PHYSICAL REVIEW D VOLUME 35, NUMBER 2 15 JANUARY 1987

Renormalized coupling constants by Monte Carlo methods

A. Gonzalez-Arroyo* and M. Okawa
National Laboratory for High Energy Physics (KEK), Tsukuba, Ibaraki 305, Japan
(Received 31 January 1986)

A new method is proposed to determine the evolution of coupling constants under the renormalization-group study, by a single Monte Carlo simulation. The method makes use of Schwinger-Dyson equations and has several advantages compared to other alternatives. Application to scalar field theory in three dimensions is presented. The fixed point of this theory is determined. The case of gauge field theory is also discussed briefly.

I. INTRODUCTION

The renormalization-group method[1] is one of the most important concepts in quantum field theory (QFT) and critical phenomena.[2] There has been considerable interplay between both fields during the last 15 years. The lattice formulation of quantum field theories[3] does in fact open the way to nonperturbative methods often borrowed from statistical mechanics. One such technique is that of Monte Carlo simulation[4] which has led to considerable success in application to field theories.[5] In addition to direct simulation of lattice theories there has been increasing interest in combining the renormalization-group approach with Monte Carlo simulations following the pioneering work of Ma and Swendsen.[6] The so-called Monte Carlo renormalization-group method[7] seems more powerful in determining critical properties or renormalized quantities than ordinary simulations.

The most important applications of the Monte Carlo renormalization-group method have been in determining the critical exponents of several lattice systems.[8] In addition to these quantities it is desirable to study the renormalization of the coupling constants themselves and the position of the fixed point. Some advantages are expected and reported[9] by performing simulations along the renormalized trajectory in the vicinity of the fixed point. Basically the scaling behavior of observables is simpler on this trajectory and given by the underlying continuum field theory. Considerable deviations from asymptotic scaling have been observed in gauge theories when approaching the critical surface along other trajectories.

This paper presents a new method of determining the renormalization of coupling constants by Monte Carlo techniques. The method makes use of the Schwinger-Dyson equations and, thus, is of considerable generality. The simplest case, that of scalar field theories, is explicitly developed in Sec. II. To illustrate the idea and show its practical applicability we carried out the study of three-dimensional scalar field theory. This case is in itself interesting for statistical mechanics and our data are presented in Sec. III. In Sec. IV we summarize our results and discuss how to include gauge interactions.

Before presenting the details of our method it is interesting to point out its advantages with respect to other proposals. The first works on this subject used a comparison of two simulations and two lattices.[10] This requires more computer time than methods using one simulation and furthermore statistical errors are amplified. Swendsen[11] proposed a method which is free of this difficulty based on an idea by Callen.[12] An application of it to four-dimensional scalar theory was done by Lang.[13] This method is, however, less simple than ours. The equations that determine the couplings are nonlinear and have to be solved by iterative use of the linearized local form of the equations. This fact suggests that Swendsen's method might consume more time than our method. This might turn out to be compensated or enhanced if less statistics is required in one method or other to obtain the same accuracy for the couplings. For theories with unconstrained fields our method has the definite advantage of requiring essentially no more time than the one required to obtain the critical exponents alone.

There are other methods proposed by several authors[14] but in general the one presented in this paper seems superior for one or another of the following reasons.

(a) The physical principle underlying the method is simple: Use Schwinger-Dyson equations to determine the couplings.

(b) Only one Monte Carlo simulation is required to obtain both the original couplings and the renormalized ones.

(c) The set of equations which determine the couplings is in most cases linear and thus easy to solve.

(d) The expectation values required are for scalar theories, the same ones used in computing critical exponents.

(e) The method is valid for any lattice theory although with varying degrees of simplicity.

(f) There are more equations than unknowns. This allows a number of checks to be done on the approximations and other systematic effects.

II. GENERAL FRAMEWORK

A. Notation and conventions

Consider the set of all operators integrated over the whole lattice of volume V. We label each operator by an index and denote it by $O_i^{(0)} \equiv O_i^{(0)}(\phi)$. The action can then be written as a linear combination of these operators:

$$S^{(0)} = \sum_i \beta_i^{(0)} O_i^{(0)} , \tag{2.1}$$

where $\beta_i^{(0)}$ are the different couplings.

The expectation values of these operators as well as their connected correlations are denoted by

$$A_i^{(0)} = \langle O_i^{(0)} \rangle_{\beta^{(0)}} , \tag{2.2a}$$

$$\underline{H}_{ij}^{(0,0)} = \langle O_i^{(0)} O_j^{(0)} \rangle_{\beta^{(0)}}^c , \tag{2.2b}$$

where the superscript c stands for connected part and the subscript $\beta^{(0)}$ means that the averages are taken with respect to the action (2.1).

