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Periodic generalizations of static, self-dual SU(2) gauge fields

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Linear pairs are used to inject time dependence (periodic or not) in a particularly simple fashion into static, self-dual SU(2) gauge fields. A canonical formalism is proposed for periodic generalizations of Euclidean versions of monopoles of arbitrary charge. The cases whose static limit corresponds to charges 1 and 2 are studied in some detail. Finite actions over one period are obtained. As nonperiodic examples Witten's solution for arbitrary index is derived in the context of our formalism with one single pole and a class of possible generalizations is indicated. A further generalization of our formalism, in another direction, is sketched. It is based on a coordinate transformation leading to a "time" with finite domain. This paper is principally concerned with periodic solutions. Such generalizations of only spherically and axially symmetric static solutions are considered here. The basic formalism, however, is not thus limited.

I. INTRODUCTION

Static, self-dual Bogomolny-Prasad-Sommerfield (BPS) solutions are well known. They are the monopoles and their Euclidean versions (Higgs scalar $\Phi \rightarrow$time component A_t of the gauge potential). Here a systematic method giving periodic generalizations of such solutions is presented. (Once time enters one cannot, of course, go back to monopoles through $A_t \rightarrow \Phi$.) Euclidean periodic solutions are usually considered in the context of nonzero temperature.[1–3] The possible roles of periodic solutions have been discussed, for example, in Ref. 2. Available, really explicit, solutions of this type are limited. The PS monopole can be given a periodic form through a gauge transformation.[1–5] There are certain related ones[1] and one class can be obtained through "heating transformations"[3] of Witten's solutions. The latter ones do not necessarily have finite action over one period.

Here a canonical method is presented for constructing *periodic generalizations of* (Euclidean versions of) *monopoles of arbitrary charge.* The present study is limited to spherical and axial symmetry. Only the cases of charges 1 and 2 will be studied explicitly. But the formalism is fully forged for the general case. Moreover our aim is to ensure *finite action over one period.* Our generalization of charge-1 static solution can be looked at from the point of view of Ref. 3. But even there two new features are present. First, of course, the method used permits iterations to higher charges. Second, the *Ansatz* introducing periodicity [Eq. (2.21)] ensures a finite action over a period. It turns out to be the simplest member of a canonical hierarchy. Our solutions are authentically periodic. Periodicity is not a gauge artifact as for the PS case.[1–3] We demand that the invariant $\mathrm{Tr}F^2$ must be periodic.

It will be seen that for our solutions the asymptotic situation ($r \rightarrow \infty$) does not differ from that of the limiting static case but *the core pulsates periodically.*

An interesting feature of our solutions is that from the *Ansätze* [(2.21), (3.29), and (6.6)] one can easily extract *linear periodic fluctuations* about static limits (by making certain parameters small). Small oscillations are relevant to the construction of propagators and spinor solutions, for example. Our solutions, of course, give a particular class of linear deformations. We will, indeed, use linear developments for a study of regularity properties for the charge-2 case. Apart from that, this aspect will not be pursued in this paper.

Our technique is not limited to periodic generalization. It permits one to inject both periodic and nonperiodic time dependence in a particularly simple and systematic fashion. The periodic case will be the main object of study in the following sections. A different, quite interesting, possibility will be briefly indicated. It will be shown how Witten's solutions can be very easily obtained and how they can possibly be generalized to higher Atiyah-Ward classes (Sec. IV).

Finally another possible generalization will be pointed out. The periodic solutions, to be presented, have the static BPS solutions as limits (for zero values of certain parameters). In a series of previous papers (Refs. 6–9 and sources cited there) I have constructed sequences of instantons ("instanton chains" or "hyperbolic monopoles"[9]) which also yield the BPS solutions very simply through a scaling limit. They are studied for their own interest and give the BPS solutions as by-products. There static techniques are used to obtain finite action (in four dimensions) through the transformation (A34) of the Appendix: namely,

$$r+it=\tanh\tfrac{1}{2}(\rho+i\tau)\ ,\quad \tau\in[-\pi,\pi]\ .$$

The "τ-static" solutions are neither static nor periodic in terms of t, the standard Euclidean time. One can generalize such a solution by introducing τ dependence. The corresponding formalism will be briefly discussed (Sec. V and the Appendix).

The method of linear pairs will be used. The formalism of Belinskii and Zakharov[10] was adapted to static monopole solutions by Forgács, Horváth, and Palla.[11,12] In Refs. 6–9 the generalizations leading to finite-action solutions can be found. Sec. V of Ref. 6 is most directly relevant. This is recapitulated at the end of the Appendix [A34)—(A48)]. This contains the limiting case spelled out at the beginning of the Appendix [(A1)—(A33)]. This simpler case, presented explicitly for the first time, will be used in this paper (except Sec. V). In the Appendix the results are presented without derivations. It is hoped, however, that this is sufficient to make the paper reasonably self-contained. If the reader accepts the expressions derived for λ and ζ [(2.10), (2.11) and (3.6), (3.7)], since they can be verified through direct substitution in the self-duality equations, even the Appendix is unnecessary. That however cannot, evidently, give any understanding of the structures involved.

II. NOTATIONS AND 1-POLE SOLUTIONS

Let (t,r,θ,ϕ) be the usual spherical coordinates and

$$\begin{aligned} z&=\tfrac{1}{2}(r+it), \quad \bar{z}=\tfrac{1}{2}(r-it)\ , \\ y&=e^{i\phi}\tan\frac{\theta}{2}, \quad \bar{y}=e^{-i\phi}\tan\frac{\theta}{2}\ . \end{aligned} \tag{2.1}$$

The flat Euclidean line element is

$$\begin{aligned} ds^2&=dt^2+dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2) \\ &=4\,dz\,d\bar{z}+4r^2(1+y\bar{y})^{-2}dy\,d\bar{y}\ . \end{aligned} \tag{2.2}$$

Only SU(2) gauge fields will be considered. Let λ be a real function of the coordinates and ζ be another one, in general, complex. Define

$$D=\lambda^{-1/2}\begin{bmatrix}\lambda & 0\\ \zeta & 1\end{bmatrix}, \tag{2.3}$$

$$\begin{aligned} A_\mu&=(i\partial_\mu D)D^{-1} \\ &=i(2\lambda)^{-1}\begin{bmatrix}\partial_\mu\lambda & 0\\ \partial_\mu\zeta & -\partial_\mu\lambda\end{bmatrix}\quad(\mu=y,z)\ , \end{aligned} \tag{2.4}$$

$$\begin{aligned} A_{\bar{\mu}}&=(i\partial_{\bar{\mu}}D^{\dagger-1})D^{\dagger} \\ &=-i(2\lambda)^{-1}\begin{bmatrix}\partial_{\bar{\mu}}\lambda & \partial_{\bar{\mu}}\bar{\zeta}\\ 0 & -\partial_{\bar{\mu}}\lambda\end{bmatrix}\quad(\bar{\mu}=\bar{y},\bar{z})\ , \end{aligned} \tag{2.5}$$

and

$$G=D^{\dagger}D=\lambda^{-1}\begin{bmatrix}\lambda^2+\zeta\bar{\zeta} & \bar{\zeta}\\ \zeta & 1\end{bmatrix}. \tag{2.6}$$

The self-duality constraints for the gauge potentials $A_\mu, A_{\bar{\mu}}$ reduce (with $G_z\equiv\partial_z G$ and so on) to

$$r^2(G_zG^{-1})_{\bar{z}}+(1+y\bar{y})^2(G_yG^{-1})_{\bar{y}}=0\ . \tag{2.7}$$

Let c be a constant parameter (real for the present case) and R and ω, respectively, the radial distance and the axial angle with the origin at $(0,0,c)$ on the 3-axis. Then,

$$\begin{aligned} R&=(r^2+c^2-2cr\cos\theta)^{1/2}\ , \\ R\cos(\omega-\theta)&=r-c\cos\theta, \\ R\sin(\omega-\theta)&=c\sin\theta\ . \end{aligned} \tag{2.8}$$

