PHYSICAL REVIEW D VOLUME 35, NUMBER 2 15 JANUARY 1987

Brief Reports

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Heat-kernel regularization of gauge theory

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(Received 21 April 1986)

We discuss the heat-kernel option in the new covariant-derivative regularization program. The exponential regularization is generally simpler and more systematic than the previously studied power-law regulators. As a result, for example, we are able to verify the vanishing gluon mass in all dimensions at once.

Covariant-derivative regularization of gauge theory to all orders in d dimensions has recently been reported.[1–3] In particular, a dimension-dependent finiteness condition $n \geq n(d)$ has been established for power-law regulators $R^{(n)}=(1-\Delta/\Lambda^2)^{-n}$, so there is no doubt that heat-kernel (exponential) regularization will succeed in arbitrary dimension. Use of the heat kernel makes contact with existing literature,[4] and may also be superior for nonperturbative analysis. Here we discuss the heat-kernel Schwinger-Dyson (SD) rules, finding that they are, in general, even simpler and more systematic than the rules for power-law regularization.

We begin with the γ family of regularized SD equations given in Ref. 3:

$$0=\left\langle \left[\int (dx) \left[-\frac{\delta S_{\rm YM}}{\delta A_\mu^a(x)} + D_\mu^{ab}(x) Z^b(x) \right] \frac{\delta}{\delta A_\mu^a(x)} + \triangle(\gamma) \right] F \right\rangle . \tag{1}$$

Here $(dx)=d^dx$, $F[A]$ is an arbitrary functional of the gauge field,

$$S_{\rm YM}=\tfrac{1}{4}\int (dx) F_{\mu\nu}^a F_{\mu\nu}^a$$

is the usual Yang-Mills action in d dimensions, and Z^a is the Zwanziger gauge-fixing function,[5] which we specify as $\alpha Z^a=\partial\cdot A^a$. The γ family of regularized functional Laplacians $\triangle(\gamma)=\triangle_0+\gamma E$,

$$\triangle_0 \equiv \int (dx)(dy)(R^2)_{xy}^{ab} \frac{\delta^2}{\delta A_\mu^a(x)\delta A_\mu^b(y)} , \tag{2a}$$

$$E \equiv \int (dx)(dy)(dz) R_{yz}^{bc} \frac{\delta R_{yx}^{ba}}{\delta A_\mu^c(z)} \frac{\delta}{\delta A_\mu^a(x)} , \tag{2b}$$

will be taken with the heat-kernel regulator

$$R_{xy}^{ab}(\Delta)=[\exp(\Delta/\Lambda^2)]_{xy}^{ab} , \tag{3}$$

where Δ is the covariant Laplacian, as discussed in Refs. 1 and 3. The regularized SD diagrammatic rules, given in Ref. 3 for power-law regularization, remain exactly the same in this case, except for modification of the regulator expansion.

The weak-coupling expansion of the heat kernel proceeds in several easy steps. First define

$$\frac{\Delta}{\Lambda^2} \equiv \frac{\Box}{\Lambda^2}+V , \quad V \equiv g\Gamma_1+g^2\Gamma_2 , \tag{4}$$

where

$$(\Gamma_1)_{xy}^{ab}=f^{abc}[\partial_\mu^x A_\mu^c(x)+A_\mu^c(x)\partial_\mu^x]\delta^d(x-y) , \tag{5a}$$

$$(\Gamma_2)_{xy}^{ab}=f^{acg}f^{cbe}A_\mu^g(x)A_\mu^e(x)\delta^d(x-y) , \tag{5b}$$

and $\Box$ is the ordinary Laplacian. Next, introduce an interaction picture

$$R(\Delta)=\exp\left[\frac{\Box}{\Lambda^2}\right] T_\sigma \exp\left[\int_0^1 d\sigma\, \tilde{V}(\sigma)\right] \tag{6}$$

in which T_σ is "time ordering" in the parameter σ, and

$$\tilde{V}(\sigma) \equiv \exp\left[-\sigma\frac{\Box}{\Lambda^2}\right] V \exp\left[\sigma\frac{\Box}{\Lambda^2}\right] . \tag{7}$$

Finally, a simple change of variables gives

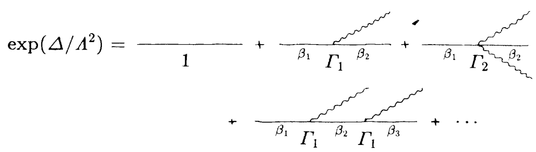

FIG. 1. Expansion of the heat kernel as regulator strings.

FIG. 2. The zeroth-order gluon propagator.

$$R(\Delta)=\exp\left[\frac{\Box}{\Lambda^2}\right]+\sum_{n=2}^{\infty}\int_0^1\left[\prod_{j=1}^{n}d\beta_j\right]\delta\left[1-\sum_{k=1}^{n}\beta_k\right]\exp(\beta_1\Box/\Lambda^2)V\exp(\beta_2\Box/\Lambda^2)V\cdots V\exp(\beta_n\Box/\Lambda^2)\ . \tag{8}$$

The diagrammatic interpretation of this expansion as a sum of regulator strings is shown in Fig. 1. As in Ref. 3, the wavy lines are gauge fields, while the three- and four-point vertices represent Γ_1 and Γ_2, respectively. In the present case, however, each straight line carries a free regulator propagator $\exp(\beta\Box/\Lambda^2)$, with its own weight β. All the β's of a given string are integrated from zero to one, subject to the constraint that their sum is unity.

