Effective gauge action on a finite-size lattice

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We present evidence to support the idea that finite-size effects for Monte Carlo simulations of gauge theories on a lattice may be represented by an effective action which could be measured for the particular lattice at hand. For SU(2) lattice-gauge-theory simulations this implies a shift in the coefficients (inverse coupling parameters) of the character expansion of the action being simulated. This shift can be measured and is observed to occur as expected in the crossover region and beyond into the weak-coupling domain.

I. INTRODUCTION

It is expected, and indeed already observed, that any simulation of a gauge theory on a finite-size lattice would show "finite-size effects" in the measurements of all operators on the lattice under consideration. These effects are expected to be most pronounced for regions of the coupling where the correlation length grows beyond the size of the lattice; and a theory for how particular operator expectation values scale with lattice size is known.[1] For example, the average of the simplest observable, the trace of the Wilson plaquette, shows such behavior and is known to obey these scaling laws.[2] Shifts in the expected location of known phase-transition points have been observed on finite lattices for models where such location is calculable exactly.

It is also implicitly assumed, based on general theorems, that the parameters of the action being simulated in a Monte Carlo calculation provide a correct description of the configurations generated by this simulation for a finite lattice. However, in view of the observed dependence of various physical quantities on the size of the lattice this assumption may not be strictly true. Any measurement of these parameters would reflect these finite-size effects and lead to an "effective action" which, through modified parameters, would account for these finite-size effects. We propose to test this possibility here using a method, introduced earlier, which allows us to determine the action directly from the Monte Carlo–generated configurations.[3]

Consider, in light of the above, the character expansion of the Boltzmann factor of the action. The Boltzmann factor is

$$F(U_p)=\exp[S(U_p)]\ , \tag{1}$$

where U_p indicates the product of gauge group elements, defined over links, taken over simple four-sided plaquettes. The action S is defined by

$$S=\sum_r \beta_r X_r(U_p)\ , \tag{2}$$

where $X_r(U_p)$ represents the character of the group in the rth representation and β_r is the corresponding coupling. This is a unique expansion, so that knowledge of the coefficients β_r is equivalent to the knowledge of the action.

The Boltzmann factor for each plaquette may be also character expanded. We use the form

$$F_p=\sum_r f_r d_r X_r(U_p)\ , \tag{3}$$

where d_r is the dimension of the rth representation. Orthonormality of the characters implies that

$$f_r=\frac{1}{d_r}\int dU_p F_p(U_p)X_r(U_p)\ . \tag{4}$$

Note that the measure is over the group space of the plaquette variable. It is clear that a knowledge of the coefficients f_r is equivalent to a knowledge of the β_r as we have

$$\beta_r=\int d(U_p)\ln F_p(U_p)X_r(U_p)\ . \tag{5}$$

Consider now the Monte Carlo integral for determining the expectation value of the character of any particular plaquette on the lattice. We have

$$\langle X_r(U_p)\rangle=\frac{\int\prod d(U_l)X_r(U_p)F(U_p)}{\int\prod d(U_l)F(U_p)}\ . \tag{6}$$

Note here that the measure is over the group space of the basic link variables of the lattice. The product over the measures of link variables may however be transformed to the measure over the group space of the plaquette variables of the lattice with an appropriate Jacobian of transformation inserted.[4]

This leads to a procedure by which the coefficients of the character expansion of Eq. (3) may be determined.[3] This is based on the measurement of the ratio of expectation values of the operators indicated below

$$\frac{f_r d_r}{f_1}=\frac{\left\langle \dfrac{X_r(U_p)}{B(U_p)}\right\rangle}{\left\langle \dfrac{1}{B(U_p)}\right\rangle}\ , \tag{7}$$

where $B(U_p)$ represents the appropriate Jacobian factor mentioned above.

This determination is an operational definition of the

TABLE I. Measured character expansion coefficients $f_r d_r / f_1$, action parameters β_r, and plaquette characters $\langle \chi_r \rangle$ as a function of the lattice size. The column $L=\infty$ indicates the exact values. (a) $\beta_2=1.25$. (b) $\beta_2=1.50$.