Let M_1 and M_2 be two functions of $(R-it)$, arbitrary to start with. Define

$$\mathscr{D}=M_1\overline{M}_1\bar{e}^{R}\sin^2\tfrac{1}{2}(\omega-\theta)+M_2\overline{M}_2e^{R}\cos^2\tfrac{1}{2}(\omega-\theta)\ . \tag{2.9}$$

It can be shown that the following λ and ζ satisfy (2.7):

$$\lambda=(2\mathscr{D})^{-1}e^{r}(M_2\overline{M}_2e^{R}-M_1\overline{M}_1\bar{e}^{R})\sin(\omega-\theta)\ , \tag{2.10}$$

$$\zeta=\mathscr{D}^{-1}M_1\overline{M}_2e^{r}\ . \tag{2.11}$$

They can be extracted from the 1-pole solution (A25) of G_1(phys). [But, if one is so inclined, one can verify by direct substitution of (2.10) and (2.11) in (2.7) and ignore the derivation in the Appendix.] In this particular case the solution can be considerably simplified by making the spherical symmetry explicit with

$$c\rightarrow 0$$

and discarding a vanishing factor c in λ and the derivatives of ζ (since only ratios appear in the A's). Define

$$(M_1/M_2)_{c\rightarrow 0}=\bar{f}(\bar{z}),\quad (\overline{M}_1/\overline{M}_2)_{c\rightarrow 0}=f(z)\qquad [z=\tfrac{1}{2}(r+it)]\ , \tag{2.12}$$

where the function f is to be chosen to ensure desirable properties. It can be shown that (for $c\rightarrow 0$) one ends up with

$$\lambda=\sin\theta\frac{1}{2r}(e^{r}-\bar{f}fe^{-r})\equiv\frac{1}{2J}\sin\theta \tag{2.13}$$

and

$$\zeta=\cos\theta\tfrac{1}{2}\left[2\bar{f}-\frac{d\bar{f}}{d\bar{Z}}\right]\equiv\tfrac{1}{2}\bar{K}\cos\theta\ . \tag{2.14}$$

The relation

$$\partial_{\bar{Z}}\partial_{Z}(\ln J)=\frac{1}{r^2}(J^2K\bar{K}-1) \tag{2.15}$$

is crucial for self-duality.

In terms of (t,r,θ,ϕ) one obtains the familiar-looking expressions (σ's being Pauli matrices)

$$\begin{aligned} &A_t=\partial_r\ln J\frac{\sigma_3}{2},\quad A_r=-\partial_t\ln J\frac{\sigma_3}{2}\ ,\quad A_\theta=\tfrac{1}{2}\begin{bmatrix}0 & iJK\\ -iJ\bar{K} & 0\end{bmatrix}, \\ &A_\phi=\sin\theta\tfrac{1}{2}\begin{bmatrix}0 & JK\\ J\bar{K} & 0\end{bmatrix}-\cos\theta\frac{\sigma_3}{2}\ . \end{aligned} \tag{2.16}$$

Define

$$K=|K|e^{i\delta},\quad \bar{K}=|K|e^{-i\delta} \tag{2.17}$$

The gauge transformation

$$A_\mu=UA_\mu U^{-1}+(i\partial_\mu U)U^{-1} \tag{2.18}$$

where $U=e^{-i\delta\sigma_3/2}$ gives

$$A_t=\partial_r\ln(J|K|)\frac{\sigma_3}{2},\quad A_r=-\partial_t\ln(J|K|)\frac{\sigma_3}{2}\ , \tag{2.19}$$

$$A_\theta = -J\,|\,K\,|\,\frac{\sigma_2}{2},\quad A_\phi = \sin\theta J\,|\,K\,|\,\frac{\sigma_1}{2} - \cos\theta\frac{\sigma_3}{2}\ .$$

Here

$$\begin{aligned} J\,|\,K\,| &= r(e^r - \bar{f}fe^{-r})^{-1} \\ &\quad\times\left[\left[2f - \frac{df}{dz}\right]\left[2\bar{f} - \frac{d\bar{f}}{d\bar{z}}\right]\right]^{1/2} \\ &= r(1 - f\bar{f}e^{-2(z+\bar{z})})^{-1} \\ &\quad\times\left[\frac{d}{dz}(fe^{-2z})\frac{d}{d\bar{z}}(\bar{f}e^{-2\bar{z}})\right]^{1/2}\ . \end{aligned} \tag{2.20}$$

For $f=1$ one gets the static, charge-1 solution. For a periodic generalization set

$$f = \prod_{j=1}^{n}\left[\frac{a_j + e^{-2kz}}{\bar{a}_j + e^{2kz}}\right]\quad (2z = r + it)\ , \tag{2.21}$$

with

$$k > 0 \quad\text{and}\quad |\,a_j\,| < 1\ . \tag{2.22}$$

(For simplicity I have set each $k_j = k > 0$. This is not the most general choice possible.)

Note that for

$$a_j = 0 \quad\text{and}\quad |\,a_j\,| = 1 \quad (j = 1, \ldots, n)\ , \tag{2.23}$$

one recovers the static case ($f=1$) with rescalings of r. Otherwise f is periodic with a period

$$T = 2\pi/k\ . \tag{2.24}$$

The choice (2.21) is excluded if one demands a finite action over $t \in [-\infty, \infty]$. But now one aims at a finite action over one period only. This opens up the above possibility. Moreover *the Ansatz (2.21) turns out to be the simplest member of a canonical hierarchy.* (See the following sections.)

Define

$$h = k\sum_{j=1}^{n}\left[\frac{1}{a_j e^{ikt} + 1} + \frac{1}{\bar{a}_j e^{-ikt} + 1}\right]\ . \tag{2.25}$$

As $r \to 0$,

$$\bar{f}f \to 1 - 2hr + 2h^2 r^2 \tag{2.26}$$

and, indeed (for $|\,a_j\,| < 1$), while $\bar{f}f = 1$ for $r = 0$,

$$\bar{f}f < 1 \quad\text{for } r > 0 \text{ for all } t\ . \tag{2.27}$$

This ensures regularity as r varies. Also as $r \to 0$

$$f^{-1}\frac{df}{dz} = \frac{d}{dz}\ln\bar{f}f \to -2\partial_z(rh)\ . \tag{2.28}$$

Apart from normalization and trivial angular integrations the action density ($\approx \mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$) reduces to

$$S_d = \partial_{\bar{z}}\partial_z[-\ln J + \tfrac{1}{2}(J\,|\,K\,|\,)^2] \equiv \partial_{\bar{z}}\partial_z H\ . \tag{2.29}$$

For $f = 1$, the PS case,

$$K = 2,\quad J\,|\,K\,| = r(\sinh r)^{-1},\quad \partial_z\partial_{\bar{z}} \approx \frac{d^2}{dr^2}\ , \tag{2.30}$$

and

$$\int_0^\infty dr\, S_d = 1\ . \tag{2.31}$$

The total action integrated over $t \in [-\infty, \infty]$, of course, diverges.

For (2.21), consider the action S_T over one period

$$0 \le t \le T = 2\pi/k\ . \tag{2.32}$$

With a factor 4π from integrations over θ and ϕ

$$\begin{aligned} S_T &= 4\pi\int_0^T dt\int_0^\infty dr\, S_d \\ &= 4\pi\int_0^T dt\int_0^\infty dr(\partial_r + i\partial_t)(\partial_z H)\ . \end{aligned} \tag{2.33}$$

Since singularities in the finite region are avoided

$$S_T = 4\pi\int_0^T dt(\partial_z H)_{r=0}^{r=\infty} + 4\pi i\int_0^\infty dr(\partial_z H)_{t=0}^{t=T}\ , \tag{2.34}$$

S_T can be evaluated in several ways. The value turns out to be continuous at $a_j = 0$ (for all j), where it is very simply obtained to be

$$S_T = 4\pi T(2nk + 1)\quad \left[T = \frac{2\pi}{k}\right]\ . \tag{2.35}$$

There is a discontinuity at $|\,a_j\,| = 1$, where

$$S_T = 4\pi T(nk + 1)\ . \tag{2.36}$$

This is a frontier feature. For $|\,a_j\,| > 1$, divergences appear.

