Strings and other distributional sources in general relativity

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This paper deals with two broad issues: the formulation of a mathematical framework for concentrated sources in general relativity and its application to strings. We isolate a class of those metrics whose curvature tensors are well defined as distributions. It is shown that shells of matter—but neither point particles nor strings—can be described by metrics in this class. This conclusion is examined in more detail for the case of strings. We estimate the errors inherent in certain determinations of the mass per unit length of a cosmic string, and in certain calculations of the gravitational radiation from such a string.

I. INTRODUCTION

A useful idealization in physics is that in which some smooth distribution of sources is replaced by a "concentrated source." One might have in electromagnetism, for example, some charge distribution which is confined to a region of space small compared with other distances in the problem, and whose internal structure is irrelevant. It can be replaced by a point charge. One thus avoids introducing internal structure that is only to be ignored anyway.

There is in the case of electromagnetism a natural mathematical framework for this idealization. Let both the Maxwell field and the charge-current density be, rather than smooth tensor fields, distributions. Recall that linear operations, including differentiation, make sense when applied to distributions. Hence Maxwell's equations, by virtue of their linearity in both fields and sources, make sense as equations on these distributions. As an example, let the charge density in electrostatics be the distribution of strength σ confined to some two-dimensional shell in space. Then these Maxwell equations require of the electric field distribution that the jump in the normal component of the electric field on crossing this shell be equal to σ. This mathematical framework means that the machinery of distribution theory is available in electromagnetism. For example, this framework provides a detailed sense in which a smooth charge density must approximate our surface charge distribution above in order that the electric field it produces be close to the corresponding field distribution. This framework is our guarantee that distributional Maxwell fields with distributional charge-currents make physical sense.

A similar idealization to concentrated sources would be useful in general relativity. But here the mathematical framework[1] cannot be as simple as for electromagnetism, for Einstein's equation, being nonlinear, does not make sense as an equation on a distributional metric and distributional stress-energy. We shall be concerned in this paper with two broad issues: the formulation of a mathematical framework for concentrated sources in general relativity and its application to strings. We shall isolate the class of metrics appropriate to such sources and discuss the fact that metrics for strings do not fall within this class.

There have been attempts[2] to introduce into general rel- ativity sources to represent gravitating point particles, i.e., sources concentrated on one-dimensional surfaces in space-time. One goal of this work was to find equations of motion for such point particles. To this end, there was introduced a class of metrics, specified by their behavior on approaching a singular world line, to describe the near-field of such a particle. It now appears, however, that the metrics in this class may not be physically realistic, for one expects that such a concentration of matter would result in collapse through a horizon, and that inside this horizon there will be further structure. Indeed, it now seems likely that there is in general relativity no mathematical framework whatever for matter sources concentrated on one-dimensional surfaces in space-time.

More successful has been the introduction[3] into general relativity of sources to represent thin shells, i.e., of sources concentrated on three-dimensional surfaces in space-time. Fix a smooth three-dimensional submanifold S of a smooth four-dimensional manifold. Introduce a Lorentz metric that is smooth up to and including S from each of its sides and is continuous across S; but they may there have a discontinuity in its first derivative. The resulting space-time is regarded as having curvature, and therefore stress-energy, concentrated on S. To obtain the formula giving the magnitude of the concentrated curvature in terms of the jump in the first derivative of the metric on crossing S, one writes out and interprets the usual formula for the components of the curvature tensor in terms of the components of the metric and their first two derivatives. For S timelike, this arrangement represents a thin shell of matter. The concentrated stress-energy acts, through its relation to the jump in the first derivative of the metric, as a source. For S null, this arrangement can represent[4] a thin shell

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of gravitational radiation. The Weyl tensor concentrated on S yields the amplitude of the radiation field, while the concentrated stress energy is required to vanish.

To summarize, while there is apparently no viable treatment of point particles as concentrated sources in general relativity, there is a satisfactory treatment of thin shells of matter or radiation. In the case of the latter, there has been isolated a class of metrics sufficiently broad to encompass the phenomena of interest, yet sufficiently narrow that the curvature tensors of these metrics make sense.

The intermediate case is that of strings, i.e., of sources concentrated on two-dimensional surfaces in space-time. Is there, or is there not, a satisfactory treatment in general relativity of such concentrated sources? Are strings more like shells or like point particles? The following example[5,6] will illustrate these issues. Let the metric be

$$-dt^2+dz^2+dr^2+\beta^2(r)d\phi^2 \ , \tag{1}$$

where t and z range from $-\infty$ to $+\infty$, ϕ from 0 to 2π (identified), and r from 0 (the axis) to $+\infty$. This space-time is static and cylindrically symmetric. Now let $\beta(r)$ be given by

$$\beta(r)=\begin{cases}(l/\gamma)\sin(\gamma r/l), & r\le l \ ,\\ [r-l+(l/\gamma)\tan\gamma]\cos\gamma, & r>l \ ,\end{cases} \tag{2}$$

where $l>0$ and $\gamma\in(0,\pi/2)$ are constants. This metric is C^1 across $r=l$, and C^∞ elsewhere (including the axis). For $r\le l$, the Einstein tensor, and through Einstein's equation,[7] the stress energy, have a mass density in the t direction, $-T^t{}_t$, and pressure in the z direction, $T^z{}_z$, given by

$$-T^t{}_t=T^z{}_z=\gamma^2/l^2 \ , \tag{3}$$

with all other components zero. For $r>l$, the space-time is flat, consisting of a position of Minkowski space-time with ϕ-angular deficit of $2\pi(1-\cos\gamma)$. Thus, this space-time consists of a static, massive fluid cylinder with a flat, "conical" exterior metric. Note that the mass per unit length of the cylinder, defined as the integral of this mass density over the two-surfaces of constant t,z, is also $2\pi(1-\cos\gamma)$—the same as the angular deficit.

Now fix γ, and consider the limit $l\to 0$. Then the exterior metric becomes Minkoski space-time with ϕ-angular deficit $2\pi(1-\cos\gamma)$; and one can think of the source as approaching a "line mass density" of mass per unit length also $2\pi(1-\cos\gamma)$. One might thus regard this limit as representing a source in general relativity concentrated on a two-dimensional surface in space-time. The equality here between the angular deficit and the mass per unit length would be analogous to that, in electrostatics, between the jump in the normal component of the electric field and the surface charge density. Both relate the external field to the strength of the source.

This example certainly does suggest that sources concentrated on two-dimensional surfaces in space-time will be meaningful in general relativity. In fact, such string sources have been introduced in certain applications. However, such isolated examples can be misleading, as is illustrated by the following.[8] Consider Newtonian gravitation, so we have, in Euclidean three-space, mass density ρ and gravitational potential U, satisfying $\nabla^2 U=-\rho$. Let ρ vanish outside some solid cylinder. Then one might expect that the mass per unit length of the cylinder will be reflected in the behavior of U outside the cylinder. But, we claim, there exists an example with positive mass per unit length—in fact, with $\rho\ge 0$ everywhere in the cylinder—and yet $U=0$ everywhere outside the cylinder. To obtain such an example, fix a positive constant l and a smooth function $f(r)$, defined for $r\ge 0$, so is positive for $r<l$ and zero for $r\ge l$. Set, in cylyindrical coordinates z,r,ϕ, $U=e^{\alpha z}f(r)$, where α is a constant, so this potential vanishes outside the cylinder of radius l. One now checks directly that the corresponding mass density, $\rho=-\nabla^2 U$, satisfies $\rho\ge 0$ everywhere provided the constant α is chosen sufficiently large. By using a limiting family of such examples, one might conclude that a line distribution of mass in Newtonian gravitation results in zero external field. What is needed for this example—as well as for that of Eqs. (1) and (2) in general relativity—is detailed rules as to the allowed limiting behavior of sources and fields.

In Sec. II we introduce a mathematical framework for concentrated sources in general relativity. The key step is the introduction of a class of metrics, called regular metrics, for which the curvature tensor makes sense as a distribution. It is the regular metrics that can arise from a distributional source. The definition requires that the metric be locally bounded with locally bounded inverse, and have locally square-integrable weak first derivative. In turns out that the metrics for thin shells of matter or radiation, described above, are regular. But the class is much more general than this, admitting, e.g., certain metrics that are not even continuous. The main theorem asserts that, for a regular metric with source concentrated on some submanifold of space-time, that submanifold must be of dimension three. Thus, point particles and strings—which correspond to dimensions one and two—are not permitted as sources. This result is closely related to the fact that the energy density of the Newtonian gravitational field of a massive shell is locally integrable, but not for a point particle or string. We also obtain an approximation theorem, which gives the sense in which a smooth metric must approximate a regular metric in order that the curvature tensor of the former be close to the curvature distribution of the latter.

In Sec. III we consider in more detail the case of strings, i.e., of sources in general relativity concentrated on two-dimensional surfaces in space-time. Minkowski space-time with angular deficit—the space-time that results as the limit of the family (1) and (2) of static, cylindrically symmetric solutions—turns out not to be regular in the sense of Sec. II. Thus, we cannot in any natural way regard this as the external metric for a distributional source. What happens, then, if we introduce some general source for the external metric consisting of Minkowski space-time with angular deficit, and then take the limit as that source becomes concentrated on the axis?

The Newtonian example above suggests that, if we wish to recover in the limit a fixed relationship between source and external field, then we shall have to impose some restrictions on the character of the source or the nature of the limit. We argue, however, that there are no such restrictions that are fully satisfactory. If, for example, one merely imposes an energy condition, then for given angular deficit one can obtain in the limit a variety of values for the mass per unit length of the source. Even fixing an equation of state for the matter will not do. It turns out, for example, that for almost every choice of equation of state there exists *no* source whatsoever made of that matter and having for its external metric a portion of Minkowski space-time with angular deficit. Even when there does exist such a source—e.g., for the matter of Eqs. (1) and (2)—further difficulties arise. For example, if one relaxes the static and cylindrical symmetries, then it is questionable whether one can recover any simple relationship between source and field.

