_2 we propose to use any of the following gauge conditions:

$$w_\beta=k-\beta M^i\partial_i\phi=0\ , \quad (15)$$

where β is a real number. Now

$$B_2=\tfrac{1}{2}\beta(2-\beta)(M^i\partial_i\phi)^2\ . \quad (16)$$

Choosing $\beta\in[0,2]$ we obtain $B_2\geq 0$. (The method used in Ref. 3 corresponds to the special choice $\beta=1$.)

This ends the proof of the positivity provided we are able to show that the condition $w_\beta=0$ actually may be satisfied.

Because of the formula $-k=M^i{}_{|i}$ and $M^i=g^{3i}(g^{33})^{-1/2}$ we have

$$\begin{aligned}(g^{33})^{\beta}(\det g)^{1/2}w_\beta&=\partial_i[(\det g)^{1/2}(g^{33})^{\beta-1/2}g^{3i}]\\&=(\det g)^{1/2}\nabla_i(\,|\nabla\tau|^{2\beta-1}\nabla^i\tau)\\&=0\ .\end{aligned} \quad (17)$$

We see that our gauge condition (15) is equivalent to the elliptic equation for the function τ. This equation can be derived from the variational principle

$$\delta\int_\Sigma\mathcal{L}=0\ , \quad (18)$$

where $\mathcal{L}=(\det g)^{1/2}\,|\nabla\tau|^{2\beta+1}$.

There follow special cases. For $\beta=0$ our equation reduces to $k=0$ which means that the leaves of our foliation are minimal surfaces of the metric g_{ij}. For $\beta=\frac{1}{2}$, Eq. (17) is linear ($\Delta\tau=0$) and our coordinate $\tau=x^3$ is a harmonic function. For $\beta=1$, Eq. (17) is conformally invariant as a consequence of conformal invariance of $\mathcal{L}$ given by (18) and the condition $w_1=0$ is precisely the one used in previous papers.[3] This equation was thoroughly investigated by Chruściel.[5] In all other cases we do not know at the moment whether or not Eq. (17) has appropriate solutions.

We rewrite our energy formula in a more explicit way. Because of Eq. (15) the right-hand side of (5) reads

$$E=(16\pi)^{-1}\int_\Sigma[g^{33}(\lambda)^{-1}P^{ij}P_{ij}+\lambda(k^{AB}k_{AB}-\tfrac{1}{2}k^2)+\tfrac{1}{2}\lambda\tilde{g}^{AB}(\partial_A\phi)(\partial_B\phi)+\tfrac{1}{2}\beta(\beta-2)(M^i\partial_i\phi)^2+(g^{33})^{1/2}\mathcal{E}_{\rm mat}]\ . \quad (19)$$

It is worthwhile to notice that $E=0$ implies that $P^{ij}=0$, $\phi=$const (and therefore $g^{33}=1$ due to boundary conditions at infinity). Moreover $k_{AB}=0$. This means that g_{ij} is flat. Therefore our space is flat and empty ($\mathcal{E}_{\rm mat}=0$).

III. RADIAL FOLIATIONS

We assume now that $\tau=x^3=r$ is a radial coordinate and surfaces $r=$const are topologically 2-spheres. The spherical analog of condition (15) is

$$w_\beta=k-\beta M^i\partial_i\phi+2\,|\nabla r|\,/r\ , \quad (20)$$

where again $|\nabla r|=(g^{33})^{1/2}=M^3$. Using this condition and constraint (8) we may rewrite Eq. (5) as

$$\begin{aligned}&\partial_i(2\lambda M^i k+2\lambda a^i)+2\lambda(1-\beta)M^3M^i(\partial_i\phi)/r+2\lambda g^{33}/r^2+\lambda R\\&\quad=g^{33}(P^{ij}P_{ij}-\tfrac{1}{2}P^2)/\lambda+\lambda(k_{AB}k^{AB}-\tfrac{1}{2}k^2)+\tfrac{1}{2}\lambda\tilde{g}^{AB}(\partial_A\phi)(\partial_B\phi)+\tfrac{1}{2}\beta(2-\beta)(M^i\partial_i\phi)^2+(g^{33})^{1/2}\mathcal{E}_{\rm mat}\ .\end{aligned} \quad (21)$$