III. 2-POLE SOLUTIONS

This corresponds to the solution G_2 described in the Appendix from (A28)–(A33). Let c be a complex parameter (eventually restricted to be purely imaginary) and define

$$\begin{aligned} R &= (r^2 + c^2 - 2cr\cos\theta)^{1/2}\ , \\ \bar{R} &= (r^2 + \bar{c}^2 - 2\bar{c}r\cos\theta)^{1/2}\ , \end{aligned} \tag{3.1}$$

$$\mu = -\frac{c\sin\theta}{R + r - c\cos\theta}\ ,\quad \bar{\mu} = -\frac{\bar{c}\sin\theta}{\bar{R} + r - \bar{c}\cos\theta}\ , \tag{3.2}$$

In terms of these and two functions

$$f_1(R - it),\quad f_2(\bar{R} - it) \tag{3.3}$$

define

$$M = (1 + \bar{\mu}\mu)^{-2}M_{11}M_{22} - (1 + \bar{\mu}^2)^{-1}(1 + \mu^2)^{-1}M_{12}M_{21} \tag{3.4}$$

with

$$\begin{aligned} M_{jj} &= 1 + \bar{f}_j f_j e^{-(R + \bar{R})}\quad (j = 1, 2)\ , \\ M_{12} &= 1 - \bar{f}_1 f_2 e^{-2\bar{R}}\ , \\ M_{21} &= \bar{M}_{12} = 1 - f_1\bar{f}_2 e^{-2R}\ . \end{aligned} \tag{3.5}$$

From (A31)–(A33), substituting in (2.6), one can extract

$$\lambda^{-1}=e^{-r}\left[\bar{\mu}\mu-M^{-1}\left[(1+\bar{\mu}\mu)^{-1}(M_{11}+M_{22})-(1+\bar{\mu}^2)^{-1}\frac{\bar{\mu}}{\mu}M_{12}-(1+\mu^2)^{-1}\frac{\mu}{\bar{\mu}}M_{21}\right]\right], \tag{3.6}$$

$$\frac{\zeta}{\lambda}=M^{-1}\left[\left[(1+\bar{\mu}\mu)^{-1}M_{11}-(1+\mu^2)^{-1}\frac{\mu}{\bar{\mu}}M_{21}\right]f_2e^{-\bar{R}}-\left[(1+\bar{\mu}\mu)^{-1}M_{22}-(1+\bar{\mu}^2)^{-1}\frac{\mu}{\bar{\mu}}M_{12}\right]f_1e^{-R}\right]. \tag{3.7}$$

That this pair (λ,ζ) satisfies (2.7) can again, in principle, be verified by direct substitution. Such a verification will, presumably, be far from simple.

For

$$f_1=f_2=1 \tag{3.8}$$

and

$$c=\frac{i\pi}{2}=-\bar{c}\ , \tag{3.9}$$

one obtains the Euclidean version of a static monopole of charge (Refs. 13 and 14).

As a generalization of (2.21) introduce the *Ansatz* $(k>0)$

$$f_1=\sum_{j=1}^{n}\left[\frac{a_j+e^{-k(R-it)}}{\bar{a}_j+e^{k(R-it)}}\right], \tag{3.10}$$

$$f_2=\prod_{j=1}^{n}\left[\frac{b_j+e^{-k(\bar{R}-it)}}{\bar{b}_j+e^{k(\bar{R}-it)}}\right]. \tag{3.11}$$

As before, this is not the most general possibility, but sufficient to illustrate nontrivial periodicity.

Case 1. A rescaled and gauge transformed static solution $(a_j=b_j=0,\ j=1,\dots,n)$: This is *not* a new solution. But the reason for displaying certain features of this case will be appreciated in the following subsection. Little unnecessary repetition will be involved.

For $a_j=0,\ b_j=0\ (j=1,\dots,n)$,

$$f_1=e^{-2nk(R-it)},\quad f_2=e^{-2nk(\bar{R}-it)}\ . \tag{3.12}$$

Substituting in (3.4)—(2.7)

$$M=(1+\bar{\mu}\mu)^{-2}(1+e^{-(p+1)(R+\bar{R})})^2-(1+\mu^2)^{-1}(1+\bar{\mu}^2)^{-1}(1-e^{-(p+1)2\bar{R}})(1-e^{-(p+1)2R})\quad (p=2nk)\ , \tag{3.13}$$

$$\lambda^{-1}=e^{-r}\left[\bar{\mu}\mu-M^{-1}\left[2(1+\bar{\mu}\mu)^{-1}(1+e^{-(p+1)(R+\bar{R})})-(1+\bar{\mu}^2)^{-1}\frac{\bar{\mu}}{\mu}(1-e^{-(p+1)2\bar{R}})\right.\right.$$
$$\left.\left.-\ (1+\mu^2)^{-1}\frac{\mu}{\bar{\mu}}(1-e^{-(p+1)2R})\right]\right], \tag{3.14}$$

$$\zeta/\lambda=e^{-ipt}M^{-1}\left[(1+\bar{\mu}\mu)^{-1}(1+e^{-(p+1)(R+\bar{R})})(e^{(p+1)\bar{R}}-e^{-(p+1)R})\right.$$
$$\left.+\ (1+\bar{\mu}^2)^{-1}\frac{\bar{\mu}}{\mu}(1-e^{-(p+1)2\bar{R}})e^{-(p+1)R}-(1+\mu^2)^{-1}\frac{\mu}{\bar{\mu}}(1-e^{-(p+1)2R})e^{-(p+1)\bar{R}}\right]$$
$$\equiv e^{-ipt}\eta\ . \tag{3.15}$$

A gauge transformation by

$$U=e^{ipt\sigma_3/2} \tag{3.16}$$

gives an explicitly static form of the gauge potentials. It can be verified that it is entirely a rescaled version of (3.8).

One has, for the transformed static A's (with $\lambda_r\equiv\partial_r\lambda$, etc.),

$$2\,\mathrm{Tr}A_t{}^2=\left[\frac{\lambda_r}{\lambda}+p\right]^2+\frac{1}{\lambda^2}[\eta\bar{\eta}+\eta_r\bar{\eta}_r+p(\eta\bar{\eta}_r+\bar{\eta}\eta_r)] \tag{3.17}$$

and

$$(2\,\mathrm{Tr}A_t{}^2)_{r\to\infty}=\left[p+1-\frac{2}{r}\right]^2+O(r^{-3})\ . \tag{3.18}$$

Assuming absence of singularity for finite r, the three-dimensional integral for the Euclidean energy (E) can be evaluated as a surface integral in a well-known fashion and gives

$$E=2(p+1)4\pi\quad (p=2nk)\ . \tag{3.19}$$

The factor 2 indicates charge 2. The factor $(p+1)$ plays the role of a scale factor which (along with 4π) is absorbed in the definition of the topological charge. For

$$p=2nk=0$$

one gets back the standard charge-2 case.

For (3.19) to be valid, one must ensure regularity for finite r. Two critical regions need careful study:[13,14]

$$\theta=0\ \text{ and }\ \theta=\frac{\pi}{2}\ .$$

Without going into details we recapitulate some crucial features useful for the periodic case to follow.

(i) $\theta=0$. From (3.1), (3.2), and (3.13) with $c=ic_0$ (c_0 real),

$$M=M_0+O(\theta^2),$$

where

$$M_0=(1+e^{-2(p+1)r})^2 -(1-e^{-2(p+1)(r+ic_0)})(1-e^{-2(p+1)(r-ic_0)}) . \quad (3.20)$$

Hence for

$$(p+1)c_0=\frac{\pi}{2} \quad (p=2nk), \quad (3.21)$$

$$M_0=0, \quad M=O(\theta^2), \quad (3.22)$$

and

$$\lambda=O(\theta^2) \text{ as } \theta\to 0 . \quad (3.23)$$

This condition turns out to be crucial to eliminate a singularity at the origin. It is known from the $p=0$ case.[13,14] [The condition $f_0=0$ of Eq. (5.33) of Ref. 7 is related.]