In Sec. IV we consider applications to cosmic strings. There are at least two situations, in work on cosmic strings, in which the string is idealized as a concentrated, gravitating source. The first is that in which the mass per unit length of a string is inferred from external geometrical effects, e.g., from the formation of a double image of a quasar. The second is that in which the amount of gravitational radiation from such a string is computed. Both of these situations involve precisely the idealization that appears to be problematic in general relativity. We attempt, for these two situations, to estimate the errors resulting from this idealization.

II. METRICS WITH DISTRIBUTIONAL CURVATURE

A smooth metric in general relativity gives rise to a smooth curvature tensor, and hence requires for its source a smooth stress-energy tensor. But concentrated sources are anything but smooth. So, if we wish to admit concentrated sources in general relativity, we shall have to admit metrics less well behaved than smooth. The wider the class of metrics thus allowed, the wider will be the class of sources with which one is able to deal. But we cannot, on the other hand, admit metrics so badly behaved that their curvature tensors are not meaningful. So, we are led to seek the widest class of metrics whose curvature tensors make sense.[1] We shall find an appropriate class, called the regular metrics, in this section. The curvature tensor of a regular metric will turn out to make sense in general only as a distribution. Hence, the source for a regular metric is a distributional stress-energy. It will turn out that not every distributional stress energy is permitted as a source. Disallowed, in particular, is any such stress energy concentrated on a submanifold of space-time of dimension less than three.

We first recall a few facts about distributions.[9] Fix a smooth (C^∞), four-dimensional manifold M. By a *test field* on M we mean a smooth tensor density, $t^{a\cdots c}{}_{b\cdots d}$, of weight -1, having compact support on M. Note that this weight[10] is such that the integral over M of a scalar test field can be carried out without any additional volume element on M. The test fields of given index structure form a vector space. The contraction of a test field, its outer product with any smooth tensor field, and its derivative via any smooth derivative operator all yield, again, test fields.

A *distribution* on M is a linear mapping from the vector space of test fields of a given index structure to the real numbers satisfying the following continuity condition. The result of applying this linear mapping to each of a sequence of test fields, ${}_it^{a\cdots c}{}_{b\cdots d}$ $(i=1,2,\ldots)$, must converge to that of applying it to $t^{a\cdots c}{}_{b\cdots d}$ provided (i) these test fields all have support in a common compact set and (ii) the ${}_it$ converge uniformly to t, as do all the corresponding derivatives. We shall adopt the following index notation for distributions: A distribution α applicable to test fields of the form $t^{a\cdots c}{}_{b\cdots d}$ will be denoted $\alpha_{a\cdots c}{}^{b\cdots d}$, and the action of α on t will be denoted $\alpha_{a\cdots c}{}^{b\cdots d}*t^{a\cdots c}{}_{b\cdots d}$. Note that this is a number, not a field.

There is an important class of distributions, a class which motivates the definition above as well as those that will follow. Let $\mu_{a\cdots c}{}^{b\cdots d}$ be any smooth tensor field on M. Then this field gives rise to a distribution, which we write $\hat{\mu}_{a\cdots c}{}^{b\cdots d}$, with the following action:

$$\hat{\mu}_{a\cdots c}{}^{b\cdots d}*t^{a\cdots c}{}_{b\cdots d}=\int_M \mu_{a\cdots c}{}^{b\cdots d}t^{a\cdots c}{}_{b\cdots d}\,. \tag{4}$$

Note that the integral on the right-hand side makes sense (since t is a density of the appropriate weight) and converges (since t has compact support). The mapping on text fields so obtained is linear, and satisfies the continuity condition—and so does indeed define a distribution. Thus, every smooth tensor field on M gives rise, via (4), to a distribution—and, indeed, it was to make this true that we defined a distribution as we did. Of course, not every distribution arises in this way.

Linear operations on tensor fields can generally be extended to operations on distributions. The sum of two distributions with the same index structure, defined as the sum of the corresponding linear maps, is again a distribution. We define the contraction of a distribution thus. For example, for $\alpha_{ac}{}^{bd}$ a distribution, its contraction $\alpha_{mc}{}^{bm}$ is the distribution with action

$$\alpha_{mc}{}^{bm}*t^c{}_b=\alpha_{ac}{}^{bd}*(t^c{}_b\delta^a{}_d)\,. \tag{5}$$

The outer product of a distribution and a smooth tensor field is defined similarly. Fix *any* smooth derivative operator ∇_m. Then, for $\alpha_{a\cdots c}{}^{b\cdots d}$, a distribution, its derivative, $\nabla_m\alpha_{a\cdots c}{}^{b\cdots d}$, is defined by[11]

$$(\nabla_m\alpha_{a\cdots c}{}^{b\cdots d})*t^{ma\cdots c}{}_{b\cdots d}=-\alpha_{a\cdots c}{}^{b\cdots d}*(\nabla_m t^{ma\cdots c}{}_{b\cdots d})\,. \tag{6}$$

All of these operations on distributions—addition, contraction, outer product with a tensor field, and derivative—reduce, for a distribution arising via (4) from a smooth tensor field, to the corresponding operators on the tensor field. For example, for $\mu_{a\cdots c}{}^{b\cdots d}$ a smooth tensor field, we have

$$\nabla_m\hat{\mu}_{a\cdots c}{}^{b\cdots d}=(\nabla_m\mu_{a\cdots c}{}^{b\cdots d})^\wedge\,, \tag{7}$$

i.e., that the derivative of the distribution defined by μ equals the distribution defined by the derivative of μ. Indeed, it is to make such things true that the operations are defined as they are. Furthermore, these operations on distributions generally satisfy the same properties as do the corresponding operations on smooth tensor fields. But note that there is no operation on smooth tensor fields not available for distributions: There is no such thing in general as the outer product of two distributions.

A tensor field $\mu_{a\cdots c}{}^{b\cdots d}$ on M is said to be *locally integrable* if, for every test field $t^{a\cdots c}{}_{b\cdots d}$, the scalar density $\mu_{a\cdots c}{}^{b\cdots d}t^{a\cdots c}{}_{b\cdots d}$ on M is (Lebesque) measurable, and its (Lebesque) integral converges. Since our test fields have compact support, this is truly a local property of μ. For example, any continuous tensor field is necessarily locally integrable. Equation (4), defining the distribution associated with smooth tensor field $\mu_{a\cdots c}{}^{b\cdots d}$, works also for μ merely locally integrable We again denote the resulting distribution by $\hat{\mu}_{a\cdots c}{}^{b\cdots d}$. The *weak derivative* of locally integrable tensor field $\mu_{a\cdots c}{}^{b\cdots d}$ is a locally integrable tensor field $\nu_{ma\cdots c}{}^{b\cdots d}$, if one exists, such that

$$\hat{\nu}_{ma\cdots c}{}^{b\cdots d}=\nabla_m\mu_{a\cdots c}{}^{b\cdots d}\,. \tag{8}$$

That is, the weak derivative is a tensor field ν giving rise to the distribution that is the derivative of the distribution defined by μ. For example, for the tensor field $\mu_{a\cdots c}{}^{b\cdots d}$ C^1, its weak derivative does exist, and is in fact precisely the tensor field $\nabla_m\mu_{a\cdots c}{}^{b\cdots d}$. But, the weak derivative exists also for far less-well-behaved tensor fields μ.

We now return to the issue at hand: to find the widest class of metrics whose curvature tensors make sense as distributions. One might be tempted to conclude that any distributional metric should be acceptable, for the curvature tensor involves the second derivative of the metric, while the second derivative of any distribution makes sense. However, this argument fails, for the curvature tensor is also nonlinear in the metric, and products of distributions do not in general make sense. So, the appropriate conditions on the metric should require more than that it be a mere distribution, but less, e.g., than that it be C^2.

Fix a smooth derivative operator ∇_m. For g_{ab} a smooth metric, its curvature tensor $R_{abc}{}^d$ can be written

$$R_{abc}{}^d=\rho_{abc}{}^d-2\Gamma^d{}_{m[a}\Gamma^m_{b]c}-2\nabla_{[a}\Gamma^d_{b]c}\ , \tag{9}$$

where

$$\Gamma^a{}_{bc}=\tfrac{1}{2}g^{am}(2\nabla_{(b}g_{c)m}-\nabla_m g_{bc}) \tag{10}$$

and $\rho_{abc}{}^d$ is the curvature tensor of ∇_m. The idea is to continue to use Eqs. (9) and (10) to define the curvature tensor even for a less-well-behaved metric g_{ab}. Specifically, we shall require of g_{ab}, not that it be smooth, but rather only that it be such that each term on the right-hand side in (9) can be interpreted as a distribution. The first term on the right-hand side as a smooth tensor field, can always be interpreted as a distribution. The second term can be interpreted as a distribution provided the square of Γ is locally integrable. The third term can be interpreted as the derivative of a distribution—and so as a distribution in its own right—provided only that Γ itself can be so interpreted. Thus, the right-hand side of Eq. (9) becomes a distribution provided only that the square of Γ is locally integrable and Γ can be regarded as a distribution. But this $\Gamma^a{}_{bc}$, in turn, is given in Eq. (10) as the product of the first derivative of the metric and an expression algebraic in the metric. These considerations motivate the following definition.

A symmetric tensor field g_{ab} on M will be called a *regular metric* provided (i) its inverse g^{ab} exists everywhere, and both g_{ab} and g^{ab} are locally bounded, and (ii) the weak first derivative of g_{ab} exists and is locally square integrable. The first condition means that, for any test fields t^{ab} and u_{ab}, the scalar densities $g_{ab}t^{ab}$ and $g^{ab}u_{ab}$ are bounded. The second means, writing μ_{abc} for the weak derivative of g_{bc}, that $\mu_{abc}\mu_{def}$ is locally integrable.