Now consider a renormalization-group transformation with a scale factor 2. The block-spin variable $\phi^{(1)}$ is a function of the original spin variable ϕ. For example, for a linear renormalization-group transformation, $\phi^{(1)}$ is given by

$$\phi^{(1)}(\text{block}) = \frac{\lambda}{2^d} \sum_{x \in \text{block}} \phi(x) , \tag{2.3}$$

where d is the space-time dimension and λ is a parameter which has to be tuned in order to have a fixed point. The renormalized operators are defined as

$$O_i^{(1)}(\phi) = O_i^{(0)}(\phi^{(1)}) . \tag{2.4}$$

We will assume for simplicity that the operators O_i are homogeneous functions of the field with dimension d_i:

$$O_i(\mu\phi) = \mu^{d_i} O_i(\phi) . \tag{2.5}$$

The renormalized action is specified by the renormalized couplings $\beta_i^{(1)}$,

$$S^{(1)} = \sum_i \beta_i^{(1)} O_i^{(0)} , \tag{2.6}$$

and is defined as

$$\langle F(\phi) \rangle_{\beta^{(1)}} = \langle F(\phi^{(1)}) \rangle_{\beta^{(0)}} . \tag{2.7}$$

where F is any functional (operator) of the field ϕ.

In a similar fashion we can consider iterating the renormalization-group transformation and label $O_i^{(a)}, \beta_i^{(a)}$ the corresponding renormalized operators and couplings after a (1,2,3, etc.) iterations. In what follows we will be concerned with the expectation values

$$A_i^{(a)} = \langle O_i^{(a)} \rangle_{\beta^{(0)}} , \tag{2.8a}$$

$$\underline{H}_{ij}^{(a,b)} = \langle O_i^{(a)} O_j^{(b)} \rangle_{\beta^{(0)}}^c . \tag{2.8b}$$

In Swendsen's method of determining the critical exponents a major role is played by the transition matrices

$$\underline{T}_{ij}^{(a,b)}(\beta^{(0)}) = \frac{\partial \beta_i^{(a)}}{\partial \beta_j^{(b)}} \tag{2.9}$$

which can be computed in terms of the correlation matrices $\underline{H}$ as follows:

$$\underline{T}^{(a,b)} = (\underline{H}^{(c,a)})^{-1} \underline{H}^{(c,b)} \tag{2.10}$$

with $c \geq a,b$.

B. Schwinger-Dyson equations

The problem of determining the renormalized couplings in terms of the expectation values has been considered many times in the literature. Here we present a simple method which can be applied to any statistical system including gauge theories. The method takes its simplest form for spin theories with unconstrained continuous fields, i.e., $-\infty < \phi(x) < +\infty$. In this case the same quantities which enter Swendsen's determination of the critical exponents suffice to determine the renormalized couplings. The method is based on Schwinger-Dyson equations and we will now describe it for the case of scalar field theories, with no internal degrees of freedom.

We consider the expectation value of the field $\langle \phi(x) \rangle_{\beta^{(0)}}$ at some given point x on the lattice. We can obtain a Schwinger-Dyson equation by changing variables in the path integral from $\phi(x) \to \phi(x) + \epsilon$ and equating to zero the terms linear in ϵ:

$$\left\langle \frac{\delta \phi(x)}{\delta \phi(x)} \right\rangle - \left\langle \phi(x) \frac{\delta S^{(0)}}{\delta \phi(x)} \right\rangle = 0 . \tag{2.11}$$

Notice that the change of variables is performed at the same point x where the field is averaged. Summing over x and using (2.1) and (2.5) we obtain

$$V = \sum_i A_i^{(0)} d_i \beta_i^{(0)} . \tag{2.12}$$

This relation is a simple consequence of the properties of the path integral under a scale transformation on the field.

In a similar way applying the same steps to the renormalized actions and using (2.7) we get

$$V^{(a)} = \sum_i A_i^{(a)} d_i \beta_i^{(a)} , \tag{2.13}$$

where $V^{(a)} = V/2^{da}$ is the volume of the renormalized lattice.