(ii) $\theta=\pi/2$. For $r<c_0$,

$$R=i(c_0{}^2-r^2)^{1/2}\equiv i\delta(=-\bar{R}) \quad (3.24)$$

and

$$\mu=\frac{-ic_0}{r+i\delta}=-i\left[\frac{r-i\delta}{r+i\delta}\right]^{1/2}=\bar{\mu}^{-1} . \quad (3.25)$$

Substituting in (3.14) and (3.15)

$$\lambda=e^r\frac{r\sin(p+1)\delta-\delta\cos(p+1)\delta}{r\sin(p+1)\delta+\delta\cos(p+1)\delta}, \quad 0\le\delta\le c_0 \quad (p=2nk), \quad (3.26)$$

$$\zeta=0 . \quad (3.27)$$

Since logarithmic derivatives are involved, for regularity, the numerator and the denominator of λ should not be separately zero. The choice

$$(p+1)c_0=\frac{\pi}{2}$$

again plays a crucial role. For this choice

$$r\sin(p+1)\delta\mp\delta\cos(p+1)\delta$$

both vanish only for

$$r=0$$

and (3.28)

$$\delta=0 \text{ or } r=c_0 ,$$

and then they vanish simultaneously their ratio being finite. (One can start with $r>c_0$ and come to similar conclusions.)

These two domains ($\theta=0,\pi/2$) fix the parameters and no difficulty arises elsewhere.[13,14]

Case 2. Periodic solutions. Consider now nonzero values of a's and b's in

$$f_1=\prod_{j=1}^{n}\left[\frac{a_j+e^{-k(R-it)}}{\bar{a}_j+e^{k(R-it)}}\right],$$

$$f_2=\prod_{j=1}^{n}\left[\frac{b_j+e^{-k(\bar{R}-it)}}{\bar{b}_j+e^{k(\bar{R}-it)}}\right] \quad (k>0) . \quad (3.29)$$

In (2.21) it was sufficient to impose (2.22) ($|a_j|<1$) to avoid divergences, since $|e^{2kz}|=e^{2kr}\ge 1$. Now there is a difference. Setting (with $c=ic_0$, c_0 real)

$$R=(r^2-c_0{}^2-2irc_0\cos\theta)^{1/2}=R_1+iR_2 \quad (3.30)$$

the real part R_1 can be negative. Thus, for

$$r\ll c_0, \quad R_1=r\cos\theta+O(r^3) \quad (3.31)$$

which is negative for $\theta>\pi/2$. For $r>c_0$, R_1 can always taken to be positive and since

$$\frac{R_1{}^2}{c_0{}^2}\left[\frac{R_1{}^2}{c_0{}^2}+1-\frac{r^2}{c_0{}^2}\right]=\frac{r^2}{c^2}\cos^2\theta \quad (3.32)$$

for

$$r<c_0, \quad R_1{}^2<c_0{}^2 . \quad (3.33)$$

Hence it is sufficient to choose

$$|a_j| \text{ and } |b_j|<e^{-kc_0} \quad (3.34)$$

to have f_1, f_2 finite everywhere. Divergences in f_1, f_2 need not be unacceptable for all types of solutions. But (3.34) will be assumed in the following.

Let us examine now the two critical domains *for all values of t.*

(i) $\theta=0$. It was pointed out, following (3.23) that the condition

$$M_{(\theta=0)}=0 \quad (3.35)$$

turns out to be necessary to avoid a singularity at $r=0$.

Now for $\theta=0$ ($\mu=0=\bar{\mu}$), from (3.4) and (3.5),

$$M=M_{11}M_{22}-M_{12}M_{21} =(f_1e^{-R}+f_2e^{-\bar{R}})(\bar{f}_1e^{-\bar{R}}+\bar{f}_2e^{-R}) . \quad (3.36)$$

For $\theta=0$,

$$R=r-ic_0, \quad \bar{R}=r+ic_0 .$$

Hence, if for $\theta=0$,

$$f_1e^{ic_0}+f_2e^{-ic_0}=0 , \quad (3.37)$$

(3.35) is satisfied. A solution is

$$(2nk+1)c_0=\frac{\pi}{2}, \quad a_je^{-ikc_0}=b_je^{ikc_0} \quad (j=1,\ldots,n) . \quad (3.38)$$

(ii) $\theta=\pi/2$. The situation is now to be compared to that analyzed in (3.24)–(3.28). Using (3.24)

$$f_1=\prod_j\left[\frac{a_j+e^{-ik(\delta-t)}}{\bar{a}_j+e^{ik(\delta-t)}}\right]\equiv e^{i\omega_1} \quad (\omega_1 \text{ real}) , \quad (3.39)$$

$$f_2=\prod_j\left[\frac{b_j+e^{+ik(\delta+t)}}{\bar{b}_j+e^{-ik(\delta+t)}}\right]\equiv e^{i\omega_2}\quad(\omega_2\ \text{real})\ . \tag{3.40}$$

Define (for $\theta=\pi/2$),

$$e^{i2\psi}=\bar{f}_1 f_2 e^{i2\delta}=e^{i2(\delta+\omega_2-\omega_1)}\ . \tag{3.41}$$

The fact that μ [see (3.25)], f_1, and f_2 are *all phases* leads, remarkably enough, to the result that for

$$\theta=\pi/2,\ r<c_0,$$

$$\lambda=e^{r}\frac{r\sin\psi-\delta\cos\psi}{r\sin\psi+\delta\cos\psi}\ , \tag{3.42}$$

$$\zeta=0\ . \tag{3.43}$$

As compared to (3.26) and (3.27), the only change is that $(2nk+1)\delta$ is replaced by

$$\psi=\delta+\omega_2-\omega_1\ . \tag{3.44}$$

Define

$$\frac{r}{c_0}=\cos\beta,\quad \frac{\delta}{c_0}=\sin\beta\ . \tag{3.45}$$

when $0\leq\beta\leq\pi/2$. For the numerator or the denominator of λ to vanish one must have (with upper and lower signs, respectively)

$$e^{i2\psi}=\bar{f}_1 f_2 e^{i2\delta}=\frac{r\pm i\delta}{r\mp i\delta}=e^{\pm i2\beta}\ . \tag{3.46}$$

It has to be assured that they vanish (if at all) simultaneously leaving λ nonzero and finite.

To analyze the situation further, consider the simpler forms (with $n=1$)

$$f_1=\left[\frac{a+e^{-k(R-it)}}{a+e^{k(R-it)}}\right],\qquad f_2=\left[\frac{b+e^{-k(\bar{R}-it)}}{\bar{b}+e^{k(\bar{R}-it)}}\right]\ . \tag{3.47}$$

Now (3.38) imposes

$$(2k+1)c_0=\frac{\pi}{2}$$

and

$$ae^{-ikc_0}=be^{ikc_0}\ , \tag{3.48}$$

and (3.46) reduces to

$$\left[\frac{\bar{a}+e^{ik(\delta-t)}}{a+e^{-ik(\delta-t)}}\right]\left[\frac{b+e^{ik(\delta+t)}}{\bar{b}+e^{-ik(\delta+t)}}\right]=e^{i2(\pm\beta-\delta)}\ . \tag{3.49}$$

Assume further

$$|a|\ \text{and}\ |b|\ll 1\ . \tag{3.50}$$

Up to first order

$$1+(\bar{a}e^{-ik\delta}-\bar{b}e^{ik\delta})e^{ikt}-(ae^{ik\delta}-be^{-ik\delta})e^{-ikt}=e^{i2[\pm\beta-(2k+1)\delta]}\ . \tag{3.51}$$

Hence, from the real parts

$$1=\cos2[\pm\beta-(2k+1)\delta]\ . \tag{3.52}$$

For $\delta=0$ ($r=c_0$) $\beta=0$ and for $\delta=c_0$ ($r=0$), $\beta=\pi/2$

$$\left[c_0=(2k+1)^{-1}\frac{\pi}{2}\right]\ .$$

Because of (3.48) again $0\leq(2k+1)\delta\leq\pi/2$. Thus as for the static case, zeros can arise only for

$$r=0\ \text{and}\ r=c_0\ . \tag{3.53}$$

Now the imaginary, time-dependent part must also be considered. Using (3.48), without any further real restriction one can set

$$ae^{-ikc_0}=\bar{a}e^{ikc_0}=be^{ikc_0}=\bar{b}e^{-ikc_0}=|a|\ . \tag{3.54}$$

Then from (3.51) and (3.52)

$$4|a|\sin(\delta+c_0)\cos kt=-\sin2[\pm\beta-(2k+1)\delta]=0\ . \tag{3.55}$$

Hence

$$\cos kt=0\ . \tag{3.56}$$

Thus, *up to first order in the parameters,* the situation remains [apart from the additional condition (3.56)] the same as for the static case. The numerator and the denominator of λ vanish only and simultaneously for $r=0$ and $\delta=0$ (and $kt=\pm\pi/2$), leaving λ finite and nonzero. The properties of ψ are difficult to analyze for the full expressions (3.39), (3.40), and even for (3.47). This will not be attempted in this paper.

