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Virasoro conditions, vertex operators, and string dynamics in curved space

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We present the perturbatively renormalized expression of a scalar vertex operator for strings in a background metric and dilaton field. The equations of motion for the background fields and the wave equation for the vertex function emerge upon imposing Virasoro conditions on the vertex operator.

In the last few years there has been great interest in two-dimensional (2D) conformally invariant field theory (CIFT). These studies have shed light on the classification of critical phenomena for systems of interest in condensed-matter physics.[1] The subject of 2D CIFT had its origins in the dual-string model.[2] Not surprisingly there is much interest in this area from the string viewpoint.

In string theory one of the central problems is that of string compactification, or equivalently to find the ground state. This is inherently a nonperturbative problem, and it would be desirable to set up a Hartree-type framework to address this question. In its absence it has become fruitful to explore consistent propagation of strings coupled to background fields which are condensates of its massless modes. For the bosonic string these are the metric, the dilaton, and the antisymmetric Kalb-Ramond field. Conformal invariance is the guiding principle that constrains these backgrounds.[3] More explicitly, one is in some sense using the representation theory of the Virasoro or conformal algebra to make statements about the background fields. In this paper we pursue this point of view and present an explicit perturbative construction of a vertex operator which creates string states that carry a representation of the Virasoro algebra. This happens when the background fields satisfy the classical equations of motion, and the vertex function, which is the wave function of the particle emitted from the string, satisfies a linear differential equation. As is well known, the classical equations of motion follow from an action principle. The linear wave equation is a consequence of a second-order variation of the action. This construction generalizes for curved space the tachyon vertex of free strings. We also present the nilpotent Becchi-Rouet-Stora-Tyutin (BRST) charge and the physical states it annihilates.

Strings and 2D CIFT. We start with an abstract formulation of the closed bosonic string: the Virasoro conditions. These state that the first-quantized bosonic string states satisfy

$$L_0 \,|\,\psi\rangle = |\,\psi\rangle,\; L_n \,|\,\psi\rangle = 0,\; n \geq 1 \;, \tag{1}$$

where L_n, $n \in \mathbb{Z}$ are the holomorphic generators of the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{1}{12}c(n^3-n)\delta_{n,-m} \;. \tag{2}$$

Identical equations are satisfied by the antiholomorphic generators $\bar{L}_n$ and there is an additional constraint $L_0 \,|\,\psi\rangle = \bar{L}_0 \,|\,\psi\rangle$. The central charge is chosen $c=26$ for string theory. A string state is created from the vacuum by a vertex operator, $|\,\psi\rangle = V(0)\,|\,0\rangle$, and the scattering amplitudes are world-sheet integrals of a correlation function of vertex operators:

$$\int \left[\prod_i d^2 z_i\right] \left\langle 0 \left| \prod_i V_i(z_i) \right| 0 \right\rangle \;.$$

Here we restrict ourselves to vertex operators local on the world sheet. Since the Virasoro conditions (1) determine the allowed vertex operators, in a sense they completely characterize the string theory, apart from the central charge and the requirement that the state space also carry a representation of the modular group.

In 2D CIFT, a primary conformal field of dimension $(h,\bar{h})$ is one which, under infinitesimal conformal transformation $z \to z + \varepsilon(z)$, $\bar{z} \to \bar{z} + \bar{\varepsilon}(\bar{z})$, transforms as

$$\delta_\varepsilon V = \left[h\frac{d\varepsilon}{dz} + \varepsilon\frac{d}{dz}\right]V + \left[\bar{h}\frac{d\bar{\varepsilon}}{d\bar{z}} + \bar{\varepsilon}\frac{d}{d\bar{z}}\right]V \;. \tag{3}$$

(Henceforth we shall omit the second term.) (1) says that the physical string states are highest-weight states of the conformal algebra with dimension (1,1), or equivalently, that the vertex operators of string theory are primary conformal fields of dimension $(h,\bar{h}) = (1,1)$. Hence

$$\delta_\varepsilon V = (d/dz)[\varepsilon(z)V(z,\bar{z})]$$

is a total derivative and the S matrix is conformally invariant.