A new set of Schwinger-Dyson equations can be obtained by applying the same change of variables $[\phi^{(b)}(x) \to \phi^{(b)}(x) + \epsilon]$ to the quantities $\langle O_i^{(a)} \phi^{(b)}(x) \rangle$ with $a \geq b$. We obtain

$$d_i A_i^{(a)} = \sum_j \underline{H}_{ij}^{(a,b)} d_j \beta_j^{(b)} . \tag{2.14}$$

It is important to distinguish the cases $a > b$ and $a = b$. The former case is only valid if the renormalization-group transformation is linear. Formula (2.14) for $a = b$ is, however, valid for arbitrary renormalization-group transformations and can be obtained by differentiating (2.13) with respect to $\beta^{(a)}$. In both cases, one can solve for the couplings

$$\beta_j^{(b)} = \frac{1}{d_j} \sum_k (\underline{H}^{(a,b)})^{-1}_{jk} d_k A_k^{(a)} \tag{2.15}$$

provided the matrices $\underline{H}^{(a,b)}$ are invertible.

The relation (2.13) is not necessary to obtain the couplings and provides a cross-check of the validity of the truncations. Formula (2.15) can be used to reobtain the original couplings ($b = 0$) and thus measure the statistical and systematic errors in the quantities.

Combining (2.15) for different values of a, we obtain

$$d_j \beta_j^{(b)} = \sum_i \underline{T}_{ji}^{(b,a)} d_i \beta_i^{(a)} \tag{2.16}$$

involving the transition matrices $\underline{T}^{(b,a)}$. Therefore, given

these matrices and the original couplings one can obtain the renormalized couplings through a simple linear relation. Notice, however, that (2.16) is only valid for linear renormalization-group transformations.

Additional sets of Schwinger-Dyson equations involving correlations of three operators can be derived but we will not use them in what follows.

C. Truncation

In all numerical studies of the renormalization group one is necessarily forced to restrict oneself to a finite number of operators O_i. It is, therefore, necessary to consider what is the effect of this truncation on the quantities of interest.

Suppose that we restrict ourselves to a set of operators $O_1,\ldots,O_N$ and we assume that $\beta_j=0$ for $j>N$. It is then possible to solve for $\beta_1,\ldots,\beta_N$ using (2.14) for $a=b$ restricted to $i,j=1,\ldots,N$. The resulting couplings $\beta'_1,\ldots,\beta'_N$ depend on the assumed truncation in coupling-constant space. Let us denote by S' the action computed with these couplings. Then, we may rewrite Eq. (2.14) for $a=b$ as

$$\langle O_i(\dot{S}-\dot{S}')\rangle^c=0,\quad i=1,\ldots,N\ , \tag{2.17}$$

where S is the true action and

$$\dot{S}=\sum_x \phi(x)\frac{\delta S}{\delta\phi(x)}\ . \tag{2.18}$$

From these relations the interpretation of S' follows: of all truncated actions (containing only the first N operators) S' is the "closest" to the true action S.

In the previous statement we have implicitly introduced a distance operator on functional space. It is the one corresponding to the norm

$$||A(\phi)||^2\equiv\langle[\dot{A}(\phi)]^2\rangle^c\ , \tag{2.19}$$

where the overdot is defined as in (2.18). Now it is easy to see that the equations expressing the extremum condition for $||S-S'||^2$ coincide with (2.17).

For (2.19) to define a norm, it must satisfy that the only functional with zero norm is the zero functional. This is only true if we define functionals modulo additive constants, as is physically natural for an action functional. Furthermore it assumes that there is no functional of the field with zero covariance. This assumption turns out to be false for theories in the large-N limit (N being the range of the internal index). However, this difficulty is unavoidable in any method which computes the couplings in terms of expectation values, and furthermore replacing S by S' would have no effect on the quantum properties of the theory.

Returning to our truncated action S' we stress that a nice advantage of our method is that it provides the *optimal* approximation to the action with respect to the norm (2.19). It is, however, desirable to check how good the approximation (truncation) is, i.e., how small is $||S-S'||$. Upon integration by parts and using (2.17) we obtain

$$||S-S'||^2=\langle\ddot{S}-\ddot{S}'\rangle\ . \tag{2.20}$$

Unfortunately, it is impossible to calculate (2.20) without knowing the exact action S. There is one exception, namely, when $S-S'$ has a definite dimension D as for the Gaussian model.[15] In this case we can use expression (2.13) since

$$V-\sum_i A_i d_i\beta'_i=\langle\dot{S}-\dot{S}'\rangle=\frac{1}{D}\langle\ddot{S}-\ddot{S}'\rangle\ . \tag{2.21}$$

As a corollary we notice that for a Gaussian theory only the exact solution satisfies the finite set of equations (2.13) and (2.17). For interacting theories expression (2.21) may be approximated for a given truncation with some noninteger D.

In our practical application we will employ two methods to check how good is a given truncation. First, one can check how well equation (2.13) is satisfied. Second, we will agree to say that S' is a good approximation of S whenever $S'-S$ is a small operator in the sense that

$$\langle(S-S')^2\rangle^c\ll V\ . \tag{2.22}$$

Although (2.22) cannot be directly checked, it is at least possible to compare subsequent truncations. This criterion seems more reasonable than comparing the values of the couplings themselves, since different couplings could affect expectation values in the same way as explicitly shown in Sec. III B.