Action over one period. Consider now the action. For static self-dual solutions the action density is

$$\sqrt{g}\,\tfrac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}=r^2\sin\theta\left[\frac{1}{r^2}\partial_r r^2\partial_r+\frac{1}{r^2\sin^2\theta}(\partial_\theta\sin\theta\partial_\theta+\partial_\phi^2)\right](T_rA_t^2)\ . \tag{3.57}$$

This no longer holds for periodic solutions. But since the solutions do not depend on ϕ (the azimuth) using self-duality one can show

$$\sqrt{g}\,\tfrac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}=\left[\frac{1}{r^2}\partial_\theta\frac{1}{\sin\theta}\partial_\theta+\frac{1}{\sin\theta}(\partial_r^2+\partial_t^2)\right](T_rA_\phi^2)\ . \tag{3.58}$$

(For solutions both static and axially symmetric the two expressions can be shown to give identical results.) For our 2-pole solutions (3.58) is still too complicated to use as such. In addition, we have not fully explored the regularity constraints on the parameters, which is necessary to make eventual surface integrations valid. So we fall back on the first-order development considered following (3.49). After the discussion for the critical regions ($\theta=0,\pi/2$) it is not difficult to convenience oneself that (up to first order in a and b) there is no special problem elsewhere. Moreover, now the integration of the action over one period $T(=2\pi/k)$ presents no problem. Set

$$f_1=e^{-2k(R-it)}(1+ae^{k(R-it)}-\bar{a}e^{-k(R-it)}) \tag{3.59}$$

and develop similarly f_2. The terms of first order in a and b being all periodic will be annihilated by the time integration giving [using (3.19)]

$$\begin{aligned} S_T &= T \quad \text{(static action over three dimensions)} \\ &= 4\pi T 2(2k+1)\ . \end{aligned} \tag{3.60}$$

For n factors

$$S_T=4\pi T2(2nk+1) \quad (T=2\pi/k)\ . \tag{3.61}$$

This is of course to be expected, since the action, when finite, should be independent of the parameters (a,b). Now using the argument of continuity one can say that the value (3.60) should hold for a certain domain of the values of a and b until singularities are encountered. We have already introduced the restrictions (3.34) and (3.38). For $\theta=\pi/2$ our study has been limited to linear deformations. Our results remain incomplete in this sense.

IV. NONPERIODIC EXAMPLES: WITTEN'S SOLUTIONS AND A POSSIBLE GENERALIZATION

As remarked in the Introduction our technique is not limited to periodic solutions. For the 1-pole case, instead of (2.21), set

$$f=e^{2z}\prod_{j=1}^{n}\frac{a_j-2z}{\bar{a}_j+2z} \quad (2z=r+it) \tag{4.1}$$

with

$$a_j+\bar{a}_j>0 \quad (j=1,\dots,n)\ .$$

Substituting in (2.19) and (2.20) one gets immediately Witten's solution[15] for index $(n-1)$. This arbitrary index is obtained in *one single step* (with 1 pole at the origin) solely through the freedom of choice of f. Compare the results of Refs. 16. There the index of the 't Hooft solution obtained through an analogous method is related to the number of poles or iterations.

For the corresponding *Ansatz* with 2 poles one can try, instead of (3.10) and (3.11) [and remembering (3.33)]

$$f_1=e^{(R-it)}\prod_{j=1}^{n}\left[\frac{a_j-(R-it)}{\bar{a}_j+(R-it)}\right] \quad (a_j+\bar{a}_j>2c_0)\ , \tag{4.2}$$

$$f_2=e^{(\bar{R}-it)}\prod_{j=1}^{n}\left[\frac{b_j-(\bar{R}-it)}{\bar{b}_j+(\bar{R}-it)}\right] \quad (b_j+\bar{b}_j>2c_0)\ , \tag{4.3}$$

where $R=(r^2-c_0{}^2-2ic_0 r\cos\theta)^{1/2}$. Substituting in (3.6), (3.7), and using (3.5) it is seen that all terms periodic in t disappear except an overall factor e^{-it} in ζ which can be gauge transformed away [as after (3.15)]. A study of this *Ansatz* will not be undertaken in this paper. One can conjecture that this provides a direct generalization of Witten's solutions to Atiyah-Ward[17] (AW) class 2.

V. A FURTHER GENERALIZATION

In the Introduction a generalization[6–9] of static BPS solutions was mentioned. It uses the transformation

$$r+it=\tanh\tfrac{1}{2}(\rho+i\tau),\quad \rho\in[0,\infty],\ \tau\in[-\pi,\pi]\ . \tag{5.1}$$

Static, finite-action solutions (in four dimensions) can be constructed which are "τ static" (independent of τ). But τ dependence can also be considered. The necessary linear pair formalism is summarized in the Appendix [(A34)–(A48)].

To generalize the 1-pole solution (2.19) define

$$J=\frac{\sinh\rho}{e^{\alpha\rho}-\bar{f}(\bar{z})f(z)e^{-\alpha\rho}}\ , \tag{5.2}$$

$$K=\left|2\alpha f-\frac{df}{dz}\right|,\quad \bar{K}=\left|2\alpha f-\frac{d\bar{f}}{d\bar{z}}\right|\ . \tag{5.3}$$

Here α is a real parameter and

$$f=f(z)=f(\rho+i\tau)\ . \tag{5.4}$$

The formalism of the Appendix leads to

$$\begin{aligned} &A_\tau=\partial_\rho\ln(J\,|K\,|)\frac{\sigma_3}{2},\quad A_\rho=-\partial_\tau\ln(J\,|K\,|)\frac{\sigma_3}{2}\ , \\ &A_\theta=-J\,|K\,|\frac{\sigma_2}{2},\quad A_\phi=\sin\theta J\,|K\,|\frac{\sigma_1}{2}-\cos\theta\frac{\sigma_3}{2}\ . \end{aligned} \tag{5.5}$$

For

$$f=1,\quad \alpha=2,3,4,\dots, \tag{5.6}$$

one gets instantons (or "hyperbolic monopoles") of action[6–9]

$$S=8\pi^2(\alpha-1)\ . \tag{5.7}$$

(For noninteger values of α, there are either divergences or branch points in the A's.)

The rescaling (A44),

$$\begin{aligned} &\rho\to r'/\alpha,\quad \tau\to t'/\alpha\ , \\ &A_\tau\to\alpha A_{t'},\quad A_\rho\to\alpha A_{r'}\quad (\alpha\to\infty)\ , \end{aligned} \tag{5.8}$$

gives back very simply the monopole of charge 1. In this sense one can attempt a generalization of the periodic *Ansatz* (2.21), setting

$$f=\prod_{j=1}^{n}\left[\frac{a_j+e^{-k\alpha(\rho+i\tau)}}{\bar{a}_j+e^{k\alpha(\rho+i\tau)}}\right]. \tag{5.9}$$

Since τ anyhow has a range 2π one can integrate over the whole four-dimensional domain (taking integer k for example) without being restricted to one period T as before.

One can next go on to analogous generalizations of multiple *Ansätze* such as (3.10) and (3.11).