Let g_{ab} be a regular metric. We define its curvature distribution as follows. Interpret the derivatives on the right-hand side of Eq. (10) as weak derivatives, so this equation defines a tensor field $\Gamma^a{}_{bc}$. From regularity and the fact that the outer product of a locally bounded field and a locally integrable field is locally integrable, it follows that this $\Gamma^a{}_{bc}$ is locally square integrable and locally integrable. Now interpret the first term on the right-hand side of Eq. (9) as the distribution resulting from a smooth tensor field, the second term as the distribution resultng from a locally integrable tensor field (for Γ is locally square integrable), and the third term as the distribution resulting from the derivative of Γ regarded as a distribution (for Γ is locally integrable). The result, the curvature distribution of the regular metric g_{ab}, is of course independent of the choice of the smooth derivative operator ∇_m.

We emphasize that a regular metric g_{ab} is to be specified *as a tensor field*—i.e., as an assignment of a tensor to each point of M—and not as a distribution. Thus, for example, one is not permitted to have the metric g_{ab} already "concentrated on surfaces." It is only in this way that one can negotiate the products in Eqs. (9) and (10). One could as well have required merely that the tensor field g_{ab} be specified almost everywhere, and then, in the first condition, that the inverse exist almost everywhere and that g_{ab} and g^{ab} be merely almost everywhere locally bounded. Note that regularity treats the metric and its inverse on an equal footing. From regularity of g_{ab} it follows that the weak derivative of the inverse metric g^{ab} exist and is locally square integrable.

A regular metric is not suitable for raising or lowering the indices of a general distribution, for the outer product of a regular metric and a distribution is not in general well defined as a distribution. Nonetheless, we *can* interpret as a distribution the outer product of any number of metrics and inverse metrics with a single curvature tensor. This is done by writing an equation analogous to Eq. (9), but with the term involving the second derivative of the metric expressed as a total derivative. For example, we have, from Eq. (9),

$$g^{pq}R_{abc}{}^{d}=g^{pq}\rho_{abc}{}^{d}-2g^{pq}\Gamma^{d}{}_{m[a}\Gamma^{m}{}_{b]c}$$
$$-2\nabla_{[a}(g^{pq}\Gamma^{d}_{b]c})+2(\nabla_{[a}g^{pq})\Gamma^{d}{}_{b]c} \; . \qquad (11)$$

Each term on the right-hand side can now be interpreted as a distribution. (For the last term, we must use the fact that the weak derivative of g^{pq} exists and is locally square integrable, and that the outer product of two locally square-integrable tensor fields is locally integrable.) So, this equation defines $g^{pq}R_{abc}{}^{d}$ as a distribution. Using the fact that any contraction of a distribution is a distribution, we now obtain the Einstein tensor, Weyl tensor, etc., as distributions. These distributions have the expected symmetries. In particular, then, it is meaningful with a regular metric to write Einstein's equation with distributional stress-energy.

It is hard to see how one could find any conditions on a metric weaker than regularity that would still yield a distributional curvature tensor. It is hard to see how the individual terms in Eq. (9), while failing to be distributions, could conspire to cancel to yield a sum that is a distribution. Yet, there may be other routes to defining a curvature distribution. Consider, for example, a positive-definite metric g_{ab} on a two-dimensional manifold M, such that (i) the inverse g^{ab} exists everywhere and both g_{ab} and g^{ab} are locally bounded, and (ii) the weak first derivative of g_{ab} exists. These conditions are strictly weaker than regularity, for we do not now require that the derivative of g_{ab} be square integrable. The metric of a cone, for instance, satisfies these conditions, but is not regular. Under these conditions on the metric g_{ab}, one can define as a distribution the curvature combination $R\epsilon_{ab}$, i.e., the combination that appears in the Gauss-Bonnet formula. To see how, fix, in addition to the smooth derivative operator ∇_a, a nowhere vanishing[12] vector field v_a, and consider the formula

$$R\epsilon_{ab}=\nabla_{[a}[(\nabla_{b]}v_c-\Gamma^{m}{}_{b]c}v_m)\epsilon^{cd}v_d(4/g^{pq}v_pv_q)] \; . \qquad (12)$$

For g_{ab} smooth, this formula holds (and so, in particular, the right-hand side is independent of the choices of ∇_a and v_a). But, the right-hand side makes sense as a distribution whenever the vector in brackets does, while this vector is locally integrable, and hence interpretable as a distribution, under the conditions above on g_{ab}. We thus define $R\epsilon_{ab}$ as a distribution. Note, however, that this argument does not work to make distributions out of other curvature combinations, such as R, $R\epsilon^{ab}$, or $R_{abc}{}^{d}$.

Let g_{ab} be a regular metric. We wish to introduce the derivative operator ${}^{g}\nabla_m$ compatible with this metric. Its action, e.g., on the vector field μ_b, is to be

$${}^{g}\nabla_a\mu_b=\nabla_a\mu_b-\Gamma^{m}{}_{ab}\mu_m \; , \qquad (13)$$

where ∇_m is our fixed smooth derivative operator and Γ is the locally square-integrable tensor field given by Eq. (10). How well behaved must the field μ_b be in order that the right-hand side of Eq. (13) make sense as a distribution? The answer is that μ must be locally square integrable. Indeed, μ locally square integrable implies that the second term on the right-hand side of Eq. (13) is locally integrable, and hence that this term can be interpreted as a distribution. In addition, μ locally square integrable implies that μ is also locally integrable—thus, interpretable as a distribution—and hence that the first term on the right-hand side of Eq. (13) can be interpreted as the derivative of a distribution. Thus, our operator ${}^{g}\nabla_m$ can be applied to any locally square-integrable tensor field, yielding a distribution. In particular, the equation ${}^{g}\nabla_m g_{ab}=0$ makes sense—and, in fact, holds. In the case in which the field μ_b is smooth, the distribution on the right-hand side of Eq. (13) arises from some locally square-integrable field. Hence, in this case the second ${}^{g}\nabla_m$ derivative of μ makes sense. Thus, for μ smooth both sides of the equation

$${}^{g}\nabla_{[a}{}^{g}\nabla_{b]}\mu_c=\tfrac{1}{2}R_{abc}{}^{d}\mu_d \qquad (14)$$

make sense as distributions. It follows directly from Eq. (9) that this equation does hold. We could in fact have employed (14) from the beginning as our definition of the curvature tensor. But note that the Bianchi identity,

$${}^{g}\nabla_{[a}R_{bc]de}=0 \; , \qquad (15)$$

does not make sense. Since the curvature of a regular metric is a distribution, and not a locally square-integrable tensor field, we cannot apply the operator ${}^{g}\nabla_m$ to it. Of course, one could impose further conditions on the metric to force a meaning for the left-hand side of Eq. (15).

The following will illustrate the nature of the condition that a metric be regular. Fix a submanifold S, of dimension d $(=0,1,2,3)$, of the four-dimensional manifold M. Consider a metric field g_{ab} that is smooth at points other than on S, but some of whose components become infinite, on approaching S, at rate r^{-s}, where r is some characteristic distance from S (defined, e.g., using a smooth, positive-definite metric on M). Now the derivative of g_{ab} has behavior r^{-s-1}, while the volume element has behavior $r^{3-d}dr$. Hence, the weak derivative of g_{ab} will exist and be locally square integrable provided[13] $s<1-d/2$. Note that the metric is allowed to grow more quickly on approaching a lower-dimensional singular surface than a higher-dimensional one.

For $d=0$ (an isolated singular point), the limiting allowed growth rate is r^{-1}. A metric with components growing at nearly this rate would, while having locally square-integrable weak derivative, not be locally bounded. But we can obtain examples that are locally bounded by "replacing growth by oscillation." Thus, the metric[14]

$$ds^2=[2+\sin(t^2+x^2+y^2+z^2)^{-1/4}]$$
$$\times(-dt^2+dx^2+dy^2+dz^2) \qquad (16)$$

is regular. Note that this metric is discontinuous at the origin. For $d=1$ (a singular world line) the limiting rate is $r^{-1/2}$. This rate is such that, e.g., the Schwarzschild metric, with a world line attached to the space-time manifold at either $r=2m$ or $r=0$, is not regular.[14] Thus, one apparently cannot treat a point particle as satisfying Einstein's equation with a distributional stress-energy confined to a world line. For $d=2$ (a singular string), the limiting rate is r^{0}. We shall show in the following section that the metric consisting of Minkowski space-time with angular deficit is, though just barely, not regular.[14] Thus, we cannot assign to this metric a distributional stress-energy to represent the "string." Finally, consider $d=3$ (a singular shell). The metrics for shells, described in Sec. I, are seen immediately to be regular. Indeed, regularity admits metrics in this case somewhat more badly behaved than these.

There is a simple and precise result that reflects the behavior of the regular metrics illustrated above. Let g_{ab} be a regular metric, so its Einstein tensor and, through Einstein's equation, its stress-energy source T_{ab}, makes sense as a distribution. The essence of the distributional character of T_{ab} is its ability to represent sources concentrated on surfaces of lower dimensions. What "lower dimensions" are possible? Restrictions may arise from the fact that the curvature distribution is of the special form (9). Indeed, this distribution is the sum of one distribution arising from a locally integrable field [first two terms on the right-hand side of (9)], and another the derivative of a distribution arising from a locally square-integrable field [third term on the right-hand side of (9)]. At issue is what restrictions this form imposes on a distribution. The answer is provided by the following.

Theorem 1. Let S be a submanifold, of dimension d $(=0,1,2,3)$, of the four-dimensional manifold M. Let $\alpha_{a\cdots c}{}^{b\cdots d}$ be a nonzero distribution which (i) has support[15] on S and (ii) is the sum of one distribution arising from a locally integrable tensor field and another the derivative of a distribution arising from a locally square-integrable field. Then [16] $d=3$.

