# Two important invariant identities

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Two important invariant identities about the products of the Riemann tensor $R_{\alpha\beta\gamma\lambda}$ and the Ricci tensor $R_{\alpha\beta}$ and the scalar curvature $R$ are derived by the technique of Weyl decomposition of the Riemann tensor and by the spinor formalism. These identities are very useful in four dimensions for simplifying the final expression of the $a_3$ coefficient of the scalar fields and for simplifying the evaluation of the vacuum-polarization energy-momentum tensor. This result is of relevance to the work of Jack and Parker on the summed form of the heat kernel.

## I. INTRODUCTION

Gilkey[1] has calculated the $a_3$ coefficient of spinor fields by geometric methods and reasons. The final expression involves 46 fundamental invariants (the coefficients of three terms are equal to zero). It reduces to 17 so-called fundamental invariants for scalar fields. We have calculated the $a_3$ coefficient of the scalar fields by proper-time-expansion methods.[2] The primary expression involves 25 invariants, and the 25 invariants can be reduced to 17 invariants by using many invariant identities. The final expression agrees with the formula given by Gilkey for the $a_3$ coefficient of the scalar fields.

The following questions then arise. Are Gilkey's 17 invariants really fundamental? Can it be further simplified? We derive two important identities in this article. Using these identities we can reduce the expression for the $a_3$ coefficient from 17 invariants to 15 invariants. The problems are not solved completely. Does another invariant identity exist? What is the number of the independent invariants?

Our results are very useful for simplifying the evaluation of the vacuum-polarization energy-momentum tensor of scalar fields in curved spacetime.[3]

Our results mean that in four dimensions the coefficient of the $R$-dependent terms in $a_3$ can be written in more than one way. This will be relevant to the work of Jack and Parker.[4]

We use the sign convention $(-+-)$ as in Carmeli's book[5] and Plancks units $\hbar=c=G=1$.

## II. THE TWO IMPORTANT INVARIANT IDENTITIES

The decomposition of the Riemann tensor in a four-dimensional manifold is

$$R_{\alpha\beta\gamma\lambda}=C_{\alpha\beta\gamma\lambda}+\tfrac{1}{2}(g_{\alpha\gamma}R_{\beta\lambda}-g_{\alpha\lambda}R_{\beta\gamma}+g_{\beta\lambda}R_{\alpha\gamma}-g_{\beta\gamma}R_{\alpha\lambda}) -\tfrac{1}{6}R(g_{\alpha\gamma}g_{\beta\lambda}-g_{\alpha\lambda}g_{\beta\gamma})\,. \tag{1}$$

From (1) we can, after a lengthy calculation, obtain the relations

$$C_{\alpha\beta}{}^{\gamma\sigma}C_{\gamma\sigma}{}^{\lambda\tau}C_{\lambda\tau}{}^{\alpha\beta}=R_{\alpha\beta}{}^{\gamma\sigma}R_{\gamma\sigma}{}^{\lambda\tau}R_{\lambda\tau}{}^{\alpha\beta}-\tfrac{1}{2}RR_{\alpha\beta\gamma\lambda}R^{\alpha\beta\gamma\lambda}-6R^{\alpha\gamma}R^{\beta\lambda}R_{\alpha\beta\gamma\lambda}-6R_{\alpha}{}^{\beta}R_{\beta}{}^{\gamma}R_{\gamma}{}^{\alpha}+7RR_{\alpha\beta}R^{\alpha\beta}-\tfrac{17}{18}R^{3}\,, \tag{2}$$

$$\begin{aligned}R_{\alpha\beta}{}^{\gamma\sigma}R^{\alpha\lambda}{}_{\gamma\tau}R_{\lambda}{}^{\beta\tau}{}_{\sigma}&=C_{\alpha\beta}{}^{\gamma\sigma}C^{\alpha\lambda}{}_{\gamma\tau}C_{\lambda}{}^{\beta\tau}{}_{\sigma}-3R_{\lambda\sigma}C^{\lambda\alpha\beta\gamma}C^{\sigma}{}_{\gamma\beta\alpha}+\tfrac{1}{2}RC_{\lambda\alpha\beta\gamma}C^{\lambda\gamma\beta\alpha}-R_{\alpha}{}^{\beta}R_{\beta}{}^{\gamma}R_{\gamma}{}^{\alpha}+\tfrac{3}{4}RR_{\alpha\beta}R^{\alpha\beta}-\tfrac{1}{9}R^{3}\\&=C_{\alpha\beta}{}^{\gamma\sigma}C^{\alpha\lambda}{}_{\gamma\tau}C_{\lambda}{}^{\beta\tau}{}_{\sigma}-\tfrac{1}{8}RC_{\lambda\alpha\beta\gamma}C^{\lambda\alpha\beta\gamma}-R_{\alpha}{}^{\beta}R_{\beta}{}^{\gamma}R_{\gamma}{}^{\alpha}+\tfrac{3}{4}RR_{\alpha\beta}R^{\alpha\beta}-\tfrac{1}{9}R^{3}\,,\end{aligned} \tag{3}$$