The τ-static solutions provide explicit construction of instanton sequences in successive AW classes, the class being given by the number of poles. For n poles

$$S=8\pi^2 n(\alpha-n)\quad (\alpha=n+1,n+2,\dots)\ . \tag{5.10}$$

Thus for each class one has an infinite sequence. But for a given α no free parameters are left.

The τ-dependent generalizations will provide a further explicit solution in the higher AW classes. The maximal number of parameters one can incorporate through this particular approach and their domains compatible with finite action should then be studied.

VI. REMARKS

Our study of even the 2-pole solutions (Sec. III) remains incomplete. But the canonical structure emerging is easy to see. Define

$$\begin{aligned} R_j&=r^2+c_j{}^2-2c_j r\cos\theta\ ,\\ \bar{R}_j&=r^2+\bar{c}_j{}^2-2\bar{c}_j r\cos\theta\quad (j=1,2,\dots)\ , \end{aligned} \tag{6.1}$$

and

$$\mu_j=-\frac{c_j\sin\theta}{R_j+r-c_j\cos\theta}\ . \tag{6.2}$$

For static axially symmetric solutions[11,13,8] the nonzero values of c_j appear in purely imaginary conjugate pairs with quantized values. For an odd number of poles,

$$c_1=0,\quad \bar{c}_{2j}=\bar{c}_{2j+1}=i\pi j\quad (j=1,2,\dots)\ . \tag{6.3}$$

For an even number of poles

$$c_{2j-1}=\bar{c}_{2j}=i(2j-1)\frac{\pi}{2}\quad (j=1,2,\dots)\ . \tag{6.4}$$

For our case (as in Sec. III) we have to rescale such sequences suitably. An example is (3.21), which gives [instead of (6.4) with $j=1$]

$$c_1=\bar{c}_2=\frac{i\pi}{2(2nk+1)}\ . \tag{6.5}$$

Generalizing (2.21), (3.10), and (3.11) define for m poles

$$f_j=\prod_{i=1}^{n}\left[\frac{a_i^{(j)}+e^{-k(R_j-it)}}{\bar{a}_i^{(j)}+e^{k(R_j-it)}}\right]\quad (j=1,2,\dots,m) \tag{6.6}$$

which is only a particular, relatively simple possibility. The problem of making the regularity constraints explicit will increase with the number of poles. But one can at least study linear deformations (first order in the a's) about static solutions as was done in Sec. III. Moreover properties such as (3.37), (3.42), and (3.43) are encouraging. One can reasonably hope that for an arbitrary number of poles certain aspects will remain (for our *Ansatz*) formally close to the static limit ($a_i=0$). One can envisage similar generalizations also for the formalisms sketched is Secs. IV and V. Other possible interesting choices of f_j's should, of course, be investigated. Time-dependent poles should also be explored.

It is known that breaking axial symmetry by introducing ϕ dependence[11,12] involves considerable computational difficulties. To start with, ϕ dependence enters into the poles in a quite complicated fashion. Our idea was that, keeping axial symmetry, introducing time dependence might be simpler. Indeed it has turned out that one can obtain new, interesting solutions by preserving the simple pole structure (apart from rescalings) and introducing time solely through the row vectors $M^{(k)}$ of (1.13) [or rather through such ratios f_j as in (A30)]. Moreover this t dependence can be injected in a remarkably simple fashion thanks to the feature [see (A20)]

$$\begin{aligned} \beta_1(\mu_j)&=\tfrac{1}{2}(c_j-R_j+it)\ ,\\ \beta_2(\mu_j)&=\tfrac{1}{2}(-c_j-R_j+it)\ . \end{aligned} \tag{6.7}$$

Dropping the ϕ-dependent B in (A20) we have taken the $M^{(k)}$'s to be functions of β_1 and β_2 and hence of (R_j-it) for μ_j. We know of no systematic method of finding solutions of (A9) and (A12). The results finally obtained are showing agreeable properties. The generalizations (A48) are also noteworthy. Our formalism (the Appendix) also permits periodic generalizations of ϕ-dependent (separated) monopoles.[11,12] Explicit constructions will be much more complicated. Linear pairs for SU(N) [Refs. 12(b), 12(c), and 7] ($N>2$) can also be generalized to include time dependence in an analogous fashion. This will lead to some new features and additional difficulties.

In this paper we have used throughout the language of linear pairs. There are other well-known formalisms for construction of static self-dual monopoles. (See the talks by Atiyah, Corrigan, Nahm, and Ward in Ref. 11.) It would be quite interesting to see the counterpart of our generalizations in the context of other approaches. In particular, the twistor approach[18,19] is closely related to that of linear pairs. Periodic twistor solutions corresponding to our hierarchy can lead to a better understanding through comparison.

Nahm has pointed out that his adaptation of the Atiyah-Drinfeld-Hitchin-Manin formalism[20] to monopoles[21] can be extended to periodic solutions or "calorons."[22] He has also related them to spectral curves of Hitchin.[23] To see what may correspond to our explicit constructions in such formalisms one should presumably start with linear pairs based on the standard quaternionic combinations of coordinates ($x_1\pm ix_2, x_3\pm ix_0$) rather than (2.1) and (A1). This will complicate even the 1-pole solution of Sec. II. (A comparison of the two approaches can be found in Ref. 9.) But even after such a reformulation, pinning down explicit solutions in certain other formalisms may remain difficult.

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APPENDIX: CONSTRUCTION AND SOLUTIONS OF LINEAR PAIRS

The construction is based on the method of Belinskii and Zakharov.[10] Forgács, Horváth, and Palla[11,12] adapted such methods to the explicit construction of static monopoles. Our technique will be related to theirs. But due to our choice of coordinates the structure of the linear pair and particularly that of the solutions of the pole equations (to be introduced later) will be rather different. Moreover time dependence will enter in a quite particular fashion. (See Sec. V of Ref. 6.) The essential results are summarized below.

The self-duality constraints for SU(2) gauge fields can be expressed in terms of the 2×2 matrix G, defined in (2.6), as (with $G_z=\partial_z G$ and so on)

$$r^2(G_zG^{-1})_{\bar z}+(1+y\bar y)^2(G_yG^{-1})_{\bar y}=0\ . \tag{A1}$$

Here, in terms of the standard spherical coordinates

$$z=\tfrac{1}{2}(r+it),\quad \bar z=\tfrac{1}{2}(r-it)\ ,$$

$$y=\tan\frac{\theta}{2}e^{i\phi},\quad \bar y=\frac{\theta}{2}e^{-i\phi}\ .$$

Consider the linear pair (ψ being a 2×2 matrix and Λ a complex spectral parameter)

$$D_1\psi\equiv r^{-1}[r\partial_z-\Lambda(1+y\bar y)\partial_{\bar y}-(1+y\Lambda)\Lambda\partial_\Lambda]\psi$$
$$=(G_zG^{-1})\psi\ , \tag{A2}$$

$$D_2\psi\equiv(1+\bar y y)^{-1}[\Lambda r\partial_{\bar z}+(1+y\bar y)\partial_y+(\Lambda-\bar y)\Lambda\partial_\Lambda]\psi$$
$$=(G_yG^{-1})\psi\ ; \tag{A3}$$

D_1 and D_2 commute and their consistency condition

$$[D_1,D_2]\psi=D_1(G_yG^{-1}\psi)-D_2(G_zG^{-1}\psi)=0 \tag{A4}$$

implies (A1).

Seed solutions of (A1), (A2), and (A3) to be used are

$$G_0=\begin{bmatrix} e^r & 0\\ 0 & \epsilon e^{-r}\end{bmatrix},\quad \psi_0(\Lambda)=\begin{bmatrix} e^{h(\Lambda)} & 0\\ 0 & \epsilon e^{-h(\Lambda)}\end{bmatrix}\qquad (\epsilon=\pm1) \tag{A5}$$

with

$$h(\Lambda)=\tfrac{1}{2}r[(1+\Lambda y)^{-1}-\bar y(\Lambda-\bar y)^{-1}]\ . \tag{A6}$$

($\epsilon=+1$ or -1, respectively, for an even or odd number of iterations to be performed. This will be illustrated below.) Now that $h(\Lambda=0)=r$ so that $\psi_0(\Lambda=0)=G_0$. Also

$$h(\Lambda)+\overline{h(-\bar\Lambda^{-1})}=r\ ,$$

the overbar denoting complex conjugation.