The zeroth-order gluon propagator, shown in Fig. 2,

$$\langle A_\mu^a(x)A_\nu^b(y)\rangle=\int\frac{d^dp}{(2\pi)^d}e^{-ip\cdot(x-y)}[T_{\mu\nu}(p)+\alpha L_{\mu\nu}(p)]\delta^{ab}\frac{\exp(-2p^2/\Lambda^2)}{p^2}\ , \tag{9}$$

already exhibits the exponential regularization, but loop computations are necessary to appreciate the simplicity of the heat kernel. As an illustration, we discuss a computation of the one-loop gluon mass for all dimensions at once. In contrast, power-law regularization essentially requires a separate computation for each dimension.

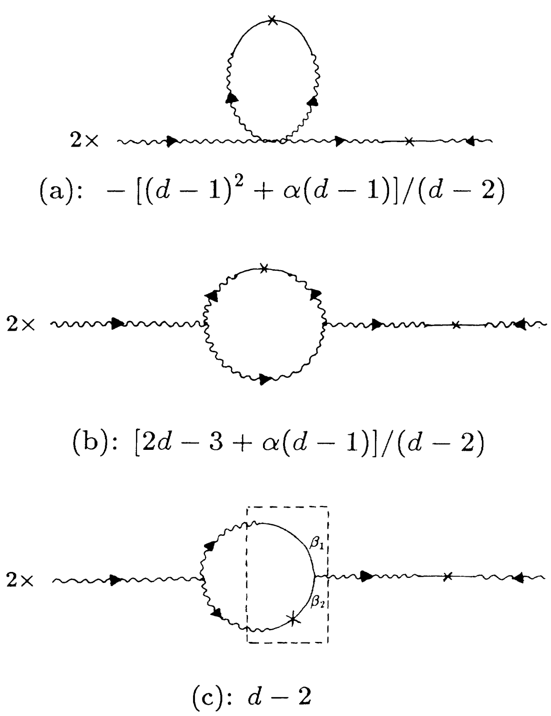

FIG. 3. Second-order diagrams which contribute to the gluon mass.

For the computation, we choose the case $\gamma=0$, thus eliminating, as noted in Ref. 3, the regulator vertex clusters with one incoming line (RVC_1's). Then, only three types of diagrams contribute to the gluon mass as shown in Fig. 3. The number of diagrams of each type (trivially related by symmetry to the representative diagram shown) is given with each diagram. Diagrams 3(a) and 3(b) are "ordinary" since they contain only Zwanziger gauge-fixed Yang-Mills vertices, while diagram 3(c) is an additional diagram with a regulator vertex. The dotted box in the diagram indicates, as in Ref. 3, a regulator vertex cluster (with two incoming lines), which consists of two regulator strings contracted at the cross.

Because it exhibits more of the heat-kernel structure, we sketch the explicit evaluation of diagram 3(c), shown in Fig. 4 with all relevant indices. For simplicity we explicitly discuss only the Feynman-Zwanziger gauge $\alpha=1$, though we have checked that the diagram is in fact independent of α. The sequence of SD "pictures" for the representative diagram 3(c) is shown in Fig. 5, in which the vertex factors are given by

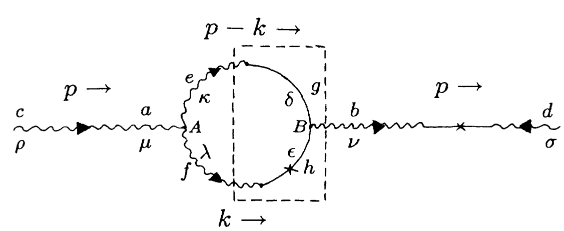

FIG. 4. Diagram 3(c) with indices. The only valid ordering is *AB*.

$$V^{abc}_{\mu\kappa\lambda}(p_1,p_2,p_3)=-\frac{i}{2}gf^{abc}[(p_1-p_2)_\lambda\delta_{\kappa\mu}+(p_2-p_3)_\mu\delta_{\lambda\kappa}+(p_3-p_1)_\kappa\delta_{\mu\lambda}]-\frac{i}{2\alpha}gf^{abc}[(p_3)_\lambda\delta_{\kappa\mu}-(p_2)_\kappa\delta_{\mu\lambda}]\ , \tag{10a}$$

$$(\Gamma_1)^{abc}_{\mu\kappa\lambda}(p_1,p_2,p_3)=igf^{abc}(p_1-p_3)_\kappa\delta_{\mu\lambda}/\Lambda^2\ . \tag{10b}$$