D. Fixed point

Here we list several consequences of the set of linear relations which provide the renormalized couplings in terms of the original ones. Let $\boldsymbol{\omega}^{(0)}$ be the vector orthogonal to the critical surface at point $\beta^{(0)}$, and $\boldsymbol{\omega}^{(a)}$ the corresponding one at the renormalized points. Then, since the critical surface is mapped onto itself under the renormalization group, we conclude that

$$\boldsymbol{\omega}^{(a)}\underline{T}^{(a,b)}=\mu\boldsymbol{\omega}^{(b)} \tag{2.23}$$

and therefore at the fixed point $\boldsymbol{\omega}^*$ is a left eigenvector of the $\underline{T}$ matrix.[16] It must necessarily correspond to the largest eigenvalue (we have assumed that there is only one vector orthogonal to the critical surface at each point).

Furthermore, as a consequence of (2.16) we have that the vector $\mathbf{v}^*$,

$$(\mathbf{v}^*)_i=d_i\beta_i^*\ , \tag{2.24}$$

is a right eigenvector of the $\underline{T}$ matrix with eigenvalue 1 and orthogonal to $\boldsymbol{\omega}^*$. The existence of a unit eigenvalue is a consequence of the invariance under rescaling of the field which ensures that if (β_1^*) is a fixed point, then $(\mu^{d_i}\beta_i^*)$ is also a fixed point for any value of μ. Thus the eigenvector corresponding to the marginal operator gives the fixed-point location.

In fact, on the critical surface we must have $\boldsymbol{\omega}^{(a)}\cdot\mathbf{v}^{(a)}=0$, where

$$(\mathbf{v}^{(a)})_i=d_i\beta_i^{(a)}\ . \tag{2.25}$$

This relation is consistent with the fact, substantiated by the data, that the divergent part of the correlation ma-

trices $\underline{H}^{(a,b)}$ has the form

$$\underline{H}^{(a,b)} \sim \xi^{(a,b)} \boldsymbol{\omega}^{(a)} \times \boldsymbol{\omega}^{(b)} , \tag{2.26}$$

where $\xi^{(a,b)}$ diverges as we approach the critical surface. However, since $\boldsymbol{\omega}^{(b)}$ is orthogonal to $\mathbf{v}^{(b)}$ it follows that the vector $\mathbf{A}^{(a)}$ stays finite [see (2.14)]. Therefore, $\boldsymbol{\omega}^{(a)} \cdot \mathbf{v}^{(a)}$ measures how far we sit from the critical surface and our data show that this quantity is small and grows as a grows, as expected.

To conclude this section we notice that the fixed-point couplings can be computed from the data. A first method, mentioned previously below (2.24), gives the fixed-point couplings in terms of the eigenvector of unit eigenvalue of the $\underline{T}$ matrix at the fixed point. Couplings are determined only modulo a normalization factor. The second method is to solve the linear relation (approximation)

$$\beta^{(a+1)} - \underline{T}^{(a+1,a)} \beta^{(a)} = (I - \underline{T}^{(a+1,a)}) \beta^* , \tag{2.27}$$

where the couplings are determined modulo the addition of an eigenvector of unit eigenvalue. In principle, both methods put together allow a complete determination of the fixed-point couplings. In practice, the determination of the fixed-point couplings faces the difficulty of the extrapolation of the $\underline{T}$ matrix to the fixed point and the problem of higher-order corrections to formula (2.27). In addition, if we sit away from the fixed point one may question which of the infinitely many fixed points is selected. All that one can say is that it must be close enough to $\beta^{(a)}$ and $\beta^{(a+1)}$.

III. APPLICATION TO THREE-DIMENSIONAL SCALAR THEORY

A. Monte Carlo data

We have analyzed the lattice theory of a single scalar field in three dimensions, using the method presented in the previous section. For definiteness, we have adopted the linear renormalization-group transformation (2.3).

We only used eight operators in our analysis which are listed in Table I. In order to locate the fixed point we took data at three different regions in coupling-constant space. The second and third regions were selected on the basis of the previous runs aiming to get closer to the fixed point. The three regions are as follows.

Region I: $\beta_1 = -1.19, -1.2, \beta_2 + 3\beta_1 = -0.1, \beta_3 = -1,$
$\beta_4 = 1$;

Region II: $\beta_1 = -4.5, \beta_2 + 3\beta_1 = -1.002, 1, \beta_3 = 2.5$,
$\beta_4 = 2.5$;

Region III: $\beta_1 = -1, \beta_2 + 3\beta_1 = -0.244, \beta_3 = 0.18,$
$\beta_4 = 0$.