Proof. Let, without loss of generality, α be a scalar distribution. Then by (ii) the action of α on test field t is

$$\alpha(t)=\int_M (\mu t+\nu^a\nabla_a t)\ , \tag{17}$$

where μ is a locally integrable scalar field and ν^a is a locally square-integrable vector field. Fix any test field t and any positive-definite metric g^+_{ab} on M. For any sufficiently small $\epsilon>0$, denote by U_ϵ the neighborhood of S of g^+ radius ϵ; and let h_ϵ be a smooth, non-negative function on M that vanishes in a neighborhood of S, has value one outside U_ϵ, and whose gradient has g^+ norm not exceeding $2/\epsilon$ in the support of t. Then we have

$$\begin{aligned}\left|\int_M (\mu t+\nu^a\nabla_a t)h_\epsilon\right| &= \left|\alpha(h_\epsilon t)-\int_M t\nu^a\nabla_a h_\epsilon\right| \\ &= \left|\int_{U_\epsilon} t\nu^a\nabla_a h_\epsilon\right| \le \left[\int_{U_\epsilon}|t|\,\nu^a\nu^b g^+_{ab}\right]^{1/2}\left[\int_{U_\epsilon}|t|\,(\nabla_a h_\epsilon)(\nabla_b h_\epsilon)g^{+ab}\right]^{1/2} \\ &\le \left[\int_{U_E}|t|\,\nu^a\nu^b g^+_{ab}\right]^{1/2}\left[\frac{4}{\epsilon^2}\int_{U_\epsilon}|t|\right]^{1/2} .\end{aligned} \tag{18}$$

In the first step we used the definition of α, in the second that α has support on S, in the third the Schwarz inequality, and in the fourth the bound on $\nabla_a h_\epsilon$. Now let $\epsilon\to 0$. The left-hand side of Eq. (18) approaches $|\alpha(t)|$. The first factor on the right-hand side approaches zero (since ν^a is locally square integrable), while the integral in the second factor is bounded by a multiple of ϵ^{4-d}. Hence, were $d<3$ then the right-hand side of Eq. (18) would approach zero, which would yield $\alpha=0$. ■

We conclude that a regular metric in general relativity can have its curvature concentrated on a submanifold only of dimension three (a shell of matter or radiation). It appears that neither a point particle nor a string has a general formulation with distributional source. Of course, the curvature distribution of a general regular metric can be quite complicated. It need not be "concentrated" on any submanifold at all.

The key feature that makes this theorem work is the requirement that the vector field ν^a on the right-hand side of Eq. (17) be locally *square* integrable. With the weaker condition of local integrability this equation would still define a distribution α, but the conclusion of Theorem 1 would fail. In the proof, local square integrability is used near the end to bound the first factor on the right-hand side of Eq. (18). Local square integrability is the appropriate condition on ν^a because ν in Eq. (17) plays the role analogous to Γ in the last term on the right-hand side of Eq. (9), and Γ is locally square integrable as a consequence of regularity.

As an example, consider the case in which the dimension of the manifold M is two. Then the theorem would require $d=1$, i.e., it would permit line, but not point, concentrations of curvature. But, we remarked earlier that one can have, via Eq. (12), certain point concentrations of curvature in two dimensions, provided one introduces a weaker version of regularity. This weakening, in turn, consists essentially of allowing Γ to be merely locally integrable instead of locally square integrable.

This feature also has the following physical interpretation. Think of the metric g_{ab} as analogous to the Newtonian gravitational potential, so Γ is, by Eq. (10), analogous to the gravitational field. The requirement of local square integrability of Γ, and so of ν, is analogous to the requirement that the integral for the energy of the gravitational field converge locally. Thus, the special dimension $d=3$ singled out in Theorem 1 can be regarded as arising because the integral for Newtonian gravitational energy converges locally for a shell of matter, but not for either a string or point particle.

These remarks can be made more concrete. Fix a smooth metric g_{ab} for space-time, and consider a conserved test stress-energy, i.e., a symmetric tensor field T^{ab} satisfying $\nabla_b T^{ab}=0$. Since this equation of conservation is linear, it clearly makes sense when T^{ab} is an arbitrary distribution. That is, a test stress energy can be concentrated in any way we wish. Suppose next that we try to solve the linearized Einstein equation for the first-order perturbation in the metric, ${}^1h_{ab}$, with this T^{ab} as source perturbation. Since this equation is linear, it will in general admit a solution ${}^1h$ that is also a distribution. That is, even linearized general relativity imposes no restrictions on how sources may be concentrated. We next attempt to pass to second order in perturbation theory. Einstein's equation, at this order, equates a linear operator acting on the second-order metric perturbation ${}^2h$, to an expression quadratic in the first derivative of ${}^1h$. But, for ${}^1h$, a distribution, this expression involves a product of distributions, and so does not in general make sense. To force this expression to be meaningful, we must require that ${}^1h$ be a tensor field (not merely a distribution), and that its weak first derivative exist and be locally square integrable. That is, we must require that ${}^1h$ be a perturbation that is "to a metric that is regular to first order." This requirement on ${}^1h$, in turn, imposes on the original source T^{ab} essentially the conditions of the hypothesis of Theorem 1. Thus, it is at second order in perturbation theory—the order at which "the energy density of the field contributes to the field"—that our conditions on sources in general relativity arise.

One regards regular metrics and distributional sources as idealizations, i.e., as approximations to fields and sources that are more well behaved. It is of interest, therefore, to understand how these approximations operate. This is the subject of the remainder of this section.

We first introduce a kind of "distance" between fields. Let $\mu^{a\cdots c}{}_{b\cdots d}$ and $\nu^{a\cdots c}{}_{b\cdots d}$ be locally square integrable tensor fields. For $t_{a\cdots ce\cdots f}{}^{b\cdots dg\cdots h}$ a test field, set

$$\rho_t(\mu,\nu)=\int(\mu^{a\cdots c}{}_{b\cdots d}-\nu^{a\cdots c}{}_{b\cdots d})(\mu^{e\cdots f}{}_{g\cdots h}-\nu^{e\cdots f}{}_{g\cdots h})t_{a\cdots be\cdots f}{}^{c\cdots dg\cdots h}\ , \tag{19}$$

noting that the integral converges. We say that a sequence of locally square integrable fields, $\mu_1,\mu_2,\ldots$ is *locally Cauchy* if, for every $\epsilon>0$ and every test field t, there exists number N such that $|\rho_t(\mu_i,\mu_j)|\le\epsilon$ whenever $i,j\ge N$. We say that the μ_i converge to μ *locally in square integral* if, for every test field t, $\lim_{i\to\infty}\rho_t(\mu,\mu_i)=0$. Thus, for example, every locally Cauchy sequence converges locally in square integral by the standard proof in analysis of completeness of L^2.

The following theorem merely gives the sense in which a sequence of regular metrics must converge to another in order that their curvature distributions also converge.

Theorem 2. Let ${}_ig_{ab}$ $(i=1,2,\ldots)$ and g_{ab} be regular metrics. Let (i) the ${}_ig_{ab}$ and ${}_ig^{ab}$ be locally uniformly bounded,[17] and (ii) the ${}_ig_{ab}$, ${}_ig^{ab}$, and $\nabla_a{}_ig_{bc}$ (weak derivative) converge locally in square integral to g_{ab}, g^{ab}, and ∇_ag_{bc}, respectively. Then the corresponding curvature distributions, ${}_iR_{abc}{}^d$, converge to $R_{abc}{}^d$, in the following sense: For any test field $t^{abc}{}_d$,

$$\lim_{i\to\infty}({}_iR_{abc}{}^d*t^{abc}d)=R_{abc}{}^d*t^{abc}{}_d\ . \tag{20}$$

Proof. Fix the text field, and rewrite Eq. (9) in the form

$${}_iR_{abc}{}^d*t^{abc}{}_d=\int_M(\rho_{abc}{}^dt^{abc}{}_d-2\,{}_i\Gamma^d{}_{ma}\,{}_i\Gamma^m{}_{bc}t^{abc}{}_d+2\,{}_i\Gamma^d_{bc}\nabla_at^{abc}{}_d)\ , \tag{21}$$

where ${}_i\Gamma^d{}_{bc}$ is given in terms of ${}_ig_{ab}$ by Eq. (10). We see from this expression, using the Schwarz inequality on the last term on the right, that it suffices to show that the ${}_i\Gamma$ converge to Γ locally in square integral. It follows from the conditions given on the ${}_ig_{ab}$ and ${}_ig^{ab}$ that, in the support of t, these fields converge in measure[18] to g_{ab} and g^{ab}, respectively. Using this and condition (i) we conclude: the result of taking any smooth function of ${}_ig_{ab}$ and ${}_ig^{ab}$ is a locally uniformly bounded sequence that, in support of t converges in measure to the corresponding function of g_{ab} and g^{ab}. But any product of a uniformly bounded sequence of fields converging in measure with a sequence converging in square integral is a sequence converging in square integral. Equation (10), however, represents the ${}_i\Gamma$ and Γ as precisely such products. Hence, the ${}_i\Gamma$ converge locally in square integral in Γ. ■

We remark that the conclusion of the theorem—that the ${}_iR_{abc}{}^d$ converge to $R_{abc}{}^d$—holds also after taking any outer products with the metric or its inverse, and after any contractions. Hence, this conclusion holds as well for the Weyl tensor, Einstein tensor, etc. So, for example, if one wishes to approximate a regular metric with its distributional source by some smooth metric and source, then the approximation of metrics should be in the sense of Theorem 2.

Suppose that one is simply given a sequence of regular (or even smooth) metrics. How can one tell whether this sequence converges, in the sense of Theorem 2, to *some* regular metric?

Theorem 3. Let ${}_ig_{ab}$ $(i=1,2,\ldots)$ be a sequence of regular metrics. Let (i) the ${}_ig_{ab}$ and ${}_ig^{ab}$ be locally uniformly bounded and (ii) the sequences ${}_ig_{ab}$, ${}_ig^{ab}$, and $\nabla_a{}_ig_{bc}$ be locally Cauchy. Then there exists a regular metric g_{ab} to which this sequence converges in the sense of Theorem 2.