$$C_{\alpha\beta\gamma\lambda}C^{\alpha\beta\gamma\lambda}=R_{\alpha\beta\gamma\lambda}R^{\alpha\beta\gamma\lambda}-2R_{\alpha\beta}R^{\alpha\beta}+\tfrac{1}{3}R^{2}\,. \tag{4}$$

In spinor formalism we have

$$C_{\alpha\beta}{}^{\gamma\sigma}C^{\alpha\lambda}{}_{\gamma\tau}C_{\lambda}{}^{\beta\tau}{}_{\sigma}=C_{A\dot{B}C\dot{D}}{}^{E\dot{F}G\dot{H}}C^{A\dot{B}L\dot{M}}{}_{E\dot{F}N\dot{P}}C_{L\dot{M}}{}^{C\dot{D}N\dot{P}}{}_{G\dot{H}}\,, \tag{5}$$

$$C_{\alpha\beta}{}^{\gamma\sigma}C_{\gamma\sigma}{}^{\lambda\tau}C_{\lambda\tau}{}^{\alpha\beta}=C_{A\dot{B}C\dot{D}}{}^{E\dot{F}G\dot{H}}C_{E\dot{F}G\dot{H}}{}^{L\dot{M}N\dot{P}}C_{L\dot{M}N\dot{P}}{}^{A\dot{B}C\dot{D}}\,, \tag{6}$$

$$C_{A\dot{B}C\dot{D}E\dot{F}G\dot{H}}=-(\psi_{ACEG}\epsilon_{\dot{B}\dot{D}}\epsilon_{\dot{F}\dot{H}}+\epsilon_{AC}\epsilon_{EG}\bar{\psi}_{\dot{B}\dot{D}\dot{F}\dot{H}})\,. \tag{7}$$

From (5) and (6) we get the formulas

$$C_{\alpha\beta}{}^{\gamma\sigma}C^{\alpha\lambda}{}_{\gamma\tau}C_{\lambda}{}^{\beta\tau}{}_{\sigma}=4(\psi_{AE}{}^{CG}\psi^{AE}{}_{LN}\psi^{LN}{}_{CG}+\bar{\psi}_{\dot{B}\dot{F}}{}^{\dot{D}\dot{H}}\bar{\psi}_{\dot{D}\dot{H}}{}^{\dot{M}\dot{P}}\bar{\psi}_{\dot{M}\dot{P}}{}^{\dot{B}\dot{F}})\,, \tag{8}$$

$$C_{\alpha\beta}{}^{\gamma\sigma}C_{\gamma\sigma}{}^{\lambda\tau}C_{\lambda\tau}{}^{\alpha\beta}=8(\psi_{AC}{}^{EG}\psi_{EG}{}^{LN}\psi_{LN}{}^{AC}+\bar{\psi}_{\dot{B}\dot{D}}{}^{\dot{F}\dot{H}}\psi_{\dot{F}\dot{H}}{}^{\dot{M}\dot{P}}\bar{\psi}_{\dot{M}\dot{P}}{}^{\dot{B}\dot{D}})\,. \tag{9}$$

Then we have

 

$$C_{\alpha\beta}{}^{\gamma\sigma}C^{\alpha\lambda}{}_{\gamma\tau}C_{\lambda}{}^{\beta\tau}{}_{\sigma}=\tfrac{1}{2}C_{\alpha\beta}{}^{\gamma\sigma}C_{\gamma\sigma}{}^{\lambda\tau}C_{\lambda\tau}{}^{\alpha\beta}\ . \tag{10}$$