We will prove that the left-hand side gives $16\pi\times$(ADM energy). For maximal surfaces $\Sigma(P=0)$ and $\beta\in[0,2]$ the above equation implies the energy positivity. To analyze condition (20) we observe that

$$r^{-2}(g^{33})^{\beta}(\det g)^{1/2}w_{\beta}=\partial_i[r^{-2}(\det g)^{1/2}(g^{33})^{\beta-1/2}g^{3i}]$$
$$=(\det g)^{1/2}\nabla_i(r^{-2}\,|\,\nabla\tau\,|^{\,2\beta-1}\nabla^i\tau)=0\ . \tag{22}$$

For $\beta\neq 0$ and $\beta\neq 1$ we introduce the function

$$\rho=\beta(\beta-1)^{-1}(r^{(\beta-1)/\beta}-1)\ . \tag{23}$$

The condition $w_\beta=0$ now reads

$$\nabla_i(\,|\,\nabla\rho\,|^{\,2\beta-1}\nabla^i\rho)=0 \tag{24}$$

which has to be satisfied outside the center x_0 (i.e., for $r\neq 0$). Equation (24) is identical with Eq. (17) but the boundary conditions for the solution ρ are different. In the case of flat foliation we looked for a solution $\tau=z+F(x,y,z)$, where (x,y,z) are asymptotically flat coordinates and F is a bounded function. Now we are looking for

$$\rho=\beta(\beta-1)^{-1}[(x^2+y^2+z^2)^{(\beta-1)/2\beta}-1]+F(x,y,z)\ , \tag{25}$$

where F is bounded. To remove the freedom in the choice of an additive constant we assume that $\rho(x,y,z)=0$ at the point where the first (Euclidean) part of (25) vanishes (i.e., at zero for $\beta>1$ and at infinity for $\beta<1$). We may formally pass to the limit $\beta\to 1$ and obtain, for $\beta=1$,

$$\rho=\ln r=\tfrac{1}{2}\ln(x^2+y^2+z^2)+F(x,y,z) \tag{26}$$

as a boundary condition for the same equation (24) which now reads

$$\nabla_i(\,|\,\nabla\rho\,|\,\nabla^i\rho)=0\ . \tag{27}$$

The above equation was investigated by Chruściel[5] who proved that it possesses appropriate solutions. For $\beta=\frac{1}{2}$ we have $r=-\rho^{-1}$, where ρ is a solution of a linear equation

$$\Delta\rho=4\pi\delta_0\ , \tag{28}$$

where δ_0 is a unit charge concentrated at the origin $r=0$ (a gauge condition of this type was also used by Jang[7]). The function r is well defined outside of the origin because $\rho<0$. For $\beta=0$ the equation $w_\beta=0$ gives a condition for external curvature of leaves:

$$\lambda k=-2r^{-1}(\det g)^{1/2}\ . \tag{29}$$

In this case there is no "homogeneous" [i.e., analogous to (24)] version of Eq. (22).