Proof. Denote by g_{ab} and g^{ab} the respective fields to which the Cauchy sequences ${}_ig_{ab}$ and ${}_ig^{ab}$ locally converge. Then each is locally square integrable by this construction, and each is locally bounded by condition (i). Replacing our sequences if necessary by subsequences converging almost everywhere, and again using condition (i), we have that g^{ab} is the inverse of g_{ab} almost everywhere. From the construction of g_{ab} and the fact that the $\nabla_{a\,i}g_{bc}$ are locally Cauchy it follows[19] that the weak derivative $\nabla_a g_{bc}$ exists and the $\nabla_{a\,i}g_{bc}$ converge to it in square integral. We have now verified all the conditions for g_{ab} to be regular, and for the ${}_ig_{ab}$ to converge to it in the sense of Theorem 2. ■

So, for example, let there be given a distributional source T^{ab}. Suppose we find a sequence of smooth sources that converges, as distributions, to T^{ab}, and a sequence of smooth metrics, ${}_ig_{ab}$, for these sources. When can we guarantee the existence of a regular metric with source T^{ab}? When the ${}_ig_{ab}$ satisfy the conditions of Theorem 3.

For both of these approximation theorems, the normal case will be that in which the approximating metrics are actually smooth. This leads to the question of which regular metrics can be suitably approximated by smooth metrics.

Theorem 4. Let g_{ab} be a continuous regular metric. Then there exists a sequence, ${}_ig_{ab}$ $(i=1,2,\ldots)$ of smooth metrics that converges to g_{ab} in the sense of Theorem 2.

Proof. Take the convolution of g_{ab} with a sequence of mollifiers,[20] and denote the resulting sequence of fields ${}_ig_{ab}$ $(i=1,2,\ldots)$. Then from this construction it follows[21] that each ${}_ig_{ab}$ is smooth, and the ${}_ig_{ab}$ and $\nabla_{a\,i}g_{bc}$ converge locally in square integral to g_{ab} and $\nabla_a g_{bc}$, respectively. But, since g_{ab} is continuous, these ${}_ig_{ab}$ actually converge locally uniformly to g_{ab}. It now follows immediately that the ${}_ig_{ab}$ are locally uniformly bounded, and that their inverses, ${}_ig^{ab}$, exist for all sufficiently large i, are themselves locally uniformly bounded, and converge locally in square integral to g^{ab}. We have thus verified all the conditions for convergence in the sense of Theorem 2. ■

Thus, those regular metrics that are continuous—probably the most interesting class physically—can be approximated in a suitable sense by smooth metrics. There do, of course, exist discontinuous regular metrics, such as that of Eq. (16). We do not know whether or not continuity can be omitted in Theorem 4. With this omission, the present proof fails badly.

To summarize, we have introduced a class of metrics whose curvature tensors make sense as distributions, and have discussed their properties. These three approximation theorems, in particular, deal with the sense in which metric fields must be close in order to guarantee that the corresponding source distributions will be close.

III. STRINGS

In this section, we consider in more detail the status of strings—of sources in general relativity concentrated on two-dimensional surfaces in space-time. The idealization to concentrated sources is generally useful only insofar as one can relate source to field, i.e., obtain a relationship analogous to that in electrostatics between the jump in the normal component of the electric field and the surface charge density. We shall therefore focus on the possible existence of such a relationship in the present case of string sources for gravitation. We shall find that it is difficult to introduce such sources, and relate them to the external field, in any physically realistic way.

Consider the metric

$$ds^2=-dt^2+dz^2+dr^2+r^2(\cos^2\gamma)d\phi^2\ , \tag{22}$$

where t and z range from $-\infty$ to $+\infty$, r range from 0 to $+\infty$, and ϕ range from 0 to 2π (identified); and where $\gamma\in[0,\pi/2)$ is a constant. This is Minkowski space-time with a ϕ-angular deficit $2\pi(1-\cos\gamma)$. As we have seen in Sec. I, this metric results as the limit of the family (1) and (2) of static, cylindrically symmetric metrics, each with source confined to a neighborhood of the axis. We interpreted these sources to give, in the limit, a source concentrated on the axis, with mass per unit length the same as the angular deficit: $2\pi(1-\cos\gamma)$.

By contrast, we proved in Sec. II a theorem to the effect that no regular metric in general relativity can have a source confined to a two-dimensional surface in space-time. How is this example, which seems to have a source concentrated on a two-dimensional surface, and this theorem, which seems to deny the possibility of such sources, to be reconciled? At one level, the reconciliation is quite simple: We merely check to see whether the metric (22) is regular.[22] This metric is indeed locally bounded, with locally bounded inverse; and, further, its weak first derivative does indeed exist. However, this derivative turns out not to be locally square integrable, for it has behavior r^{-1} on approaching the axis, while $\int_\epsilon (r^{-1})^2 r\,dr$ diverges logarithmically as $\epsilon\to 0$. Thus, the metric (22) is not regular: This example is simply not one to which the general theorem applies.

Lack of regularity of the metric (22) means only that we cannot apply to it the general framework of Sec. II to assign a distributional source and to relate that source to the external field. Yet there may well be some other framework—one more specially adapted to this type of problem—that *is* applicable. There may well be some other way to extract physics from a metric such as (22).

We might, for example, seek some procedure by which one can assign to the metric (22) a number μ, its mass per unit length, and can relate that number to the external field, characterized in this example by the angular deficit $2\pi(1-\cos\gamma)$. One promising procedure[23] is that suggested by the example of Eqs. (1) and (2): take the limit of a family of well-behaved sources. In more detail, introduce, for each $r_0>0$, a metric that agrees with (22) for $r\geq r_0$ and is smooth for $r<r_0$. This metric represents a choice of source, confined to a neighborhood of the axis, for (22) as external field. Next, compute from Einstein's equation the stress-energy of this metric and integrate over cross sections of the string. Finally, take the limit as $r_0\to 0$. This limit we take as our source for the external metric (22).

One particular instance[24] of this procedure is of course that of Eqs. (1) and (2) themselves, with constant mass density in the interior. In this case the limiting mass per unit length, μ, turns out to be precisely the angular deficit, $2\pi(1-\cos\gamma)$.

Suppose now that we apply the above procedure—introduce a family of well-behaved sources, keeping the external metric fixed, and take the limit—but now using some different family of internal metrics. Do we then still obtain, for the mass per unit length in the limit, $\mu=2\pi(1-\cos\gamma)$? In general, we do not, as the following example shows. Let, for each $l>0$, the metric be $\exp[2\lambda f(r/l)]g_{ab}$, where g_{ab} is the metric of Eqs. (1) and (2), λ is a constant, and f is any fixed smooth nonnegative function vanishing for its argument outside the closed interval $[\frac{1}{2},1]$. This metric agrees with that of Eq. (22) for $r\geq l$, and so we do have an interior metric for (22). For the mass per unit length—defined as the integral of the mass density in the t direction over the two-surfaces of constant t,z—we obtain[25]

$$\mu_l=2\pi(1-\cos\gamma)-2\pi\lambda^2\int_0^1\frac{\sin\gamma x}{x}[f'(x)]^2dx \ . \quad (23)$$

Now take the limit $l\rightarrow 0$. Fixing λ in this limit, we obtain a "source" for the external metric (22) with limiting mass per unit length μ, strictly less than the angular deficit. Alternatively, one could vary λ in the limit in such a way that the μ_l approach no limiting value at all.

This example shows, then, that our procedure—taking the limit of a family of well-behaved sources—does not in general yield a unique relationship between mass per unit length and deficit angle. So, if we are to have any hope of recovering by our procedure the "normal" relationship, $\mu=2\pi(1-\cos\gamma)$, then we shall have to impose some conditions on our allowed internal metrics.

A candidate for such a condition is suggested by the case of a shell—a source in general relativity concentrated on a three-dimensional surface S in space-time. The external metric for a shell is required to be continuous across S, but may have there a discontinuity in its first derivative. One can in this case formulate a procedure, similar to that above for a string, involving introducing a well-behaved source in a neighborhood of S and taking the limit. But one can also in this case construct examples, similar to that above, in which a variety of values of the mass per unit area of S arise in the limit for given external metric. In this case of a shell, however, there is a simple condition that will rule out all such examples. We merely demand that the first derivative of the interior metrics be locally bounded, uniformly as the thickness of the shell of matter approaches zero. This condition guarantees that the limiting surface stress-energy density on S will be given by the discontinuity in the first derivative of the metric across S, no matter what is otherwise done with the sources in the limit. This result suggests, then, that one impose a similar condition in the string case. Unfortunately, this condition is not appropriate for the external metric (22), because of the following: For $\gamma\neq 0$, there exists *no* choice of internal metrics having first derivatives locally bounded, uniformly as the thickness of the string approaches zero. This follows from the fact that the first derivative of the metric (22) itself is locally unbounded, near the axis. We remark that the fact that the above condition suffices in the case of a shell to yield a unique source in the limit is a special case of Theorem 2 of Sec. II. That the same condition is not available for the string is a special case of Theorems 3 and 1. In short, one can trace these difficulties in "building" concentrated sources for strings to the fact that our general mathematical framework excludes such sources.

Thus, the condition that yields in the case of a shell a unique relationship between source and field fails in the case of a string. Is there, then, any condition on the interior for a string that will force in the limit a unique relationship between the source and the external field (22)? One might think of imposing an energy condition on the matter. This will not suffice. Indeed, our example above satisfies the strong energy condition provided only that $|\lambda|$ is chosen sufficiently small (where this bound depends on the fixed function f, but is independent of l). One might think of imposing the condition that the symmetries of the external field—the static character and cylindrical symmetry—apply also to the interior. This will not suffice either, for our example above manifests all of these symmetries everywhere. (Note that this feature distinguishes the present example in general relativity from the Newtonian example of Sec. I. The Newtonian example relies in an essential way on a source violating the cylindrical symmetry.)