It is worth emphasizing that (10) is valid for an arbitrary four-dimensional Riemann manifold. From (3), (4), and (10) we obtain

$$\begin{aligned}R_{\alpha\beta}{}^{\gamma\sigma}R^{\alpha\lambda}{}_{\gamma\tau}R_{\lambda}{}^{\beta\tau}{}_{\sigma}&=R_{\alpha\beta\gamma\sigma}R^{\alpha\lambda\gamma\tau}R^{\beta}{}_{\lambda}{}^{\sigma}{}_{\tau}\\ &=\tfrac{1}{2}C_{\alpha\beta}{}^{\gamma\sigma}C_{\gamma\sigma}{}^{\lambda\tau}C_{\lambda\tau}{}^{\alpha\beta}-\tfrac{1}{2}RR^{\lambda\alpha\beta\gamma}R_{\lambda\alpha\beta\gamma}-R_{\alpha}{}^{\beta}R_{\beta}{}^{\gamma}R_{\gamma}{}^{\alpha}+\tfrac{3}{4}RR_{\alpha\beta}R^{\alpha\beta}-\tfrac{1}{9}R^{3}\\ &=\tfrac{1}{2}(R_{\alpha\beta}{}^{\gamma\sigma}R_{\gamma\sigma}{}^{\lambda\tau}R_{\lambda\tau}{}^{\alpha\beta}-\tfrac{1}{2}RR_{\lambda\alpha\beta\gamma}R^{\lambda\alpha\beta\gamma}-6R^{\alpha\gamma}R^{\beta\lambda}R_{\alpha\beta\gamma\lambda}-6R_{\alpha}{}^{\beta}R_{\beta}{}^{\gamma}R_{\gamma}{}^{\alpha}+7RR_{\alpha\beta}R^{\alpha\beta}-\tfrac{17}{18})R^{3})\\ &\quad-\tfrac{1}{8}R(R_{\lambda\alpha\beta\gamma}R^{\lambda\alpha\beta\gamma}-2R_{\alpha\beta}R^{\alpha\beta}+\tfrac{1}{3}R^{2})-R_{\alpha}{}^{\beta}R_{\beta}{}^{\gamma}R_{\gamma}{}^{\alpha}+\tfrac{3}{4}RR_{\alpha\beta}R^{\alpha\beta}-\tfrac{1}{9}R^{3}\\ &=\tfrac{1}{2}R_{\alpha\beta}{}^{\gamma\sigma}R_{\gamma\sigma}{}^{\lambda\tau}R_{\lambda\tau}{}^{\alpha\beta}-\tfrac{3}{8}RR_{\lambda\alpha\beta\gamma}R^{\lambda\alpha\beta\gamma}-3R^{\alpha\gamma}R^{\beta\lambda}R_{\alpha\beta\gamma\lambda}-4R_{\alpha}{}^{\beta}R_{\beta}{}^{\gamma}R_{\gamma}{}^{\alpha}+\tfrac{9}{2}RR_{\alpha\beta}R^{\alpha\beta}-\tfrac{5}{8}R^{3}\ .\end{aligned} \tag{11}$$

Similarly we derive

$$R^{\lambda}{}_{\tau}R_{\lambda\alpha\beta\gamma}R^{\tau\alpha\beta\gamma}=\tfrac{1}{4}RR_{\lambda\alpha\beta\gamma}R^{\lambda\alpha\beta\gamma}+2R^{\alpha\gamma}R^{\beta\lambda}R_{\alpha\beta\gamma\lambda}+2R_{\alpha}{}^{\beta}R_{\beta}{}^{\gamma}R_{\gamma}{}^{\alpha}-2RR_{\alpha\beta}R^{\alpha\beta}+\tfrac{1}{4}R^{3}\ . \tag{12}$$

We must point out that the important identities (11) and (12) cannot be derived by Bianchi identities and by the algebraic properties of the Riemann tensor alone.