The stress tensor $T_{\mu\nu}$ in 2D CIFT is conserved and traceless. Its traceless part $T_{zz} = T_{11} - T_{22} + 2iT_{12}$ is the generator of the conformal transformations:

$$\delta_\varepsilon V(\omega) = \oint_{c_\omega} \varepsilon(z) T_{zz} V(\omega)\, dz/2\pi i \;.$$

The statement that V is a conformal field of dimension h is contained in the operator-product expansion (OPE)

$$T_{zz}V(\omega) = \frac{hV(\omega)}{(z-\omega)^2} + \frac{\partial_\omega V(\omega)}{z-\omega} + \text{regular terms} \;, \tag{4}$$

which recovers (3) using Cauchy's formula. Thus (4) with $h=1$ is equivalent to the Virasoro conditions (1). In this paper, we present, for strings in curved space, a V that satisfies (4).

The Virasoro generators are given by

$$L_n = \oint z^{n+1} T_{zz}\, dz/2\pi i$$

and (4) is equivalent to the operator statement

$$[L_n, V(z)] = z^{n+1}(d/dz)V(z) + (n+1)hz^n V(z) .$$

This representation together with the Jacobi identity implies the closure of the Virasoro algebra (2) on the state space generated by V, up to the central extension c. In the OPE language (2) is expressed as

$$T_{zz}T_{\omega\omega} = \frac{c/2}{(z-\omega)^4} + \frac{2T_{\omega\omega}}{(z-\omega)^2} + \frac{\partial_\omega T_{\omega\omega}}{z-\omega} + \cdots , \quad (5)$$

and c is determined by the correlation

$$\langle T_{zz}T_{\omega\omega}\rangle = \tfrac{1}{2}c(z-\omega)^{-4}$$

Vertex operators for free fields. As a prelude to the curved-space case, we study the free-field representation of (4) and (5). Here $c = D$, the number of free boson fields. An example is the tachyon vertex $V_k(z) = \exp[ik\cdot x(z)]$. Dimensional regularization $(d = 2+2\varepsilon)$ is used to control ultraviolet divergences, and we introduce a mass m as an infrared cutoff. The expectation value of $V_k(z)$ in the presence of an arbitrary source j^a is

$$\langle V_k(\omega)\rangle_j = [1/z(j)] \int \mathcal{D}x \exp\left[-(1/4\pi)\int d^2\xi \tfrac{1}{2}(\partial_\mu x)^2 + \int d^2\xi\, j\cdot x + ik\cdot x(\omega)\right]$$
$$= \left\langle \exp\left[\int d^2\xi\, j'\cdot x\right]\right\rangle_0 \Big/ z(j) ,$$

where $j'(\xi) = j(\xi) + ik\delta^2(\xi - \xi_\omega)$. A simple Gaussian integration gives

$$\langle V_k(\omega)\rangle_j = \exp\left[ik^a \int d^2\xi\, \Delta(z,\omega)J^a(z) - \tfrac{1}{2}k^2\Delta_\varepsilon(0)\right] .$$

The propagator is

$$\Delta(z,\omega) = -\ln(|z-\omega|^2 e^\gamma m^2/4\pi)[1+O(m^2)] ,$$

the coincident propagator is $\Delta_\varepsilon(0) = -(1/\varepsilon + F + \varepsilon F_1 + \cdots)$, $F = \ln v$, $F_1 = \frac{1}{2}F^2 - \gamma F + \frac{1}{12}\pi^2 + \gamma^2$, $v = e^\gamma m^2/4\pi\mu^2$, μ is an arbitrary mass scale, and γ is Euler's constant.

It is clear that V_k can be multiplicatively renormalized by defining

$$V'_k(z) = \exp(-\tfrac{1}{2}k^2\varepsilon^{-1})\exp[ik\cdot x(z)] . \quad (6)$$

All matrix elements of $V'_k(z)$ are finite. The multiplicative renormalization of V is equivalent to introducing a series of additive divergent counterterms in the minimal-subtraction scheme and serves the same purpose as normal ordering in the operator formalism.