	r	$L=4$	$L=6$	$L=8$	$L=10$	$L=12$	$L=14$	$L=\infty$
					(a)			
$\frac{f_r d_r}{f_1}$	2	1.0427(6)	1.0378(4)	1.0371(1)	1.0361(4)	1.0358(4)	1.0352(1)	1.0144
	3	0.5187(5)	0.5798(3)	0.5790(2)	0.5787(2)	0.5787(3)	0.5787(3)	0.5655
	4	0.2258(8)	0.2265(5)	0.2256(4)	0.2264(4)	0.2251(3)	0.2251(4)	0.2193
	5	0.0668(8)	0.0655(4)	0.0660(5)	0.0663(3)	0.0660(3)	0.0662(5)	0.0650
	6	0.0165(8)	0.0158(4)	0.0156(5)	0.0157(3)	0.0149(2)	0.0151(5)	0.0157
	7	0.0028(8)	0.0022(4)	0.0020(5)	0.0028(3)	0.0021(2)	0.0025(5)	0.0032
β_r	2	1.396(12)	1.370(7)	1.364(6)	1.358(5)	1.348(5)	1.342(3)	1.25
	3	−0.10(1)	−0.09(1)	−0.08(1)	−0.076(6)	−0.063(6)	−0.055(5)	0
	4	0.06(2)	0.06(1)	0.05(1)	0.050(7)	0.035(7)	0.027(6)	0
	5	−0.04(2)	−0.05(1)	−0.04(1)	−0.034(7)	−0.021(7)	−0.012(7)	0
$\langle \chi_r \rangle$	2	1.3103(3)	1.3057(2)	1.3048(1)	1.3048(1)	1.3037(1)	1.30411(7)	
	3	1.0001(7)	0.9924(3)	0.9907(2)	0.9906(2)	0.9887(2)	0.9894(1)	
	4	0.5427(7)	0.5360(3)	0.5344(2)	0.5343(2)	0.5328(2)	0.5333(1)	
	5	0.2272(7)	0.2231(3)	0.2221(2)	0.2220(2)	0.2212(2)	0.2214(1)	
	6	0.0771(6)	0.0752(3)	0.0745(2)	0.0745(2)	0.0741(2)	0.0742(1)	
	7	0.0222(6)	0.0214(3)	0.0208(2)	0.0209(2)	0.0208(2)	0.0206(1)	
					(b)			
$\frac{f_r d_r}{f_1}$	2	1.118(2)	1.1183(7)	1.1158(9)	1.1161(8)	1.1157(5)	1.1142(7)	1.1358
	3	0.688(1)	0.6897(4)	0.6893(4)	0.6886(6)	0.6891(2)	0.6892(5)	0.7283
	4	0.303(1)	0.3038(5)	0.3035(6)	0.3047(1)	0.03032(4)	0.3029(7)	0.3295
	5	0.100(2)	0.1020(5)	0.1013(4)	0.1050(3)	0.1015(3)	0.1016(6)	0.1154
	6	0.025(2)	0.0270(5)	0.0271(5)	0.0318(5)	0.0272(2)	0.0272(4)	0.0329
	7	0.005(2)	0.0061(5)	0.0061(6)	0.0078(2)	0.0059(3)	0.0059(4)	0.0078
β_r	2	1.63(6)	1.59(2)	1.55(2)	1.56(2)	1.55(1)	1.52(1)	1.50
	3	−0.20(7)	−0.15(2)	−0.12(2)	−0.12(2)	−0.11(1)	−0.08(1)	0
	4	0.15(6)	0.11(2)	0.09(2)	0.08(2)	0.10(2)	0.06(1)	0
	5	−0.08(5)	−0.07(2)	−0.07(2)	−0.04(5)	−0.06(1)	−0.04(1)	0
$\langle \chi_r \rangle$	2	1.4483(3)	1.4467(1)	1.4464(1)	1.4465(7)	1.4462(6)	1.44640(6)	
	3	1.2838(6)	1.2800(3)	1.2794(2)	1.2796(2)	1.2796(1)	1.2795(1)	
	4	0.8373(8)	0.8329(3)	0.8324(2)	0.8326(2)	0.8328(2)	0.8326(1)	
	5	0.4304(8)	0.4270(3)	0.4267(2)	0.4270(2)	0.4272(2)	0.4269(1)	
	6	0.1810(7)	0.1793(3)	0.1792(2)	0.1795(2)	0.1797(2)	0.1793(1)	
	7	0.0636(7)	0.0634(3)	0.0634(2)	0.0636(2)	0.0637(2)	0.0634(1)	

TABLE II. Effective action as a function of lattice size. $L=\infty$ indicates values of the simulated action.