We thus find no general conditions on the interior matter that will allow us to regard the external metric (22) as having for its source a line density of mass per unit length $\mu=2\pi(1-\cos\gamma)$. If we wish to recover such a relationship, we shall apparently have to impose very severe additional conditions. We turn, therefore, to a condition that is both specialized and severe: the imposition of an equation of state on the matter. Let us demand that the stress-energy of the matter be given by

$$T_{ab}=\rho t_a t_b+p z_a z_b+\sigma(g_{ab}+t_a t_b-z_a z_b) \ , \quad (24)$$

where t_a is unit timelike and z_a unit spacelike, and these are orthogonal. Thus, the mass density is ρ and the principal pressures are p, σ, and σ—so this fluid is not in general isotropic. Expressed in terms of these quantities, the strong energy condition becomes that $\rho\geq 0$, $\rho+p\geq 0$, and $\rho+\sigma\geq 0$. To specify equations of state for such a fluid, one could give, e.g., the two pressures as functions of the density: $p(\rho)$ and $\sigma(\rho)$. For example, the equations of state for the material of Eqs. (1) and (2) are

$$p(\rho)=-\rho, \quad \sigma(\rho)=0 \ . \quad (25)$$

The idea, then, would be the following. First fix some equations of state. Then construct from such matter interior solutions for (22), aligning t^a along the static Killing field and z^a along the Killing field of translation up the cylinder. Finally, take the limit. Since we have now given up our freedom to vary the choice of matter in the limit, one might hope that now there will emerge in the limit equality between mass per unit length and angular deficit.

Unfortunately, this program is not as simple as suggested above. Fix equations of state. To specify a static, cylindrically symmetric solution of Einstein's equation with source (24), one must fix one additional parameter, say, the value of the density ρ on the axis. One obtains the solution by integrating outward from the axis until one reaches the boundary of the body, at which the exterior metric is attached. Now, the general[26] static, cylindrically symmetric vacuum solution of Einstein's equation can be written

$$ds^2 = -r^{2s}dt^2 + r^{-2s(1-s)}(dr^2+dz^2) + r^{2(1-s)}\cos^2\gamma \, d\phi^2 , \tag{26}$$

where t and z range from $-\infty$ to $+\infty$, r from 0 to $+\infty$, and ϕ from 0 to 2π (identified); and where s and $\gamma \in [0, \pi/2)$ are constants. This space-time is flat if and only if $s=0$ [in which case it reduces to the metric (22)] or $s=1$. In attaching our interior solution to the exterior metric (26), we use continuity across the boundary of the value of the metric and of its first derivative to determine the parameters a and γ in (26). In general, then, we shall obtain s nonzero; i.e., we shall obtain an exterior metric that is not even flat—and is certainly not Minkowski space-time with an angular deficit. Indeed, it is not difficult to show, imposing only the strong energy condition on the stress-energy (24), that $s \in [0,2]$ quite generally, with $s=0$ or 2 if and only if the first equation of state is precisely $p(\rho) = -\rho$. Thus, it is only under this further restriction on the equations of state that we obtain any flat exterior metric at all. But even for equations of state that include $p(\rho) = -\rho$, there is a further difficulty. The entire space-time—including the value of the remaining parameter γ, the size l of the source region, and the mass per unit length μ_l—is determined by a single number, which we have taken to be the value of ρ on the axis. We do not have the freedom to fix γ, i.e., fix the exterior metric, and then consider a succession of interior solutions with $l \to 0$. In general, we shall have to select different material[27] for each l value. We shall have to introduce a family of equations of state, and then impose some complicated and delicate conditions on the limiting behavior of the equations of state in this family in order to obtain in the limit a source for the external metric (22).

Thus, even the use of matter with an equation of state does not lead directly to a limiting source for the external metric (22), unless one chooses a very special equation of state. Even if we do use such special matter—for example, that of Eq. (25)–a further difficulty remains. So far, we have been dealing exclusively with space-times that are static and cylindrically symmetric. Yet one would not expect this high degree of symmetry to obtain in physically realistic situations. One may, e.g., have a "curved" string, or there may be incoming gravitational radiation or some object external to the string.

Let there be such an object or radiation. In its presence, construct, using fluid with equation of state (25), a tubular body of characteristic radius l. What happens in the limit $l \to 0$? There are really two questions here. Does a limiting space-time exist? And, if it does exist, then can one find and relate in a simple way quantities, such as the limiting mass per unit length, characteristic of the source and quantities, such as the limiting angular deficit, characteristic of the external field? We discuss these two questions in turn.

A complete answer to the question of whether there exists a limiting space-time appears to be difficult. There is one, rather weak, piece of evidence suggesting that such limiting solutions do exist: One can, in some situations, argue for a certain candidate for the limiting solution. Consider an axisymmetric space-time (neither cylindrically symmetric nor static) that is smooth at its axis, includes an axisymmetric matter distribution vanishing in a neighborhood of the axis, and may include some axisymmetric gravitational radiation. One expects on physical grounds that a variety of such solutions will exist. Introduce into such a space-time a ϕ angular deficit, just as was done for the metric (22). The result—an exact solution of Einstein's equation with a "conical singularity" on the axis—is our candidate for the limiting space-time in the case in which the gravitational radiation and the external object are both axisymmetric.[28] Of course, the mere fact that such solutions exist does not mean that these are the ones that arise in our limit. It may be possible to obtain further evidence on this issue by "linearizing in the external influences." Write down the linearized Einstein equation off Eqs. (1) and (2) as background. Introduce fixed boundary conditions at infinity, corresponding to given incoming radiation, and a fixed external source, corresponding to a given external object. The issue is whether one can find a solution to this linearized system, for each l, such that these solutions have a well-behaved limit as $l \to 0$. Even this linearized version of the question of the existence of a limiting space-time appears difficult.

Let us now suppose for a moment that we have somehow obtained, in the limit $l \to 0$, our solution of Einstein's equation. We may then turn to the second issue: whether we may describe this situation by some simple relationship between quantities characteristic of the source and of the field. We were dealing previously with just such a relationship: namely, equality between the mass per unit length and angular deficit. But these particular quantities as they stand do not make sense in the absence of symmetries. Thus, we defined the mass per unit length as the integral of a component, defined via the static Killing field, of the stress-energy over a two-surface in space-time orthogonal to that Killing field and the one of translation along the axis. We defined the angular deficit using the norm of the rotational Killing field. If we wish, lacking symmetries, to relate source and field in a similar way, then we shall have to revise these definitions.

One might define angular deficit, for example, by constructing geometrically certain "circles" about the string, and comparing proper radius and proper circumference in the limit as the circle collapses down on the string. Or, alternatively, one might define and compute the asymptotic deflection of a light ray just grazing the string. Similarly, one might define mass per unit length by integrating, for $l>0$, some specified "time component" of the stress-energy over some geometrically defined two-surfaces cutting the tube of matter, and then taking the limit $l \to 0$. It is by no means certain that there exist any general definitions whatever along these lines. Even if given such definitions, there may result no simple relationship between the source and field so defined.

To summarize, even if one specifies equations of state for one's matter, then, lacking symmetries, it is not clear whether there exists any limiting space-time for a concentrated string source. Even if such a space-time does exist, it is not clear what to do with it.

We may illustrate some of these issues with the special case in which one discards only the cylindrical symmetry, retaining the static character. Denote by γ_{ab} the three-dimensional spatial metric, and by V the norm of the timelike Killing field. We again take as our source $T_{ab}=\rho(t_a t_b - z_a z_b)$, i.e., the form (24) with equations of state (25). Then, aligning t^a along the timelike Killing field, Einstein's equation becomes the following system of three-dimensional equations on the fields V (>0), z^a (unit), ρ, and γ_{ab}:

$$D^a D_a V = 0 , \tag{27}$$

$$z^m \nabla_m z^a = \frac{1}{V}(\gamma^{am} - z^a z^m) D_m V , \tag{28}$$

$$D_a(\rho z^a) = 0 , \tag{29}$$

$$\mathcal{R}_{ab} = \frac{1}{V} D_a D_b V + \rho(\gamma_{ab} - z_a z_b) , \tag{30}$$

where $\mathcal{R}_{ab}$ denotes the Ricci tensor of γ_{ab}, and D_a its derivative operator.

Now fix some additional sources on the right in Eqs. (27)–(30), and solve[29] the resulting system for the case of a tube of fluid of characteristic size l. Does this family of solutions have a limit as $l \to 0$? This question, even here in the static case, appears difficult. Its linearized version is the following. Take as the background the family, (1) and (2), of solutions of Eqs. (27)–(30). For each l, find a solution of the linearized versions of Eqs. (27)–(30) off this background, including a fixed additional source on the right-hand sides. Does this family of linearized solutions have a limit as $l \to 0$? Even this linearized version appears difficult. However, it does have the same character as the following question. For each l, solve the equation $D^2\phi = \mu$ in the spatial metric of Eqs. (1) and (2), where one fixes the source μ, vanishing in a neighborhood of the axis, and fixes the asymptotic boundary conditions. Does the resulting family of solutions have a limit as $l \to 0$? Surely, this question can be settled.

We next turn to the issue of finding and relating quantities characteristic of source and field. In this static case there is, remarkably enough, a simple and natural notion of "mass per unit length." Consider

$$\mu = \int_S \rho z_a dS^a , \tag{31}$$

where the integral is over any two-surface S cutting the static tube of matter. By Eq. (29) this integral is independent of the choice of S. One can think of this μ as a "minimum value" for the mass per unit length, in the sense that

$$\mu \le \int_S \rho \hat{\mathbf{n}}_a dS^a \tag{32}$$

for any two-surface S cutting the tube, where $\hat{\mathbf{n}}_a$ is the unit normal to S. It is curious that this mass per unit length does not vary along the tube. Unfortunately, we do not know whether or not this μ must have a limit as $l \to 0$. It may be possible to argue that the limit does exist by using independence of the surface S to choose S in the limit far from the external object. But in any case there appears to be no similar, simple measure of angular deficit to compare with μ. Perhaps some light would be shed on these issues by studying their linearized versions.

IV. COSMIC STRINGS

We now consider applications to cosmic strings.