A simple computation with $T_{zz} = -\frac{1}{2}\partial_z x^a \partial_z x^a$ yields

$$\langle T_{zz}V'_k(\omega)\rangle_j = \tfrac{1}{2}k^2\langle V'_k(\omega)\rangle_j (z-\omega)^{-2} + \partial_\omega\langle V'_k(\omega)\rangle_j(z-\omega)^{-1} + \text{regular terms} . \quad (7)$$

Equation (7) holds for an arbitrary source $j(\xi)$ that vanishes at z, ω, and when the $O(m^2)$ terms in $\Delta(z,\omega)$ are ignored. Thus V' satisfies (4) and $h = \frac{1}{2}k^2$. Notice that (7) is satisfied even when $m \neq 0$ (but is small). In particular, $\langle V'_k(\omega)\rangle_0 = v^{k^2/2}$ and

$$\langle T_{zz}V'_k(\omega)\rangle_0 = v^{k^2/2}[\tfrac{1}{2}k^2(z-\omega)^{-2} + O(m^2)] .$$

Though both $\langle V'_k(\omega)\rangle_0$ and $\langle T_{zz}V'_k(\omega)\rangle_0$ vanish at precisely $m = 0$ (as expected from a conformal field of nonzero dimension), they satisfy

$$\langle T_{zz}V'_k(\omega)\rangle_0 = \tfrac{1}{2}k^2\langle V'_k(\omega)\rangle_0 (z-\omega)^{-2} \quad (8)$$

when m is small but not zero, and the dimension h can be correctly read off from (8). In the interacting case we will present the results for $j = 0$.

Vertex operator in curved space. The general nonlinear model in 2D [with the target space metric $G_{ab}(x)$] and with dilaton background field $\Phi(x)$ is given by

$$A = (4\pi)^{-1}\int d^2\xi\sqrt{g}\,[\tfrac{1}{2}\lambda^{-2}g^{\mu\nu}G_{ab}(x)\partial_\mu x^a \partial_\nu x^b - R^{(2)}\Phi(x)] . \quad (9)$$

$g^{\mu\nu}$ and $R^{(2)}$ are the 2D world-sheet metric and scalar curvature. The dilaton coupling was introduced by Fradkin and Tseytlin (Ref. 3). $\lambda^2 = \alpha'$ is the slope parameter. The stress tensor $T_{\mu\nu} = -4\pi\delta A/\delta g^{\mu\nu}$ is[4]

$$T_{\mu\nu} = -\tfrac{1}{2}\lambda^{-2}(G_{ab}\partial_\mu x^a \partial_\nu x^b - \tfrac{1}{2}\delta_{\mu\nu}G_{ab}\partial_\lambda x^a \partial_\lambda x^b) + \partial_\mu\partial_\nu\Phi - \delta_{\mu\nu}\partial_\lambda\partial_\lambda\Phi , \quad (10)$$

$$T_{zz} = -\tfrac{1}{2}\lambda^{-2}G_{ab}\partial_z x^a \partial_z x^b + \partial_z{}^2\Phi(x) . \quad (11)$$

In (10) and (11) we have set $g_{\mu\nu} = \delta_{\mu\nu}$. In principle $g_{\mu\nu}$ is flat only up to a conformal factor. However, at $c = 26$ the theory has Weyl invariance: The Liouville mode decouples from the dynamics. With this hindsight we henceforth use $g_{\mu\nu} = \delta_{\mu\nu}$. This is possible for the tree-level strings because the world sheet maps into the zero-genus complex plane. Thus though Φ appears in $T_{\mu\nu}$, it disappears from A.

The Virasoro equations are local equations in the configuration space of the string. Hence they can be solved locally in the neighborhood of a point in the target space. These local solutions are then patched up to obtain the general solution. (This is reminiscent of the WKB method in quantum mechanics.) Hence the implications of conformal invariance can be studied in the neighborhood of points in the target space.[5] We expand x about a fixed point X in the target space: $x^a = X^a + \lambda\pi^a$. The variable π is replaced by the Riemann[6] normal variable η (which transforms as a vector at X), and A and T_{zz} are written as perturbation series in λ. Then to order λ^2, A and T_{zz} with their one-loop counterterms in the minimal scheme included are given by

$$A = (4\pi)^{-1}\int d^2\xi[\tfrac{1}{2}\partial_\mu\eta^A\partial_\mu\eta^A + \tfrac{1}{6}\lambda^2(R_{ACDB}\eta^C\eta^D + \varepsilon^{-1}R_{AB})\partial_\mu\eta^A\partial_\mu\eta^B + \tfrac{1}{2}m^2(\delta_{AB} - \tfrac{1}{3}\varepsilon^{-1}R_{AB})\eta^A\eta^B + O(\lambda^3)] , \quad (12)$$

$$T_{zz} = -\tfrac{1}{2}\partial_z\eta^A\partial_z\eta^A - \tfrac{1}{6}\lambda^2(R_{ACDB}\eta^C\eta^D+\varepsilon^{-1}R_{AB})\partial_z\eta^A\partial_z\eta^B + \lambda D_A\Phi\,\partial_z{}^2\eta^A + \lambda^2 D_A D_B\Phi(\partial_z\eta^A\partial_z\eta_B + \eta^A\partial_z{}^2\eta^B) \ . \tag{13}$$