Purely solitonic solutions can be obtained from the "simple pole *Ansatz*"

$$\psi_N(\Lambda)=\left[I+\sum_{k=1}^{N}\frac{R_k}{\Lambda-\mu_k}\right]\psi_0(\Lambda) \tag{A7}$$

or, equivalently, iteratively from

$$\psi_N(\Lambda)=\prod_{k=1}^{N}\left[I+\frac{R_k'}{\Lambda-\mu_k}\right]\psi_0(\Lambda) \tag{A7$'$}$$

(I is the 2×2 unit matrix).

The 2×2 matrices R_k (R_k') can be constructed algebraically. R_k' will be given below in (A14). The matrix G is obtained from

$$G_N=\psi_N(\Lambda=0)\ . \tag{A8}$$

To eliminate double poles in the intermediate steps the μ_k's satisfy the "pole equations"

$$[r\partial_z-\mu(1+\bar y y)\partial_{\bar y}+(1+y\mu)]\mu=0\ ,$$
$$[\mu r\partial\bar z+(1+\bar y y)\partial_y-(\mu-\bar y)]\mu=0\ . \tag{A9}$$

Defining

$$\beta_1(\mu)=(\mu-\bar y)^{-1}(\mu z+\bar z\bar y)$$
$$=\tfrac{1}{2}[r(\mu-\bar y)^{-1}(\mu+\bar y)+it]\ , \tag{A10}$$

$$\beta_2(\mu)=(\mu y+1)^{-1}(\mu yz-\bar z)$$
$$=\tfrac{1}{2}[r(\mu y+1)^{-1}(\mu y-1)+it]\ , \tag{A10$'$}$$

$$B(\mu)=(1+\mu y)^{-1}(\mu-\bar y)\ , \tag{A10$''$}$$

one can show that

$$\beta_1(\mu)=\text{const}\ ,$$
$$\beta_2(\mu)=\text{const}\ ,$$
$$\beta(\mu)=\text{const}\ ,$$

are all solutions of (A9).

The general solution of (A9) is

$$H(\beta_1(\mu),\beta_2(\mu),B(\mu))=0\ , \tag{A11}$$

where H is any function of β_1, β_2, and B. The function H has to be chosen suitably to obtain regular, finite action solutions.

Note also that [using (A2), (A3)]

$$D_i\beta_j(\Lambda)=0,\quad D_iB(\Lambda)=0\quad (i=1,2;j=1,2)\ . \tag{A12}$$

Define the row vectors

$$m_k=M^{(k)}(\beta_1(\mu_k),\beta_2(\mu_k),B(\mu_k))\psi_{k-1}^{-1}(\mu_k)\ . \tag{A13}$$

The correct choice of the row vectors $M^{(k)}$ (functions of β_1,β_2,B) is again crucial for desirable properties.

Using (A7$'$) it can be shown that

$$\psi_k(\Lambda)=\left[I+\frac{1+\bar\mu_k\mu_k}{\bar\mu_k(\Lambda-\mu_k)}P_k\right]\psi_{k-1}(\Lambda) \tag{A14}$$

with

$$P_k=\frac{(G_{k-1}m_k^\dagger)\otimes m_k}{m_kG_{k-1}m_k^\dagger}\quad [G_{k-1}=\psi_{k-1}(\Lambda=0)]\ . \tag{A14$'$}$$

Finally, to assure unimodularity one defines

$$G_N(\text{phys})=\left[\prod_{k=1}^{N}\mu_k\bar\mu_k\right]^{1/2}G_N\ . \tag{A15}$$

The iterative structure for the G's can be shown to be

$$G_k = G_{k-1} - \frac{1+\mu_k\bar{u}_k}{\mu_k\bar{\mu}_k}\frac{(G_{k-1}m_k^\dagger)\otimes(m_kG_{k-1})}{(m_kG_{k-1}m_k^\dagger)} . \quad (A16)$$

Consider now some simple particular cases. For (A11) choose (c being a complex constant)

$$\begin{aligned}\beta_1(\mu)-\beta_2(\mu) &= r(1+y\bar{y})(\mu y-\mu^{-1}\bar{y}+1-y\bar{y})^{-1}\\ &= r[\tfrac{1}{2}(\mu e^{i\phi}-\mu^{-1}e^{-i\phi})\sin\theta+\cos\theta]^{-1}\\ &= c . \end{aligned}\quad (A17)$$

This choice (with purely imaginary values, $\pm$ in $\pi/2$, of c) appears in the construction of static, axially symmetric monopoles.[8,11–13] *The ϕ dependence is trivial and can be absorbed through a redefinition $\Lambda e^{i\phi}\to\Lambda$, $\mu e^{i\phi}\to\mu$.* Define

$$\begin{aligned}&R=(r^2+c^2-2cr\cos\theta)^{1/2} ,\\ &R\cos(\omega-\theta)=r-c\cos\theta,\quad R\sin(\omega-\theta)=c\sin\theta .\end{aligned}\quad (A18)$$

Then from (A17) (absorbing the ϕ dependence as indicated)

$$\mu e^{i\phi}\to\mu=-\tan\tfrac{1}{2}(\omega-\theta) . \quad (A19)$$

[$\mu=\cot\frac{1}{2}(\omega-\theta)$ is also a solution; only (A19) will be retained.] Let the suffix j indicate the value c_j of c. Then

$$\begin{aligned}&\beta_1(\mu_j)=\tfrac{1}{2}(c_j-R_j+it) ,\\ &\beta_2(\mu_j)=\tfrac{1}{2}(-c_j-R_j+it) ,\\ &B(\mu_j)=-e^{-i\phi}\tan\tfrac{1}{2}\omega_j .\end{aligned}\quad (A20)$$

From (A6) and (A19),

$$\begin{aligned}h(\mu_j)&=\tfrac{1}{2}r[(1+y\mu_j)^{-1}-\bar{y}(\mu_j-\bar{y})^{-1}]\\ &=\tfrac{1}{2}(r+R_j) ,\end{aligned}\quad (A21)$$

so in (A5)

$$\psi_0(\mu_j)=\operatorname{diag}(e^{(r+R_j)/2},\epsilon e^{-(r+R_j)/2}) . \quad (A22)$$

Now consider the first two steps of iteration using this restricted class of poles and in (A13)

$$m_k=M^{(k)}(\beta_1(\mu_k),\beta_2(\mu_k))\psi_{k-1}^{-1}(\mu_k) \quad (A23)$$

[$M^{(k)}$ is independent of $B(\mu_k)$]. To study "physical solutions" at the first step one starts from

$$G_0=\operatorname{diag}(e^r,-e^{-r}) . \quad (A24)$$

Now $c_1=c$ is real and

$$\begin{aligned}&\mu_1=\bar{\mu}_1=\mu=-\tan\tfrac{1}{2}(\omega-\theta) ,\\ &G_1(\text{phys})=(\mu\bar{\mu})^{1/2}G_1\\ &\quad=\tan\tfrac{1}{2}(\omega-\theta)\left[G_0-\frac{1}{\sin^2\frac{1}{2}(\omega-\theta)}\times\frac{(G_0m_1^\dagger)\otimes(m_1G_0)}{(m_1G_0m_1^\dagger)}\right] ,\end{aligned}\quad (A25)$$

$$m_1=(M_1e^{-(r+R)/2},-M_2e^{(r+R)/2}) , \quad (A26)$$

and from (A20) and (A23)

$$M^{(1)}=(M_1(R-it),M_2(R-it)) . \quad (A27)$$

Only the ratio M_1/M_2, a function of $(R-it)$, plays a role. At the end one can take, in the gauge potentials, the limit $c\to 0$. (See Sec. II.)