The data were taken on a HITAC S810/10 vector computer at KEK. We used a 32^3 lattice for the three regions and in addition a 16^3 lattice for region III. A Metropolis algorithm was used to update the configurations and the acceptance rate was about 50%. In regions I and II only the first six operators O_i were used in the analysis. The three regions of coupling constants were seen to correspond to the vicinity of the critical surface, the correlation length being roughly 9 in all cases.

TABLE I. List of the operators used in the analysis. $\hat{\boldsymbol{\mu}}$ is the unit vector in the μ direction and $\mu \neq \nu \neq \lambda$.

O_1	$\sum_{x,\mu} \phi(x)\phi(x+\hat{\boldsymbol{\mu}})$
O_2	$\sum_{x} \phi^2(x)$
O_3	$\sum_{x} \phi^4(x)$
O_4	$\sum_{x} \phi^6(x)$
O_5	$\sum_{x,\mu,\nu} \phi(x)\phi(x+\hat{\boldsymbol{\mu}}+\hat{\boldsymbol{\nu}})$
O_6	$\sum_{x,\mu,\nu,\lambda} \phi(x)\phi(x+\hat{\boldsymbol{\mu}}+\hat{\boldsymbol{\nu}}+\hat{\boldsymbol{\lambda}})$
O_7	$\sum_{x,\mu} \phi(x)\phi(x+2\hat{\boldsymbol{\mu}})$
O_8	$\sum_{x} \phi^8(x)$

The analysis of renormalized coupling constants was performed truncating the space of couplings sequentially from 1 to the maximum value (6 or 8). In all cases a few million iterations were performed initially to allow the system to reach thermal equilibrium.

Then, the matrices $\underline{H}$ and vectors $\mathbf{A}$ were computed by averaging over a given set of groups of configurations, each group consuming roughly one hour of computer time. The length of each group and number of groups is as follows.

Region I. Each group contains 1250 configurations separated by 100 sweeps. There are 20 and 13 groups for each value of the couplings Ia ($\beta_1 = -1.19$) and Ib ($\beta_1 = -1.2$), respectively.

Region II. Groups of 2000 configurations separated by 50 sweeps. There are 40 groups used in this analysis for each value of the coupling IIa ($\beta_2 + 3\beta_1 = -1.002$) and IIb ($\beta_2 + 3\beta_1 = -1$).

Region III. 60 groups of 1600 configurations separated by 50 sweeps for the 32^3 lattice (IIIa) and 40 groups of 10 000 configurations for the 16^3 lattice (IIIb).

B. Renormalization of coupling constants

The problem of the unknown value of λ in formula (2.3) which appears in the determination of the critical exponents does not show up in the determination of renormalized couplings since the change of the couplings under a change of λ is simply given by

$$\beta_i^{(a)}(\lambda) = \left[\frac{\lambda_0}{\lambda} \right]^{a d_i} \beta_i^{(a)}(\lambda_0) . \tag{3.1}$$

We choose to present the data in the form of the λ-independent ratios:

$$x_i^{(a)} = \frac{\beta_i^{(a)}}{|\beta_1^{(a)}|^{d_i/2}} \tag{3.2}$$

which corresponds to the replacement of the linear renormalization-group transformation by a nonlinear one obtained by selecting $\lambda(\beta)$ so that $\beta_1^{(a)} = -1$. For completeness we also give the ratios

$$\rho^{(a)}=\frac{\beta_1^{(a)}}{\beta_1^{(0)}} \tag{3.3}$$

determined for $\lambda=\lambda_0=1.5$. We choose to normalize our values to β_1 since this quantity has smaller errors than other couplings. However, it is to be noticed that this definition corresponds to a different normalization in terms of continuum operators for each truncation in the coupling-constant space.

The results of our analysis are presented in Tables II–V for the values starting from the initial coupling constants Ia, IIa, IIIa, and IIIb. The numbers quoted have been obtained from the data using formula (2.15) with $a=b$. For $a=0$ the value of ρ quoted gives the ratio of the determined value of $\beta_1^{(0)}$ to the true value used in the simulation.

The tables give the values of the ratios for different truncations in coupling constant space from three up to the maximum number of operators used, and for all blocking levels from zero to three. We also give the particular combination of couplings corresponding to M^2/β_1, where M^2 is the lattice mass square

$$M^2=d\beta_1+\beta_2+2\begin{bmatrix}d\\2\end{bmatrix}\beta_5+4\begin{bmatrix}d\\3\end{bmatrix}\beta_6+d\beta_7 \tag{3.4}$$

in d space-time dimensions. We did not use expression (2.13) in the determination of the couplings and in order to check it we also present the lattice volume as determined from this formula.