Work in this area may be divided into three broad categories, according to how the strings are treated. The first is that in which the gravitational field of the string itself is ignored; i.e., the string is represented by a test stress-energy distribution. An example[30] is the calculation of the evolution of a network of test strings on a Robertson-Walker background space-time. The second is that in which one deals with strings of finite size, with the internal structure determined by a given equation of state or even by an explicit field theory. Examples include the construction of straight strings from certain gauge and scalar fields,[31] and calculations[32] of the effects of loops of such strings on the growth of matter fluctuations. Finally, the third category is that in which the string is idealized as a concentrated source, and that source is then inserted, directly or indirectly, into Einstein's equation. Examples include calculations of the gravitational effects of thin strings on external matter[33] or radiation, and of the gravitational radiation from oscillating thin strings.[34] We have been concerned in this paper with the gravitational fields of concentrated sources. Thus, the present considerations are not applicable to work in the first category, in which gravitational effects are ignored, nor to work in the second, in which the sources are not "concentrated." But this paper is applicable to work in the third category.

We consider first the work involving the gravitational effects of thin strings on external matter or radiation. Here, one observes directly these gravitational effects, then attempts to infer from these effects features of the external gravitational field, and finally attempts to infer from these features properties of the string source itself. Consider, for example, the case[35] of a cosmic string lying on the line of sight between an observer and a distant quasar. Because of the gravitational bending of light by the string, one expects to see a double image of the quasar. From the observed angle θ between the two quasar images one infers,[36] assuming external metric (22), that this external metric has angular deficit 2θ. Then, assuming that the source is that of Eqs. (1) and (2), one infers that the mass per unit length of the string is also 2θ.

But we have seen in Sec. III that, for a concentrated string source, the relation between source and field is a delicate one. Suppose, for example, that the string were made of matter whose equations of state are not exactly those of Eq. (25). Then the external metric would not in general be that of Eq. (22), but rather of Eq. (26). The parameter s in this metric would be of the order of $\mu(\delta\rho/\rho)$, where $\delta\rho$ is a typical change in the density of the string matter resulting, via the Einstein-hydrodynamic equations, from our new equations of state. We must now relate the observed angle θ between quasar images to the parameters of this new external metric. We obtain,[37] to first order in s,

$$\theta=\pi(1-\cos\gamma)-2\pi s\cos\gamma \ . \qquad (33)$$

We conclude that, in the case of interest ($\cos\gamma\sim 1$), the mass per unit length of the string inferred from the angle between the images will be subject to a fractional error of the order of $\delta\rho/\rho$. This error could, depending on the new equations of state, be substantial. As a second example, suppose that an external object were introduced into the vicinity of the string. Then this object would of course have a direct effect through its own bending of light. It would thereby change the observed angle between quasar images by the order of ma/r^2, where m is some characteristic mass of the object, a a typical separation of the two light rays as they pass the object, and r a typical distance from object to light rays. But, as we have seen in Sec. III, there will also be an indirect effect arising from the influence of this object on the string itself: The presence of an external object will break the normal relationship between mass per unit length and external gravitational field of the string. Given the size of the string, one could, at least in principle, compute this effect from Eqs. (27)–(30). But, as we remarked in Sec. III, how the magnitude of this effect behaves in the limit of a concentrated source is presently unknown. It could be substantial. We conclude that the mass per unit length of the string inferred from the angle between quasar images will be subject to an error of at least the order of ma/r^2, and possibility far larger. Similar remarks should apply to other methods for determining the mass per unit length of a string from its gravitational effects, e.g., by the effects of the string on nearby free particles, or on the background microwave radiation.[38]

We consider next the work on gravitational radiation from oscillating thin strings. Consider a loop of cosmic string that undergoes oscillations, and as a result radiates gravitationally. A particularly sensitive probe for such gravitational radiation is provided by the pulsar periods,[39] since they can be measured so accurately. The amount of gravitational radiation emitted is estimated by idealizing the string to a distributional source and computing its linearized gravitational field.[34] This calculation predicts effects from cosmic strings which, for the case of the millisecond pulsar, are close to what currently would be observed.

How reliable are these estimates of the emission of gravitational radiation? As we have noted in Sec. II, the linearized Einstein equation makes sense with any distributional source, while the full Einstein equation does not seem to make sense for a source concentrated on a two-dimensional surface in space-time. That is, the first-order metric perturbation, in this case is not the linear approximation, in any meaningful sense, to some exact metric. Indeed, this problem is seen already in second order in perturbation theory. The source for the second-order metric perturbation ${}^2h$ is given by an expression of the form $(\nabla {}^1h)^2$, where ${}^1h$ is the first-order metric perturbation. But this source is not in general locally integrable, and so if we attempt to determine the second-order perturbation, say, using a Green's function, then the integral will in general diverge. We now estimate the error that results from using first-order perturbation theory and a distributional source to compute the gravitational radiation from an oscillating string. For the case of a string, the integral of the source for ${}^2h$ diverges logarithmically. Were ${}^2h$ to be computed, not for a distributional string, but rather for one of finite size l, then this divergent integral would be cut off at distances of order l from the string. We thus obtain, for the gravitational radiation of a finite string, a second-order perturbation that is the order of $\mu\ln(R/l)$ times the first-order perturbation, where μ is the mass per unit length of the string and R is a typical distance from the string at which the radiation is observed. While this expression does become infinite as $l\to 0$, it does so slowly, reflecting the fact that the metric for a concentrated string is "nearly regular." Inserting typical values for cosmic strings, $l\sim 10^{-40}$ sec, $R\sim 10^{18}$ sec, and $\mu\sim 10^{-6}$, we obtain for this number about 10^{-4}. Thus, for calculations of gravitational radiation from cosmic strings, the idealization to a concentrated source does not appear to produce significant errors.

V. CONCLUSION

We have been concerned in this paper with the status of concentrated sources in general relativity. In Sec. II, we introduced the notion of a regular metric—a metric whose curvature tensor, and therefore whose stress-energy source, makes sense as a distribution. We showed, in particular, that string sources—those concentrated on two-dimensional surfaces in space-time—are not permitted by regularity. In Sec. III, we examined this conclusion in more detail. In turns out in the case of a string to be difficult to construct a concentrated source as a limit of well-behaved sources. Finally, in Sec. IV, we discussed applications to work on cosmic strings.

We have seen in Sec. II that the Bianchi identity does not in general make sense for a regular metric. Thus, for a regular metric, one cannot in general obtain conservation of stress-energy via Einstein's equation. It seems strange that so fundamental a property of matter should be lost in this way. Is there some way to recover *some* remnant of conservation of stress-energy? A somewhat analogous issue arises in electrostatics. The force on a smooth charge density in electrostatics is the integral of the charge density times the electric field. For the case of the charge density a distribution, this formula generalizes to the result of that distribution acting on the electric field. Consider, then, a surface-charge distribution of strength σ: What is the force per unit area acting on this distribution? Formally, this question cannot be answered, for the charge distribution in this example acts only on continuous fields, while the electric field is discontinuous across the surface. Yet we know in this case that there *is* an answer to our question: The force per unit area on the charge distribution is given by the product of σ and the average of the electric fields on the two sides of the surface. Perhaps there is available something similar for certain regular metrics.

Since the stress-energy of a regular metric emerges as a distribution, the matter fields contributing to this stress-energy no longer need be smooth tensor fields. How well behaved must these matter fields be? This issue must be dealt with separately for each type of matter, the general rule being that the matter fields need only be sufficiently well behaved that the equations on these fields and Einstein's equation both make sense. Thus, for an electromagnetic field F_{ab} we have Maxwell's equations,

$$\nabla_{[a}F_{bc]}=0, \quad \nabla_{[a}(\epsilon_{bc]}{}^{de}F_{de})=0 , \tag{34}$$

and Einstein's equation,

$$R_{ab}-\tfrac{1}{2}Rg_{ab}=F_a{}^mF_{bm}-\tfrac{1}{4}g_{ab}F^{mn}F_{mn} . \tag{35}$$

Let the metric of space-time be regular. Then in order that all of Eqs. (34) and (35) make sense it is necessary that F_{ab} be a locally square-integrable tensor field. Indeed, the first equation in (34) makes sense for any distribution F_{ab}; the second, since $\epsilon_{ab}{}^{cd}$ is locally bounded, provided F_{ab} is a locally integrable tensor field. But the right-hand side of Eq. (35) makes sense as a distribution only for F_{ab} locally square integrable. It is curious that, in the case of electromagnetism, one must obtain for the stress-energy a distribution of a rather mild sort: a locally integrable tensor field. Does the resulting Einstein-Maxwell system, Eqs. (34) and (35), have an initial-value formulation? The case of a fluid source, in which the equations analogous to (34) are nonlinear, is more complicated. It is tempting to retain the four-velocity of the fluid as a smooth vector field, allowing only the density and pressure to be badly behaved. But such conditions will, presumably, conflict with the hydrodynamic equations. Is there any set of conditions on the fluid variables with a regular metric that both (i) permit a reasonable number of solutions (e.g., via an initial-value formulation) and (ii) exploit the full richness available with regular metrics?

Somewhat related to these issues is the status of the geodesics of a regular metric. the geodesic equation,

$$\xi^m\nabla_m\xi^a=\Gamma^a{}_{mn}\xi^m\xi^n , \tag{36}$$

regarded as an ordinary differential equation for a curve, will not in general have solutions. Indeed, the existence of solutions would require that $\Gamma^a{}_{mn}$ satisfy a local Lipschitz condition—something far stronger than the local square integrability that follows from regularity of the metric. Thus, a regular metric does not in general have geodesics. Yet, in some physical applications one *does* introduce curves regarded as representing the motions of free particles. For a shell of matter, for example, a "geodesic" is made to pass through the shell by matching its tangent vector continuously across the shell. Is there some larger framework into which such examples fit? One could, alternatively, introduce a "geodetic vector field," i.e., a field ξ^a satisfying Eq. (36). In order for this equation to make sense, it suffices, e.g., that ξ^a be locally bounded, and that its weak first derivative exist. (These conditions, incidentally, are too weak to ensure that integral curves of the field ξ^a exist.) Perhaps such geodetic vector fields can, in some way, be used as substitutes for geodesics.