	r	$L=4$	$L=6$	$L=8$	$L=10$	$L=12$	$L=\infty$
β_r	2	0.912(2)	0.908(1)	0.907(1)	0.907(1)	0.907(1)	0.90
	3	−0.008(3)	−0.004(2)	−0.002(1)	−0.004(1)	−0.003(1)	0.0
	4	0.006(3)	0.003(2)	0.000(1)	0.001(1)	0.001(2)	0.0
	5	−0.005(3)	−0.003(2)	−0.000(1)	−0.001(1)	−0.001(2)	0.0
β_r	2	1.186(6)	1.159(2)	1.156(2)	1.156(2)	1.157(1)	1.10
	3	−0.041(8)	−0.031(3)	−0.026(2)	−0.024(1)	−0.027(1)	0.0
	4	0.020(9)	0.017(3)	0.015(2)	0.011(2)	0.014(1)	0.0
	5	−0.012(9)	−0.011(4)	−0.010(3)	−0.006(1)	−0.008(1)	0.0
β_r	2	1.51(3)	1.46(1)	1.44(1)	1.44(1)	1.43(1)	1.35
	3	−0.14(4)	−0.11(1)	−0.08(2)	−0.08(1)	−0.08(1)	0.0
	4	0.10(4)	0.08(1)	0.06(2)	0.05(1)	0.06(1)	0.0
	5	−0.06(4)	−0.05(2)	−0.04(2)	−0.02(1)	−0.04(1)	0.0

action on the lattice under consideration. Even in the limit of infinite accuracy of measurement these coefficients should reflect the effects of the finite size of the lattice whenever these do occur. This in turn will be reflected as a finite-size dependence of the coefficients β_r of the measured action on the lattice. These coefficients may, of course, be shifted from the parameters of the action which is being simulated. Therefore, for any particular lattice simulation, we have a measurement of an "effective action" which reflects the effects of the finiteness of the lattice size and which represents the true description of the configurations generated by the simulation for this lattice. We perform these measurements for SU(2) lattice gauge theory for various values of the coupling and for lattices of sizes from 4^4 to 14^4. The main result is that a shift in the coupling parameters is observed at each lattice size as the values of β_2 approach the crossover region and beyond into the weak-coupling domain. The shift is most pronounced for small lattices and, as expected, decreases with the increase of lattice size.

We simulate SU(2) gauge theory using the heat-bath algorithm for a pure Wilson action for character coefficient η_2 up to 1.5. The effective action shows no shift for values of β_2 less than 0.9. For values larger than 0.9 the shift grows with β_2 and the effective action develops small terms of higher-dimensional characters of the group leading to an effective action that is in general of the mixed type.

II. RESULTS

We have performed measurements of the character expansion coefficients for the Boltzmann factor for the action defined by

$$S=\beta_2[\chi_2(U_p)-2]\ , \tag{8}$$

where χ_2 is the trace of the simple plaquette in the fundamental representation (Wilson action). The values of β_2 reported here are 0.9, 1.1, 1.25, 1.35, and 1.5. (Note that in this normalization our β_2 is one-half the value usually quoted in some of the literature.) Tables I(a) and I(b) show the results, at various values of the lattice size, for $\beta_2=1.25$ and 1.5, respectively. We give the results of the measurements for the ratios of $f_r d_r/f_1$ for values of r up to 7 and the corresponding measured effective action parameters for all lattice sizes considered. The last column, labeled $L=\infty$, gives the exact (infinite volume) expected values for these ratios. Also reported are the values, as measured on the various lattices, of the Monte Carlo expectation value of the characters in the rth representation of the group over the plaquettes of the lattice. These last operators have the same dimension as the coefficients measured above and hence must scale with the size of the lattice in a similar manner. This in fact can be seen in Fig. 1 where representative quantities are plotted together and where each set of data is accordingly labeled: F_r represents $f_r d_r/f_1$ and χ_r represents $\langle \chi_r \rangle$.

Figure 2 shows the variation with lattice size of the effective action as measured for the value of $\beta_2=1.25$. This plot also shows the significant components of higher di-

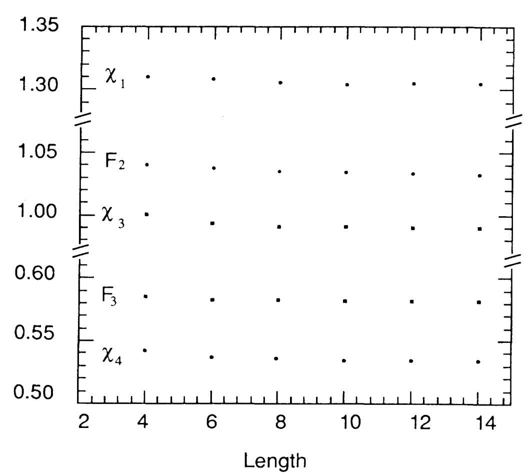

FIG. 1. Comparison of measured values of the characters of the simple plaquettes with the coefficients of the character expansion of the Boltzmann factor for $\beta_2=1.25$ as a function of lattice size.