The numbers in parentheses give the errors of the corresponding couplings and affect the last digits of the value quoted. For example, 1.868(17) means 1.868 ± 0.017. These errors have been determined by calculating each particular quantity for every group of configurations and studying the dispersion in the results, obtained for the different groups. Larger errors are obtained for the original couplings than for the renormalized ones. A full covariance analysis does in fact show strong correlations among the couplings. In particular x_5+x_6 has much smaller errors than x_5-x_6, which explains the discrepancy of the result for the zero blocking level with respect to the exact value. In fact, correlations also seem to decrease for higher blocking levels. This is presumably related to the expected and observed departure from the critical surface.

The data were taken first for region I. The results for regions Ia (Table II) and Ib are consistent with each other within errors. The values of the couplings do seem to approach a fixed point since the renormalized couplings become closer to each other for each new iteration of the renormalization group. In the space of six parameters the fixed point should be close to the values $M^2\sim0.2\beta_1$; $\beta_3\sim0.1\beta_1^2$; $\beta_4\sim-0.007\beta_1^3$.

On the basis of the previous results we selected region II as a new starting point. The results obtained [regions IIa (Table III) and IIb] are again consistent with the two values of the mass that we used in the simulation $M^2=-1.002,-1$. Already after one iteration of the renormalization group we get to a region of $M^2\sim0.13\beta_1$, $\beta_3\sim0.07\beta_1^2$, and $\beta_4<-0.003\beta_1^3$. There is no clear systematic trend after the first renormalization-group iteration. In particular the third and second iterations are not

TABLE II. Values for the ratios of couplings ρ and x [formulas (3.2) and (3.3)] for all truncations $T=3$–6 and blocking levels (BL's) $=0$–3, for region Ia: $\beta_1^{(0)}=-1.19$, $x_2^{(0)}=2.916$, $x_3^{(0)}=-0.706$, $x_4^{(0)}=0.593$. The last three columns give M^2/β_1, the volume $V^{(a)}$ computed with expression (2.13) and Δ defined in (3.8). In Tables II–IV the lattice is 32^3 and the exact volumes are $V^{(0)}=32\,768$, $V^{(1)}=4096$, $V^{(2)}=512$, $V^{(3)}=64$.

T	BL	ρ	x_2	x_3	x_4	x_5	x_6	M^2/β_1	V	Δ
3	0	0.9965	1.209	1.129				0.179	26 108	148
	1	1.509	2.437	0.351				0.562	4 034	26.6
	2	1.699	2.499	0.316				0.501	500	8.15
	3	1.758	2.54	0.325				0.460	64.3	2.53
4	0	0.9937	2.801	−0.585	0.556			0.199	32 043	60.8
	1	1.509	2.645	0.183	0.039			0.354	4 164	3.9
	2	1.699	2.596	0.239	0.0175			0.404	508	0.61
	3	1.758	2.664	0.226	0.0231			0.336	65.5	0.23
5	0	0.9862	2.818	−0.593	0.569	−0.0033		0.201	32 044	0.38
	1	1.912	2.22	0.114	0.019	0.084		0.278	4 198	4.51
	2	2.606	1.954	0.102	0.0048	0.133		0.249	524	2.89
	3	2.668	2.021	0.097	0.0064	0.131		0.192	67.9	1.05
6	0	0.995	2.795	−0.584	0.555	0.016	−0.023	0.199	32 054	1.57
		(27)	(88)	(74)	(37)	(20)	(18)	(97)	(604)	
	1	1.868	2.273	0.120	0.0208	0.0456	0.042	0.284	4 200	1.22
		(17)	(42)	(16)	(22)	(51)	(6)	(34)	(50)	
	2	2.490	2.045	0.112	0.0055	0.073	0.065	0.256	526	0.72
		(25)	(19)	(5)	(9)	(5)	(6)	(14)	(8)	
	3	2.595	2.072	0.103	0.0071	0.101	0.031	0.195	67.0	0.141
		(24)	(28)	(11)	(16)	(5)	(7)	(22)	(5)	

TABLE III. The same as Table II for region IIa: $\beta_1^{(0)}=-4.5$, $x_2^{(0)}=2.7773$, $x_3^{(0)}=0.1234$, $x_4^{(0)}=0.0274$. For very small numbers we use the notation 0.58-3(35) meaning $0.58\times10^{-3}\pm0.35\times10^{-3}$, for example.