In Regge calculus,[40] one replaces the smooth metric of space-time by a simplicial approximation, i.e., by a metric that is flat except at certain two-surfaces, on which the curvature is concentrated. Such a metric is certainly not regular in our sense, and so we cannot regard such a metric as an approximation, in the sense of Theorem 2, of a regular metric. In fact, we cannot even define, via Eqs. (9) and (10), the curvature distribution of such a metric. In what sense, then does such a "simplicial approximation" actually approximate a smooth metric? Is there any way to guarantee that the "curvature distributions" arising from the simplicial approximations converge as distributions to the curvature of some smooth metric? Can one prove that computer simulations based on Regge calculus actually reflect what would happen in full general relativity? One might think of proceeding by trying to generalize to higher dimensions Eq. (12), which defines the curvature combination $R\epsilon_{ab}$ in two dimensions for certain nonregular metrics. But that formula appears to be special to this particular curvature combination.

Is continuity actually necessary in Theorem 4 of Sec. II? Consider, as an example, the metric on $\mathbb{R}^4$ given by

$$ds^2=dx^2+dy^2+dz^2+dw^2 -4(x\,dx+y\,dy+z\,dz+w\,dw)^2/(x^2+y^2+z^2+w^2) . \tag{37}$$

This metric is smooth everywhere except at the origin, and is regular. Can it be approximated, in the sense of Theorem 4, by smooth metrics? It is not so easy. For example, there exists no smooth (or even continuous) Lorentz metric that agrees with (37) outside a bounded neighborhood of the origin, for the light cones in (37) "point toward the origin," resulting in a topological obstruction to extending them continuously inward from outside such a neighborhood. In particular, the proof of Theorem 4, applied to this example, will fail, for the ${}_ig_{ab}$ constructed in that proof will never be invertible near the origin. Yet, in spite of all this, there *do* exist smooth metrics that approximate (37) in the sense of Theorem 4. Denote by L the "wire from the origin to infinity" given by $x=y=z=0$, $w\geq 0$. Choose a sequence of smooth metrics, each coinciding with the metric (37) outside a neighborhood of L, but whose light cones within this neighborhood "turn so as to point outward along L." This can also be done in such a way to obtain convergence in the required sense. Can such "wires" be woven into a proof that Theorem 4 holds withouts its continuity condition?

What if Theorem 4, without the continuity condition, should fail? The counterexample would consist of a regular metric—necessarily discontinuous—that cannot be approximated in the required sense by smooth metrics. Then one would have at least two options. On the one hand, one could argue that general relativity is fundamentally a theory of smooth metrics, and so no metric should be considered unless it can be suitably approximated by a smooth metric. Then one would include continuity as an additional condition in the definition of a regular metric. On the other hand, one might "generalize" relativity by taking such metrics seriously. Do they suggest new physics?

The definition we have given for a regular metric was, after all, merely what seems to arise naturally from examining Eq. (9). Is there some theorem to the effect that this is the widest, physically reasonable, class of metrics for general relativity? It might, for example, read "Any class of metrics such that . . . is included in the class of regular metrics." Here, ". . ." would require essentially the existence of a distribution with properties (symmetries, dependence on the metric, locality, etc.) reminiscent of a curvature distribution. Even a partial result lending legitimacy to the definition of regularity would be interesting.

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[1]For other discussions of the use of distributional sources in general relativity, see, for example, A. Taub, J. Math. Phys. **21**, 1423 (1980); P. E. Parker, *ibid.* **20**, 1423 (1981); W. Israel, Phys. Rev. D **15**, 935 (1977); J. A. G. Vickers, report (unpublished).

[2]See, for example, J. Goldberg, in *Gravitation, An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962).

[3]W. Israel, Nuovo Cimento **44B**, 1 (1966).

[4]See, for example, K. Kahn and R. Penrose, Nature (London) **229**, 185 (1971); S. Chandrasekhar and V. Ferrari, Proc. R. Soc. London **A396**, 55 (1984).

[5]R. Gott, Astrophys. J. **282**, 4221 (1985).

[6]W. Hiscock, Phys. Rev. D **31**, 3288 (1985).

[7]We use units in which $8\pi G=1$.

[8]We wish to thank Chris Habisohn for some suggestions regarding this example.

[9]See, for example, R. A. Adams, *Sobolev Spaces* (Academic, New York, 1975), Chap. 1.

[10]Alternatively, one could have defined the test fields as tensor fields antisymmetric in four covariant indices. Thus, instead of tensor density $t^{a\cdots c}{}_{b\cdots d}$, one would introduce tensor field $t^{a\cdots c}{}_{b\cdots dmnpq}=t^{a\cdots c}{}_{b\cdots d[mnpq]}$.

[11]Even without any choice of a derivative operator, both the exterior derivative of a distributional form and the Lie derivative of a distribution by a smooth vector field are meaningful. The defining equations are analogous to Eq. (6).

[12]It is not necessary to impose on M the global condition that such a field exists. Instead, one can use Eq. (12) in overlapping patches.

[13]More generally, for the manifold M n-dimensional, this formula becomes $s<(n-2)/2-d/2$.

[14]Whether or not a given metric is regular depends in general on the differentiable structure imposed on the underlying manifold. For Eq. (16), we use the differentiable structure arising from the given chart. For the examples of the Schwarzschild space-time and Minkowski space-time with angular deficit, one can use any differentiable structure, for, by Theorem 1, there exists no differentiable structure that will render the metric regular.

[15]This means that the distribution α, applied to any test field having support not intersecting S, yields zero.

[16]More generally, for the manifold M n-dimensional, this conclusion becomes that $d=n-1$.

[17]This means that, given any test fields t^{ab} and u_{ab}, there exists a test field w, independent of i, that bounds all of the scalar densities $t^{ab}{}_i g_{ab}$ and $u_{ab\,i}g^{ab}$ ($i=1,2,\ldots$).

[18]See, for example, Adams, *Sobolev Spaces* (Ref. 9), Chap. 2.

[19]See, for example, Adams, *Solbolev Spaces* (Ref. 9), lemma 3.16. All that is needed here is that the space of square-integrable functions with square-integrable weak derivative, in the norm given by the sum of these integrals, is complete.

[20]See, for example, Adams, *Sobolev Spaces* (Ref. 9), Chap. 2.

[21]See, for example, Adams, *Sobolev Spaces* (Ref. 9), lemma 3.15.

[22]Whether or not a metric is regular depends on the underlying differentiable structure. Here, we take the differentiable structure as that in which $t,z,r\cos\phi$, and $r\sin\phi$ form a smooth chart. In fact, *no* differentiable structure will cause (22) to be a regular metric, as follows from Theorem 1.

[23]See, for example, Taub and Israel (Ref. 1). For such a procedure applied to point particles, see R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **120**, 321 (1960).

[24]That the metric (1) and (2) does have as its exterior the metric of Eq. (22) is seen by replacing r in Eq. (1) by $(r+l-l/\gamma\tan\gamma)$.

[25]The integrated pressures in directions orthogonal to the t and z directions are not in general zero in this example, but rather are given by

$$3\pi\lambda^2\int_0^1[(\sin\gamma x)/\gamma][(f')^2-f''/3\lambda]dx\ .$$

The pressure in the z direction is just minus the density in the t direction. In fact, this last property follows quite generally for the exterior metric (22) from the strong energy condition, the static character, and cylindrical symmetry. However, this property can be violated if any of these are dropped, e.g., if one discards cylindrical symmetry by allowing the function $f(r/l)$ in the metric to depend also on the coordinate z.

[26]For the most general metric, one must in fact allow the factor "$\cos\gamma$" in Eq. (26) to exceed one. We have written the metric in this form to simplify the comparison with Eq. (22).

[27]The special equations of state (25) are degenerate in this

respect, for they allow a single sample of material to be used for all l values.

[28]There also exist, by a similar construction, examples that are not axisymmetric. Begin with a metric that has, instead of the full axial symmetry, merely an n-fold rotational symmetry. Then introduce angular deficit $2\pi m/n$, with $m=1,2,\ldots,(n-1)$.

[29]The strategy for "solving" this system is as follows. First solve Eq. (27) for V, then Eq. (28) for z^a, then Eq. (29) for ρ, and finally Eq. (30) for γ_{ab}. This is an effective procedure, in the sense that at each stage one only requires fields acquired at earlier stages, except that γ_{ab} is required throughout. So, this is actually a fully coupled system.

[30]See, for example, R. Brandenberger and N. Turok, Report No. NSF-ITP-85-82 (unpublished).

[31]See, for example, D. Garfinkle, Phys. Rev. D **32**, 1323 (1985).

[32]See, for example, J. Traschen, N. Turok, and R. Brandenberger, Phys. Rev. D **34**, 919 (1986).

[33]See, for example, D. Garfinkle and C. Will, Phys. Rev. D **35**, 1124 (1987); T. Vachaspati, Bartol Research Foundation Report No. BA-86-5 (unpublished).

[34]See, for example, T. Kibble and N. Turok, Phys. Lett. **116B**, 141 (1982); N. Turok, Nucl. Phys. **B242**, 520 (1984).

[35]See Ref. 5, and A. Vilenkin, Astrophys. J. **282**, L51 (1984).

[36]This holds, of course, only for source and observer equally spaced from the string. For one closer than the other, the angular deficit will differ substantially from 2θ.

[37]This formula is for the case of source and observer equally spaced from the string. Note that the right-hand side of Eq. (33) is independent of distances from the string. This property fails at higher order in s.

[38]See, for example, N. Kaiser and A. Stebbins, Nature (London) **310**, 391 (1984).

[39]See, for example, B. Bertotti, B. J. Carr, and M. J. Rees, Mon. Not. R. Astron. Soc. **203**, 945 (1985).

[40]See T. Regge, Nuovo Cimento **19**, 558 (1961); G. Feinberg, R. Friedberg, T. D. Lee, and H. C. Ren, Nucl. Phys. **B245**, 343 (1984).