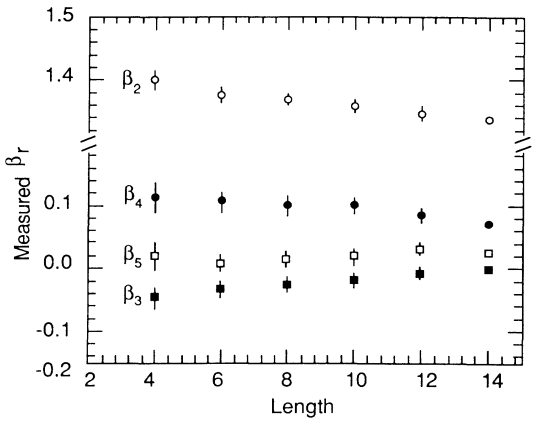

FIG. 2. Variation of the measured parameters β_r of the action as a function of lattice size for a simulation with $\beta_2=1.25$.

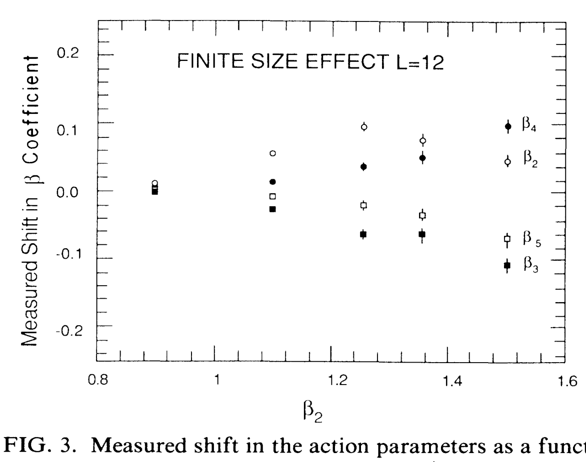

FIG. 3. Measured shift in the action parameters as a function of β_2 of the simulated action at lattice size 12^4.

mensional characters generated by the finite-size effects. Similar data for other values of β_2 are given in Table II. Note the absence of a shift for $\beta_2=0.9$ except for the lattice with size 4^4. Note also that whereas a significant shift in the coefficient of the fundamental representation persists for $\beta_2=1.25$ even for the 14^4 lattice, this shift is much smaller for the larger value of $\beta_2=1.5$ and which is farther away from the crossover region. In fact one can see a trend that indicates that the simulation for larger lattices has effective action parameters in the fundamental representation closer to the intended values for values of β_2 further away from the crossover region than for these closer to it. However, the coefficients of higher dimensional characters grow with β_2, although their actual magnitudes are smaller for larger and larger lattices as can be seen from Tables I(a) and I(b). We show this trend for the lattice 12^4 for the first four β_2 in Fig. 3.

III. DISCUSSION

The determination of the effective actions above is done using our method of Ref. 3. Several observations lead us to believe that this determination is accurate. First, for a large range in coupling there is no shift observed and our method gives a very accurate determination of the unshifted action. Second, the shift is observed and increases gradually as one approaches the crossover region where the correlation length increases and where such a shift, if it exists, is expected to be. Third, the measurement process can be effectively shifted to regions of β_2 where the method is known to be accurate and the shift is still observed. This is done by measuring the character expansion of the action being simulated minus a term whose coefficient can be tuned at will. It is clear that if one measures in the manner of Ref. 3 the expectation value over plaquette variables of $\chi_r(U_p)\exp(S')$ instead of $\chi_r(U_p)$, then one is measuring the expansion coefficients for $S+S'$. By choosing S' at will one can put this sum in the range of coupling where our method is known to be accurate. For example, if S' is taken as $-0.5[\chi_2(U_p)-2]$ for the point $\beta_2=1.25$, one, in the absence of a shift, expects to measure the resultant action as one with $\beta_2=1.25-0.5=0.75$. The result on the 8^4 lattice is however $\beta_2=0.864$, $\beta_3=-0.08$, $\beta_4=0.05$ which is the "true" shifted action as measured on that lattice directly plus the parameter $\beta_2'=-0.5$ introduced by hand. In fact, one can vary this parameter at will and in all cases the coefficients here reflected the same shift as the case with $S'=0$ indicating that S itself has this shift and it is not an error of measurement. Further, if one performs tests similar to the above in the range of β_2 where no shift is expected, no shift is observed and this assures us that the process indicated above is an accurate procedure. Fourth, the coefficients of the character expansion of the Boltzmann factor have the same dimension as the simple Wilson plaquette and hence are expected to scale with volume in the same manner. This is borne out by the data for all values of β_2 for which measurements were taken.