T	BL	ρ	x_2	x_3	x_4	x_5	x_6	M^2/β_1	V	Δ
3	0	0.9891	2.668	0.227				0.331	31939	0.331
	1	1.463	2.82	0.123				0.179	4070	0.179
	2	1.608	2.759	0.157				0.241	490	0.241
	3	1.642	2.755	0.197				0.245	63.9	0.245
4	0	0.9891	2.782	0.1326	0.0225			0.218	32462	0.218
	1	1.463	2.84	0.11	0.0027			0.16	4083	0.16
	2	1.608	2.761	0.155	0.28-3			0.239	490	0.239
	3	1.642	2.756	0.197	0.75-4			0.244	63.9	0.244
5	0	0.988	2.785	0.1329	0.0226	−0.66-3		0.218	32462	0.218
	1	1.907	2.323	0.0664	0.0012	0.092		0.124	4122	0.124
	2	2.576	2.014	0.0603	0.17-3	0.141		0.137	509	0.137
	3	2.619	2.023	0.078	−0.2-5	0.140		0.136	66.6	0.136
6	0	0.979	2.810	0.135	0.0233	−0.0207	0.0233	0.220	32464	0.220
		(11)	(45)	(17)	(45)	(71)	(65)	(25)	(204)	(25)
	1	1.840	2.411	0.0714	0.0014	0.0341	0.0640	0.128	4128	0.128
		(8)	(13)	(49)	(8)	(28)	(30)	(25)	(15)	(25)
	2	2.461	2.105	0.0664	0.16-3	0.0830	0.0640	0.142	510	0.142
		(10)	(9)	(32)	(35)	(24)	(25)	(8)	(2)	(8)
	3	2.539	2.080	0.0830	0.3-4	0.1080	0.0330	0.138	66.7	0.138
		(15)	(15)	(43)	(65)	(26)	(29)	(10)	(3)	(10)

closer to each other than the first and second ones. This is to be expected since once in the vicinity of the fixed point, the renormalization of the parameters is small and corrections due to finite volume effects and truncations in the space of parameters are not anymore negligible.

To check this last point we performed the last runs of region III increasing the number of operators up to eight and using a 16^3 lattice in addition to the 32^3 lattice used in all previous runs. The results are shown in Tables IV and V. The renormalized couplings after one iteration become close to the results of region II if we restrict ourselves to the space of six operators as before. Again, no clear tendency is observed for the second and third renormalization-group iterations, except for the fact that the renormalization of the couplings is small. The data of a 16^3 lattice show that the finite-size effect increases with the iteration as expected, since the effective lattice volume decreases. The change in the couplings with respect to the 32^3 data is of the same order as the changes of couplings from one iteration to the next. This seems to confirm our basic interpretation of the result. The best agreement between the results of regions II and III is obtained after two renormalization-group iterations. We give as best estimates of the fixed point position in the space of six operators the values:

$$\frac{M^2}{\beta_1}=0.136\pm0.007,\quad \frac{\beta_3}{\beta_1{}^2}=0.064\pm0.003\ ,\qquad \frac{\beta_4}{\beta_1{}^3}=-0.0004\pm0.0004\ . \tag{3.5}$$

The errors are purely statistical. Systematic errors are expected from finite-size effects and transient motion towards the fixed point.

The renormalization of the constants x_5 and x_6 seems quite universal for all our independent runs. It does not seem to approach any fixed point. A more thorough analysis shows that it is more natural to consider the combinations x_5+x_6 and x_5-x_6. First of all, the latter quantity has much bigger errors than the former. This is related to the fact that the corresponding operator has smaller fluctuations:

$$\langle(O_5^{(a)}-O_6^{(a)})^2\rangle^c \ll \langle(O_5^{(a)}+O_6^{(a)})^2\rangle^c\ . \tag{3.6}$$

In fact, if we restrict ourselves to the space with $x_6=0$, the value of x_5 redefines itself to keep x_5+x_6 unchanged at the expense of bigger changes in x_5-x_6 (Ref. 17). Furthermore, the renormalization of x_5+x_6 behaves as approaching a fixed point, while the difference x_5-x_6 does not. Altogether, our results show no net improvement when taking $x_5-x_6\neq x_5$.

We must now consider the effect of the truncation in the space of operators on the results of our analysis. From the tables one notices that increasing the number of operators from five to eight produces changes in ρ, x_2, x_3, and M^2/β_1 which are of the order of 5—10%. There are a few features of these changes which are worth mentioning.