Here $\eta^A = e_a^A(X)\eta^a$ where $e_a^A(X)$ is the vierbein at X, and the Riemann tensor R_{ACDB}, the Ricci tensor $R_{AB}=R_{ACCB}$, $D_A\Phi$, and $D_AD_B\Phi$ are all evaluated at X. The mass term and its counterterm in (12) regulate infrared divergences. Note that since X is independent of $z,\bar{z}$, translational invariance in the sense of 2D field theory is manifest.

A target-space scalar $V(x)$ can also be expanded:

$$\begin{aligned} V(x) &= \sum (n!)^{-1}\lambda^n\eta^{A_1}\cdots\eta^{A_n}D_{A_1}\cdots D_{A_n}V(x) \\ &= \exp(\lambda\eta^A D_A)V(x) \ . \end{aligned} \tag{14}$$

With hindsight from free fields, if $V(x)$ is a vertex operator for emission of a (mass $\neq 0$) particle, we expect $D_aV(x)\sim\lambda^{-1}$. Introducing $V_a=\lambda D_aV(x)$, $V_{ab}=\lambda^2D_aD_bV(x)$, etc., we have

$$V(x)=\sum (n!)^{-1}\eta^{A_1}\cdots\eta^{A_n}V_{A_1\cdots A_n}$$

and *each term* is of order one.

The renormalized vertex operator V_R is obtained from (14) by adding counterterms in the minimal-subtraction scheme to make all Green's functions $\langle V_R(x)\eta^{A_1}(z_1)\cdots\eta^{A_n}(z_n)\rangle$ ultraviolet finite to order λ^2. Each successive n produces a new set of counterterms, resulting in an infinite series which ultimately exponentiates. Here we present the result leaving details for Ref. 7:

$$\begin{aligned} V_R = \exp(\tfrac{1}{2}\lambda^2\varepsilon^{-1}D_AD_A)\exp(\lambda\eta^AD_A)[V(x) &- \tfrac{1}{4}\lambda^2\varepsilon^{-2}R_{AB}V_{AB}(x) \\ &+ \tfrac{1}{6}\lambda^2\varepsilon^{-1}R_{AC}V_A(x)\eta^C - \tfrac{1}{2}\lambda^2\varepsilon^{-1}R_{ACDB}V_{AB}(x)\eta^C\eta^D] \ . \end{aligned} \tag{15}$$

To obtain the analogue of (8) we compute $\langle V_R(\omega)\rangle$ and $\langle T_{zz}V_R(\omega)\rangle$ to $O(\lambda^2)$ using (13) for T_{zz}. The result is

$$\langle V_R(\omega)\rangle = v^{-\lambda^2D_AD_A/2}[V(x)-\lambda^2(\tfrac{1}{6}F+\tfrac{1}{4}F^2)R_{AB}V_{AB}(x)] \ , \tag{16}$$

$$\begin{aligned} \langle T_{zz}V_R(\omega)\rangle = (z-\omega)^{-2}v^{-\lambda^2D_AD_A/2}[&-\tfrac{1}{2}V_{AA}+\lambda D_A\Phi V_A+\tfrac{1}{2}\lambda^2(\tfrac{1}{6}F+\tfrac{1}{4}F^2)R_{AB}V_{CCAB} \\ &+(1-\ln\mu^2|z-\omega|^2)(D_AD_B\Phi-\tfrac{1}{2}R_{AB})V_{AB}] \ . \end{aligned} \tag{17}$$

Thus (4) is satisfied for a single V_k correlation function provided

$$R_{AB}=2D_AD_B\Phi \ , \tag{18}$$

and

$$\lambda^2(-\tfrac{1}{2}D_AD_AV+D_A\Phi D_AV)=hV \ . \tag{19}$$

To leading order in λ, $V_{AA}=-2hV$, because in (19) the dilaton term is of higher order.