The value of the representative diagram 3(c),

$$2\int_0^1 d\beta_1\int_0^1 d\beta_2\delta(1-\beta_1-\beta_2)\int(dk)\frac{V^{cef}_{\rho\kappa\lambda}(p,k-p,-k)(\Gamma_1)^{edf}_{\kappa\sigma\lambda}(p-k,-p,k)}{2p^4[p^2+k^2+(p-k)^2]}\times\exp\{-\beta_1(p-k)^2+(1+\beta_2)k^2+2p^2]/\Lambda^2\}\ , \tag{11}$$

is then obtained as the product of the factors from each picture times Kronecker deltas for each line in color and tensor indices, and a combinatoric factor of 2. After truncation near $p=0$, this type of diagram's contribution to the gluon vacuum polarization is

$$\Pi(c)^{ab}_{\mu\nu}(0)=4g^2N\delta^{ab}\delta_{\mu\nu}\frac{(d-2)}{d}\frac{1}{\Lambda^2}\times\int\frac{d^dk}{(2\pi)^d}\exp(-2k^2/\Lambda^2) \tag{12}$$

in the case of SU(N). Performing the final integration, we have recorded the contribution of this diagram with

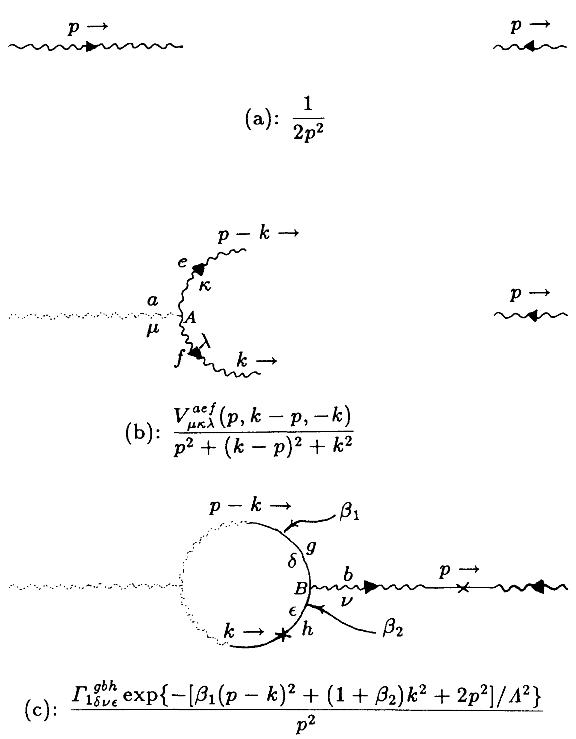

FIG. 5. Schwinger-Dyson pictures for diagram 3(c).

Fig. 3(c), in units of

$$\delta_{\mu\nu}\delta^{ab}\{2Ng^2/[d(4\pi)^d]\}(\Lambda^2/2)^{d/2-1}\ .$$

The remaining contributions to the zero-momentum vacuum polarization are similarly evaluated and recorded with their diagrams in Fig. 3. The sum of the three contributions is zero, providing a simultaneous check of gauge invariance in all dimensions[6] at once.

We also remark that the γ family of regularized[1-3] Parisi-Wu[7] Langevin systems

$$\dot{A}^a_\mu(x,t)=-\frac{\delta S_{\rm YM}}{\delta A^a_\mu}(x,t)+D^{ab}_\mu Z^b(x,t)+\int(dy)R^{ab}_{xy}\eta^b_\mu(y,t)+(\gamma-1)\int(dy)(dz)R^{bc}_{yz}\frac{\delta R^{ab}_{xy}}{\delta A^c_\mu(z)}\ , \tag{13a}$$

$$\langle\eta^a_\mu(x,t)\eta^b_\nu(y,t')\rangle=2\delta^{ab}\delta_{\mu\nu}\delta^d(x-y)\delta(t-t') \tag{13b}$$

is equivalent[8] to the γ family of regularized SD equations (1). The additional term in the Langevin equation (13a) is a one-loop counterterm for the Langevin diagrams, which generates extra RVC$_1$'s, changing their total weight, and canceling them completely at $\gamma=0$. In this connection, we note that the work of Refs. 1–3 [and Eq. (13) of this paper] have used the Stratonovich calculus. In fact, the case $\gamma=0$ in the Stratonovich calculus corresponds to the case $\gamma=1$ in the Ito calculus.[9] Put another way, if the regulator scheme of Refs. 1–3 had utilized the Ito calculus, the only change would have been that RVC$_1$'s would never have arisen in the first place.

We wish to thank H. S. Chan, D. Roekaerts, and L. Sadun for helpful discussions. This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract No. DE-AC0376SF00098 and National Science Foundation under Research Grant No. 85-15857.

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[6]It is amusing to note that our computation is defined for continuous $d > 2$, and dimensional continuation gives zero gluon mass when $d = 2$.

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[8]The equivalence is not formal for the heat kernel, nor for power-law regulators satisfying $n \geq [(d+1)/2]$. Note, however, that when $\gamma = 0$, the SD equations are finite for $n \geq [(d+3)/4]$, as in Ref. 3, while at the lower limit the Langevin system (13) generates divergent RVC_1's which cancel.

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