APPENDIX

Let metric g satisfy the condition (7). We perform the conformal transformation of g. The left-hand side of (5) may be rewritten as follows:

$$L=2\partial_i(\lambda M^i k+\lambda a^i)+\lambda R$$
$$=\partial_i(2\underline{\lambda}\underline{M}^i\underline{k})+\underline{\lambda}\underline{R}+\partial_A[\underline{\lambda}\gamma^{AB}\partial_B(\ln\gamma^{33})]$$
$$+2\partial_i(\underline{\lambda}\underline{M}^i\underline{M}^i\partial_j\Psi)-2\partial_B(\underline{\lambda}\gamma^{AB}\partial_A\Psi)\ , \tag{A1}$$

where $\underline{M}$, $\underline{k}$, λ, $\underline{R}$ are objects defined by the metric γ instead of g and $\Psi=\ln f$. Asymptotic behavior of γ and the Gauss-Bonnet theorem imply that

$$\int_\Sigma L\,d^3x=-2\int_\Sigma\partial_i(\underline{\lambda}\gamma^{ij}\partial_j\Psi)d^3x=16\pi E\ . \tag{A2}$$

We would like to stress that the function τ which satisfies Eq. (17) behaves asymptotically like

$$\tau\approx z+E(1-\beta)\cos\theta\ . \tag{A3}$$

The transition from coordinates (x,y,z) satisfying (7) to new coordinates $(\underline{x},\underline{y},\underline{z})$ where $\underline{z}=\tau$, can be analyzed in terms of spherical coordinates: $(r,\Phi,\theta)\to(\underline{r},\underline{\Phi},\underline{\theta})$. We obtain the following asymptotic behavior of the spherical angles: $\underline{\Phi}-\Phi=O(r^{-1})$ and $\theta-\underline{\theta}=O(r^{-1})$. This transformation is a "supertranslation"[8] and does not preserve the form (6) of the metric g. Nevertheless, it can be easily shown that after this supertranslation the first and the third term of (A1) cancel when integrated over Σ. Therefore (A2) remains valid even if we use new coordinates $(\underline{x},\underline{y},\underline{z})$.

For radial foliations the proof is more complicated since the left-hand side of the formula (21) depends on β. For a metric g which satisfies our strong condition (7) in a system of coordinates (x,y,z), the asymptotic behavior of the solution r of (22) at infinity is

$$r-(x^2+y^2+z^2)^{1/2}\to(1-\beta)E\ . \tag{A4}$$

Rewriting the metric in a new spherical coordinates based on this solution we obtain the following expression for the leading term $(1+2Er^{-1})\delta_{ij}$ of the metric g:

$$ds^2=(1+2Er^{-1})dr^2$$
$$+r^2(1+2Er^{-1})^{\beta}(d\theta^2+\sin^2\theta\,d\Phi^2)\ . \tag{A5}$$

Now we prove that the integral of the left-hand side of (21) over Σ gives $16\pi E$. We use the following identities which follow immediately from the definition of M^i and k:

$$-\lambda M^3M^i\partial_i\phi=(\beta-1)^{-1}[\lambda M^3w_\beta+\partial_i(\lambda M^3M^i)$$
$$-2\lambda r^{-1}M^3M^3]\ , \tag{A6}$$

$$-\lambda Mk=(\beta-1)^{-1}[\beta\partial_i(\lambda M^3M^i)+\lambda M^3w_\beta$$
$$-2\lambda r^{-1}M^3M^3]\ . \tag{A7}$$

Inserting the identities into the left-hand side of (21) we obtain the quantity

$$L=-2\beta(\beta-1)^{-1}\partial_3[r^{(\beta+1)/\beta}\partial_3(r^{-(\beta+1)/\beta}\lambda g^{33})]+\partial_A(2\lambda M^A k+2\lambda a^A)$$
$$-2\beta(\beta-1)^{-1}\partial_A[r^{-(\beta-1)/\beta}\partial_3(r^{(\beta-1)/\beta}\lambda M^3 M^A)]+2\lambda r^{-1}M^3 w_\beta+\lambda R-2(\beta-1)^{-1}\partial_3(\lambda M^3 w_\beta)\ . \quad \text{(A8)}$$

We use the identity

$$-2\beta(\beta-1)^{-1}\partial_3[r^{(\beta+1)/\beta}\partial_3(r^{-(\beta+1)/\beta}r^2)]=2\ . \quad \text{(A9)}$$