## III. APPLICATION

The old expression of the $a_3$ coefficient of the scalar fields is

$$\begin{aligned}a_3=\frac{1}{7!}\{&6(3-14\xi)R_{;\alpha}{}^{\alpha}{}_{\beta}{}^{\beta}+[84\xi(5\xi-2)+17]R_{;\alpha}R^{;\alpha}-2R_{\alpha\beta;\gamma}R^{\alpha\beta;\gamma}-4R_{\alpha\beta;\gamma}R^{\alpha\gamma;\beta}\\ &+9R_{\alpha\beta\gamma\lambda;\sigma}R^{\alpha\beta\gamma\lambda;\sigma}+28(1-6\xi)(1-5\xi)RR_{;\alpha}{}^{\alpha}-8R_{\alpha\beta}R^{\alpha\beta;\lambda}{}_{\lambda}-8(14\xi-3)R_{\alpha\beta}R^{\alpha\lambda;\beta}{}_{\lambda}\\ &+12R_{\alpha\beta\gamma\lambda}R^{\alpha\beta\gamma\lambda;\sigma}{}_{\sigma}+\tfrac{35}{9}(1-6\xi)^3R^3-\tfrac{14}{3}(1-6\xi)RR_{\alpha\beta}R^{\alpha\beta}+\tfrac{14}{3}(1-6\xi)RR_{\alpha\beta\gamma\lambda}R^{\alpha\beta\gamma\lambda}\\ &+112(\xi-\tfrac{13}{63})R_{\alpha}{}^{\beta}R_{\beta}{}^{\gamma}R_{\gamma}{}^{\alpha}-112(\xi-\tfrac{4}{21})R^{\alpha\gamma}R^{\beta\lambda}R_{\alpha\beta\gamma\lambda}-\tfrac{16}{3}R^{\lambda}{}_{\tau}R_{\lambda\alpha\beta\gamma}R^{\tau\alpha\beta\gamma}\\ &+\tfrac{44}{9}R_{\alpha\beta}{}^{\gamma\sigma}R_{\gamma\sigma}{}^{\lambda\tau}R_{\lambda\tau}{}^{\alpha\beta}+\tfrac{80}{9}R_{\alpha\beta\gamma\sigma}R^{\alpha\lambda\gamma\tau}R^{\beta}{}_{\lambda}{}^{\sigma}{}_{\tau}\}\ .\end{aligned} \tag{13}$$

We find, on substituting (11) and (12) into (13), the following new expression of $a_3$ coefficient of the scalar fields:

$$\begin{aligned}a_3=\frac{1}{7!}\{&6(3-14\xi)R_{;\alpha}{}^{\alpha}{}_{\beta}{}^{\beta}+[84\xi(5\xi-2)+17]R_{;\alpha}R^{;\alpha}-2R_{\alpha\beta;\gamma}R^{\alpha\beta;\gamma}-4R_{\alpha\beta;\gamma}R^{\alpha\gamma;\beta}\\ &+9R_{\alpha\beta\gamma\lambda;\sigma}R^{\alpha\beta\gamma\lambda;\sigma}+28(1-6\xi)(1-6\xi)RR_{;\alpha}{}^{\alpha}-8R_{\alpha\beta}R^{\alpha\beta;\lambda}{}_{\lambda}-8(14\xi-3)R_{\alpha\beta}R^{\alpha\lambda;\beta}{}_{\lambda}\\ &+12R_{\alpha\beta\gamma\lambda}R^{\alpha\beta\gamma\lambda;\sigma}{}_{\sigma}+[\tfrac{35}{9}(1-5\xi)^3-\tfrac{62}{9}]R^3-[\tfrac{14}{3}(1-6\xi)-\tfrac{172}{3}]RR_{\alpha\beta}R^{\alpha\beta}\\ &+[\tfrac{14}{3}(1-6\xi)-\tfrac{14}{3}]RR_{\alpha\beta\gamma\lambda}R^{\alpha\beta\gamma\lambda}+[112(\xi-\tfrac{13}{63})-\tfrac{416}{9}]R_{\alpha}{}^{\beta}R_{\beta}{}^{\gamma}R_{\gamma}{}^{\alpha}\\ &-112(\xi+\tfrac{1}{7})R^{\alpha\gamma}R^{\beta\lambda}R_{\alpha\beta\gamma\lambda}+\tfrac{28}{3}R_{\alpha\beta}{}^{\gamma\sigma}R_{\gamma\sigma}{}^{\lambda\tau}R_{\lambda\tau}{}^{\alpha\beta}\}\ .\end{aligned} \tag{14}$$

The terms $R^3, RR_{\alpha\beta}R^{\alpha\beta}$ and $R_{\alpha\beta\gamma\lambda}R^{\alpha\beta\gamma\lambda}$ do not vanish when $\xi=\frac{1}{6}$. Our results mean that in four dimensions, the coefficient of the $R$-dependent terms in $a_3$ can be written in more than one way. This will be relevant to the work of Jack and Parker.

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