All these measurements were done after the lattice parameters were observed to stabilize indicating that equilibration had been reached. This typically meant taking about five thousand sweeps on a 4^4 lattice, ten thousand sweeps for a 6^4 lattice, and over twenty thousand sweeps for 8^4 and 10^4 lattices and higher. In the case of $\beta=1.25$ on an 8^4 lattice there was no measurable change in the shifts even after eighty thousand sweeps, clearly indicating that the measured shift is not merely a consequence of lack of equilibration unless the equilibration rate is extremely slow. In fact, using our method of Ref. 3 we have performed a quantitative assessment of the approach to equilibration. One observes a clear approach to an equilibrium value for the measured action as the number of Monte Carlo sweeps increases. For, whereas it is a matter of 100 sweeps at $\beta_2=0.8$ for a 8^4 lattice to converge to an unshifted action, the number rapidly increases to twenty thousand at $\beta_2=1.25$ to converge to the shifted action quoted above without any further and noticeable change beyond that. A more detailed presentation of this study will be given elsewhere.

Finally a word concerning the finite-size dependence intrinsic to the method we have used. As pointed out in Ref. 3 the Jacobian of transformation from link to plaquette variables consists, in the case of a finite lattice with period boundary conditions, of two types of terms. The first type are δ functions over the boxes of the lattice, the "box" terms mentioned above and these also exist for an infinite lattice, and the second (see Ref. 5) are δ functions over a whole plane traversing the lattice, one per type of plane, and these do not exist for the infinite lattice. These planar δ functions are known strictly for Abelian gauge theory and require that the product of all plaquettes on the plane be equal to unity. Their argument is not as simple for the non-Abelian version and is not known in a simple closed form in terms of plaquette variables alone. In order to estimate the influence of such terms on our measurement. We have implemented these planar δ functions assuming the argument to be that of the Abelian version and found an insignificant change from the results obtained in their absence except in very small lattices of size 2^4. This is partly due to the strong damping factors of $(2J+1)^{-(L^2)}$ multiplying each term of spin J in the factorized form of the character expansion of such δ functions and where L^2 is the area of the plane. These quickly damp out all terms except the identity for L greater than four. Thus barring some unforeseen influence of precisely such unknown terms we do not expect them to influence the result beyond the errors we are able to get without them.

Thus, in view of the above, we interpret our results as a measurement of the "effective finite-size action" for the simulations under consideration.

IV. CONCLUSION

The main observation we can make is that the effective coupling parameters in any lattice-gauge-theory simulation are not generally those intended by the simulation except in regions of parameter space describing small correlation lengths and the impractical but realistic case of infinite-size lattice. For all finite-size lattices and values of the coupling parameters where the correlation length is expected to grow, the parameters of the effective measured action show a size-dependent shift from the intended values. We have reported here on such measurements for the range in β_2 of interest for SU(2) gauge-theory simulations. Knowledge of such shifts should be of importance primarily for calculations and data fits that depend crucially on the precise values of these parameters such as measurements of critical exponents, location of finite-temperature phase transitions, location of other phase transitions, measurement of glueball masses, and the like.

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[1] M. N. Barber, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, London, 1983), Vol. 8; M. Lüscher, Phys. Lett. **118B**, 391 (1982); Nucl. Phys. **B219**, 233 (1983); M. Lüscher and G. Münster, *ibid.* **B232**, 445 (1984).

[2] M. Nauenberg, T. Schalk, and R. Brower, Phys. Rev. D **24**, 548 (1981); R. Brower, M. Creutz, and M. Nauenberg, Nucl. Phys. **B210**, 133 (1982).

[3] K. M. Bitar, Phys. Lett. **168B**, 391 (1986).

[4] G. Batrouni, Nucl. Phys. **B208**, 467 (1982); **B208**, 12 (1982).

[5] G. Batrouni and M. B. Halpern, Phys. Rev. D **30**, 1782 (1984); see also, **30**, 1775 (1984).