We will integrate L with respect to the measure $dr\,d\theta\,d\Phi$ over $\Sigma-\{x_0\}=S^2\times R_+$. Let $\sigma=\sin\theta$ be a standard volume element on a unit sphere. Using $w_\beta=0$ and (A9) we obtain

$$L=-2\beta(\beta-1)^{-1}\partial_3\{r^{(\beta+1)/\beta}\partial_3[r^{-(\beta+1)/\beta}(\lambda g^{33}-r^2\sigma)]\}+\lambda R-2\sigma$$
$$+2\partial_A[\lambda M^A k+\lambda a^A-\beta(\beta-1)^{-1}r^{-(\beta-1)/\beta}\partial_3(r^{(\beta-1)/\beta}\lambda M^3 M^A)]\ . \quad \text{(A10)}$$

Because of the Gauss-Bonnet theorem all but the first term vanish when integrated over each sphere $r=$const. Therefore

$$\int L\,dr\,d\theta d\Phi=\lim_{r\to\infty}I(r)-\lim_{\epsilon\to 0}I(\epsilon)\ , \quad \text{(A11)}$$

where

$$I(r)=-2\beta(\beta-1)^{-1}r^{(\beta+1)/\beta}\partial_r \times\left[r^{-(\beta+1)/\beta}\int_{S(r)}(\lambda g^{33}-r^2\sigma)d\theta\,d\Phi\right]\ . \quad \text{(A12)}$$

It is easy to see that $I(\epsilon)\to 0$ for $\epsilon\to 0$. The limit at infinity is given by the leading term (A5) of the metric $[\lambda\approx r^2(1+2Er^{-1})^\beta\sigma$ and $g^{33}\approx(1+2Er^{-1})^{-1}]$:

$$\lambda g^{33}-r^2\sigma\approx r^2\sigma[(1+2Er^{-1})^{\beta-1}-1] \approx 2E\sigma r(\beta-1)\ . \quad \text{(A13)}$$

Finally

$$I(r)\approx 4E\int_{S^2}\sigma\,d\theta\,d\Phi=16\pi E\ . \quad \text{(A14)}$$

The above calculations does not work for $\beta=1$ and $\beta=0$. For $\beta=1$ we have

$$L=2\partial_i(\lambda M^i k+\lambda a^i+\lambda r^{-1}M^3M^i)+\lambda R+2\lambda M^3 w_1$$
$$=2\partial_3(\lambda M^3 M^j\partial_j\phi-\lambda r^{-1}g^{33}+r\sigma)+\lambda R-2\sigma$$
$$+2\partial_A(\lambda M^A k+\lambda a^A+\lambda r^{-1}M^3 M^A) \quad \text{(A15)}$$

(in the last equality we have assumed that $w_1=0$). Again, because of the Gauss-Bonnet theorem the integral of L is given by the leading term in the metric

$$\int_\Sigma L\,dr\,d\theta\,d\Phi=\lim_{r\to\infty}I(r)\ , \quad \text{(A16)}$$

where

$$I(r)=2\int_{S(r)}[\lambda g^{3i}\partial_i\phi+r(\sigma-\lambda r^{-2}g^{33})]d\theta\,d\Phi \underset{r\to\infty}{\longrightarrow} 16\pi E\ . \quad \text{(A17)}$$

For $\beta=0$ similar calculations give

$$L=2\partial_i(\lambda M^i w_0)+2\lambda r^{-1}M^3 w_0 -2\partial_i(\lambda r^{-1}g^{3i}-\lambda a^i)+\lambda R\ . \quad \text{(A18)}$$

Therefore

$$\int_\Sigma L\,dr\,d\theta\,d\Phi=\lim_{r\to\infty}2\int_{S(r)}r(\sigma-\lambda r^{-2}g^{33})d\theta\,d\Phi =16\pi E\ . \quad \text{(A19)}$$

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