For two-step solutions with 2 poles,

$$G_0=\operatorname{diag}(e^r,e^{-r}) . \quad (A28)$$

A complex-conjugate pair of poles are taken

$$\begin{aligned}&c_1=c,\quad c_2=\bar{c} ,\\ &\mu_1=\mu,\quad \mu_2=\bar{\mu} .\end{aligned}\quad (A29)$$

(For monopoles $c=-\bar{c}$.) Correspondingly we write

$$\begin{aligned}&R(c_1)=R(c)=R ,\\ &R(c_2)=R(\bar{c})=\bar{R} .\end{aligned}$$

Define

$$\begin{aligned}&f_1(R-it)=M_1^{(1)}/M_2^{(1)},\quad f_2(\bar{R}-it)=-M_1^{(2)}/M_2^{(2)} ,\\ &G_2(\text{phys})=(\mu_1\bar{\mu}_1\mu_2\bar{\mu}_2)^{1/2}G_2\end{aligned}\quad (A30)$$

$$\begin{aligned}&=\mu\bar{\mu}\Big[G_0-\frac{1}{N}(N_{22}V_1^\dagger\otimes V_1+N_{11}V_2^\dagger\otimes V_2\\ &\qquad -N_{21}V_1^\dagger\otimes V_2-N_{12}V_2^\dagger\otimes V_1)\Big] ,\end{aligned}\quad (A31)$$

where

$$\begin{aligned}&V_1^\dagger=\bar{\mu}^{-1}(\bar{f}_1e^{(r-\bar{R})/2},e^{-(r-\bar{R})/2}) ,\\ &V_2^\dagger=\mu^{-1}(\bar{f}_2e^{(r-R)/2},e^{-(r-R)/2}) ;\end{aligned}\quad (A32)$$

$$\begin{aligned}&N_{jj}=(1+\mu\bar{\mu})^{-1}(\bar{f}_jf_je^{-(R+\bar{R})/2}+e^{(R+\bar{R})/2})\quad (j=1,2) ,\\ &N_{12}=(1+\bar{\mu}^2)^{-1}(\bar{f}_1f_2e^{-\bar{R}}+e^{\bar{R}})=\bar{N}_{21} ,\\ &N=N_{11}N_{22}-N_{12}N_{21} .\end{aligned}\quad (A33)$$

This is used in Sec. III.

One can continue to iterate using (A16). But we now turn to a generalization of the formalism which gives back the preceding results of this appendix as limiting cases. A fuller treatment can be found in Ref. 6. Here the main features are recorded briefly for ready comparison.

Define

$$r+it=\tanh\tfrac{1}{2}(\rho+i\tau),\quad \rho\in[0,\infty],\ \tau\in[-\pi,\pi] , \quad (A34)$$

so that

$$\begin{aligned}ds^2&=dt^2+dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\\ &=(\cosh\rho+\cos\tau)^{-2}\\ &\quad\times[d\tau^2+d\rho^2+\sinh^2\rho(d\theta^2+\sin^2\theta\,d\phi^2)] .\end{aligned}\quad (A35)$$

Instead of (A1), now (with the same $y,\bar{y}$ as before)

$$\sinh^2\rho(G_zG^{-1})_{\bar{z}}+(1+y\bar{y})^2(G_yG^{-1})_{\bar{y}}=0 \quad (A36)$$

with

$$z=\tfrac{1}{2}(\rho+i\tau) ,\quad \bar{z}=\tfrac{1}{2}(\rho+i\tau) .$$

Correspondingly,

$$D_1\psi \equiv (\sinh\rho)^{-1}[\sinh\rho\partial_z - \Lambda(1+y\bar{y})\partial_{\bar{y}} - (\cosh\rho + y\Lambda)\Lambda\partial_\Lambda]\psi = (G_z G^{-1})\psi \ , \tag{A37}$$

$$D_2\psi \equiv (1+y\bar{y})^{-1}[\Lambda\sinh\rho\partial_{\bar{z}} + (1+y\bar{y})\partial_y + (\Lambda\cosh\rho - \bar{y})\Lambda\partial_\Lambda]\psi = (G_y G^{-1})\psi \ . \tag{A38}$$

Let α be an arbitrary real parameter and

$$G_0 = \text{diag}(e^{\alpha\rho}, \epsilon e^{-\alpha\rho}), \quad \psi_0 = \text{diag}(e^{h}, \epsilon e^{-h}) \ , \tag{A39}$$

with

$$e^{h(\Lambda)} = \left[\frac{(\Lambda - e^{\rho}\bar{y})(\Lambda y + e^{\rho})}{(\Lambda - e^{-\rho}\bar{y})(\Lambda y + e^{-\rho})} \right]^{\alpha/4} \tag{A40}$$

[compare with (A5) and (A6)].

The general structure of the solutions [from (A7) to (A16)] remains unaltered. But now instead of (A10), (A10′), and (A10″) one has

$$\beta_1(\mu) = (\mu e^{-z} - \bar{y}e^{\bar{z}})^{-1}(\mu e^{z} - \bar{y}e^{-\bar{z}}) = (\mu - \bar{y}e^{\rho})^{-1}(\mu e^{\rho} - \bar{y})e^{i\tau} \ , \tag{A41}$$

$$\beta_2(\mu) = (\mu y e^{-z} + e^{\bar{z}})^{-1}(\mu y e^{z} + e^{-\bar{z}}) = (\mu y + e^{\rho})^{-1}(\mu y e^{\rho} + 1)e^{i\tau} \ , \tag{A41′}$$

$$B(\mu) = (1 + \mu y e^{\rho})^{-1}(\mu e^{\rho} - \bar{y}) \ . \tag{A41″}$$

In terms of these the pole equations [compare with (A9)]

$$\begin{aligned} &[\sinh\rho\partial_z - \mu(1+y\bar{y})\partial\bar{y} + (\cosh\rho + y\mu)]\mu = 0 \ , \\ &[\mu\sinh\rho\partial_{\bar{z}} + (1+y\bar{y})\partial y - (\mu\cosh\rho - \bar{y})]\mu = 0 \ , \end{aligned} \tag{A42}$$

have the solutions

$$H(\beta_1(\mu), \beta_2(\mu), B(\mu)) = 0 \ . \tag{A43}$$

This formalism [from (A37) to (A43)] contains the preceding one [from (A1) to (A16)] as a simple scaling limit.

Define, using the parameter α of (A39) and (A40),

$$\rho = \frac{r'}{\alpha}, \quad \tau = \frac{t'}{\alpha}, \quad \text{and let } \alpha \to \infty \quad (r' \in [0,\infty],\ t' \in [-\infty,\infty]) \ . \tag{A44}$$

The preceding results are obtained with r', t' replacing r,t, respectively.

Define $\eta(c)$ and $\gamma(c)$, for an arbitrary complex c, through

$$\cosh\eta(c) = \cosh c \cosh\rho - \sinh c \sinh\rho \cos\theta \tag{A45}$$

and

$$\tan\gamma(c) = (\cosh c \sinh\rho - \sinh c \cosh\rho\cos\theta)^{-1}\sinh c \sin\theta \ . \tag{A46}$$

These (η and γ) furnish the correct generalizations of R and $(\omega-\theta)$ of (A18), respectively. [The parameter c comes out in regular solutions with the scaling factor α^{-1} incorporated in its explicit values, such as $\pm\alpha^{-1}(in\pi/2)$ instead of $\pm in\pi/2$.]

The pole equation [compare to (A17)] is now chosen to be

$$\beta_1(\mu)/\beta_2(\mu) = e^{2c} \tag{A46′}$$

and the solutions, corresponding to (A19), are

$$\mu e^{i\phi} \to \mu = -\tan\tfrac{1}{2}\gamma(c) \ . \tag{A47}$$

Substituting (A47) in (A40), (A41), and (A41′) and using (A45) and (A46), one has the remarkable simplifications

$$\begin{aligned} &e^{h(\mu)} = e^{\alpha(\rho+\eta)/2}, \quad \beta_1(\mu) = e^{c-\eta+i\tau}, \\ &\beta_2(\mu) = e^{-c-\eta+i\tau} \quad [\eta = \eta(c)] \ . \end{aligned} \tag{A48}$$

These generalize (A20) and (A21).

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