In analogy with the free case we expect that (4) is true as an OPE when (18) and (19) are satisfied. This means that for vertex functions satisfying (19) the states created by V_R carry a representation of (2), provided the backgrounds satisfy (18). It remains to compute the central charge c. This is done by directly evaluating $\langle T_{zz}T_{\omega\omega}\rangle$ using (13). We find $\langle T_{zz}T_{\omega\omega}\rangle=\frac{1}{2}c(z-\omega)^{-4}$ (for zero genus $\langle T_{zz}\rangle$ does not contribute), with[8]

$$C=D+3\lambda^2[R+4(D_A\Phi)^2-4D_AD_A\Phi]. \tag{20}$$

The BRST charge. The Virasoro conditions can be expressed in terms of the BRST charge.[9] Fixing the conformal gauge in (12) introduces a Faddeev-Popov determinant which can be written as a functional integral over anticommuting ghost fields b_{zz}, $b_{\bar{z}\bar{z}}$, c^z, $c^{\bar{z}}$. The ghost stress tensor[10] is denoted T^g_{zz} and satisfies the Virasoro algebra on its own with central charge -26. Since the determinant depends only upon $g_{\mu\nu}$, the ghosts (and hence T^g_{zz}) commute with x^a. The BRST charge is $Q_{\text{BRST}}=Q+\bar{Q}$, $Q=\oint j_z dz/2\pi i$, $j_z=c^zT^x_{zz}+\frac{1}{2}c^zT^g_{zz}+\frac{3}{2}\partial_z{}^2c^z$, and similarly for $\bar{Q}$. One can verify that

$$Q^2=\tfrac{1}{24}(c-26)\oint\partial_z^3c^z\cdot c^zdz/2\pi i \ ,$$

where c is the central charge of T^x_{zz} alone. This is a consequence simply of the fact that T^x_{zz} and T^g_{zz} satisfy (2) and that the x and ghost variables commute. Thus the BRST charge is nilpotent if (i) the state space carries a representation of the Virasoro algebra and (ii) $c=26$. In the present case to order λ^2 (i) holds for states $|\psi\rangle=V_R(0)|0\rangle$ provided (18) and (19) are satisfied and (ii) holds, for example, when $D=26$ and

$$R+4(D_A\Phi)^2-4D_AD_A\Phi=0 \ . \tag{21}$$

$V_R(0)|0\rangle$ satisfies (1) if in (19) $h=1$. When $(h,\bar{h})=(1,1)$, $|\Psi\rangle=V_R(0)|0\rangle\otimes c_1\bar{c}_1|0\rangle$ satisfies $Q_{\text{BRST}}|\Psi\rangle=0$. This is the BRST version of the Virasoro conditions.

It is well known that (18) and (21) follow from the action

$$S=\int d^{26}x\sqrt{G}\exp(-2\Phi)[R+4(D\Phi)^2] \ .$$

The physical meaning of the anomalous dimension eigenvalue equation (19) can be understood as follows: Suppose we have a background field $T(x)$ describing a tachyon condensate. The Virasoro conditions then imply that the low-energy effective action includes the tachyon coupling to G and Φ:

$$S_T=\int d^{26}x\sqrt{G}\exp(-2\Phi)[\tfrac{1}{2}(DT)^2-\lambda^{-2}T^2] \ .$$

For a fluctuation $V(X)$ around a background value $T_0(X)$ ($T=T_0+V$), the second-order term in S_T is

$$S^{(2)}=\int d^{26}x\sqrt{G}\exp(-2\Phi)V(-\tfrac{1}{2}D^2+D\Phi\cdot D-\lambda^{-2})V \ .$$

The linear operator (19) is thus precisely the small-fluctuation operator and the vertex function is interpreted as the wave function of the emitted particle. Similarly one can discuss the vertex function $h_{ab}(X)$ [which appears in the vertex $\partial_z x^a \partial_{\bar{z}} x^b h_{ab}(x)$]. Replacing G_{ab} by $G_{ab}+h_{ab}$ gives the emission amplitudes. Virasoro conditions demand that now $G_{ab}+h_{ab}$ must satisfy the background field equations. Thus $h_{ab}(X)$ satisfies the eigenvalue equation that follows from the linearization of Einstein's equations (18) and (21). The traceless part of $h_{ab}(X)$ is the graviton wave function in curved space. We will present the details in a forthcoming publication.

Note added. While this work was in progress we received the following papers prior to publication which discuss some related issues: T. Banks, D. Nemeschansky, and A. Sen, Stanford Linear Accelerator Center Report No. SLAC-PUB-3885 (unpublished); C. G. Callan and Z. Gan, Princeton University report (unpublished).

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