(a) Local operators as $\phi^2,\phi^4,\phi^6,\phi^8$ (O_2,O_3,O_4,O_8) affect each other much more than they affect bilocal operators. On the other hand, the latter ones O_5,O_6,O_7 have a very small effect on the ϕ^2,ϕ^4,ϕ^6 couplings. Notice that the changes in x_2,x_3,x_4 are accounted for by the change of β_1 alone.

(b) Small changes in the volume computed by expression (2.13) result from including more than five operators.

TABLE IV. The same as Table III for region IIIa: $\beta_1^{(0)}=-1$, $x_2^{(0)}=2.756$, $x_3^{(0)}=0.18$, $x_4^{(0)}=0$. T goes now from 3 to 8.

T	BL	ρ	x_2	x_3	x_4	x_5	x_6	x_7	x_8	M^2/β_1	V	Δ
3	0	0.988	2.816	0.157						0.184	32 774	42
	1	1.442	2.871	0.106						0.128	4 072	13.7
	2	1.569	2.805	0.146						0.195	489	6.0
	3	1.641	2.750	0.185						0.249	65	2.48
4	0	0.988	2.806	0.165	−0.0018					0.194	32 728	0.67
	1	1.442	2.864	0.111	−0.93-3					0.135	4 068	0.2
	2	1.569	2.822	0.134	0.0024					0.177	491	0.16
	3	1.642	2.791	0.158	0.0054					0.208	65.5	0.118
5	0	0.985	2.814	0.166	−0.0018	−0.0014				0.194	32 729	0.171
	1	1.917	2.308	0.0662	−0.5-3	0.0964				0.113	4 107	5.4
	2	2.559	2.019	0.0544	0.35-3	0.143				0.121	507	3.21
	3	2.633	2.023	0.0667	0.99-3	0.139				0.143	67.6	1.2
6	0	0.975	2.845	0.169	−0.0018	−0.025	0.027			0.196	32 734	1.84
		(9)	(36)	(20)	(45)	(8)	(7)			(28)	(204)	
	1	1.841	2.404	0.0726	−0.6-3	0.032	0.071			0.120	4 111	2.04
		(7)	(12)	(40)	(6)	(3)	(3)			(8)	(14)	
	2	2.433	2.119	0.0608	0.37-3	0.079	0.069			0.128	508	0.766
		(8)	(10)	(27)	(32)	(3)	(3)			(7)	(2)	
	3	2.54	2.088	0.0722	0.0011	0.102	0.038			0.149	67.6	0.176
		(1)	(13)	(42)	(5)	(3)	(3)			(11)	(3)	
7	0	0.976	2.841	0.169	−0.0018	−0.0266	0.0242	0.0085		0.197	32 731	0.69
	1	1.871	2.414	0.0717	−0.7-3	0.0112	0.0117	0.116		0.121	4 148	4.58
	2	2.496	2.148	0.0583	0.24-3	0.0532	−0.0029	0.139		0.125	519	2.31
	3	2.827	2.014	0.0576	0.77-3	0.0468	0.031	0.150		0.131	69.3	0.087
8	0	0.976	2.861	0.143	0.010	−0.027	0.0242	0.0085	−0.0018	0.176	32 800	1.02
		(9)	(52)	(51)	(23)	(8)	(72)	(44)	(33)	(46)	(242)	
	1	1.871	2.420	0.0669	0.8-3	0.011	0.0117	0.116	−0.15-3	0.116	4 151	0.16
		(7)	(14)	(78)	(22)	(3)	(28)	(3)	(21)	(12)	(16)	
	2	2.496	2.148	0.0583	0.28-3	0.053	−0.0029	0.139	−0.26-5	0.125	519	0.0019
		(8)	(10)	(38)	(86)	(3)	(25)	(2)	(70)	(8)	(2)	
	3	2.827	2.034	0.0453	0.0037	0.047	0.031	0.1500	−0.22-3	0.111	69.6	0.07
		(9)	(12)	(51)	(11)	(3)	(3)	(23)	(9)	(11)	(3)	

No clear tendency toward improvement is observed as the number of operators increases to the maximum of eight. It is important to realize that the small changes in the volume result from cancellation rather than from the small changes in the couplings themselves. For example, if we consider the change from five to six operators at the second blocking level in Table IV we observe that the volume changes from 507.3 to 508.1. The change of x_5 alone would produce a change of 75 units in the volume: 2 orders of magnitude larger.

(c) Recalling the considerations done in Sec. II C we argue that what really matters is not the change of the couplings themselves but the change in the action. In particular if $S-S'$ is such that

$$\langle (S-S')^2\rangle^c \ll V \tag{3.7}$$

all expectations computed with the action S or with S' would differ by a small amount. Thus if S_k is the action obtained in the space of k operators, the value of

$$\Delta=[\langle (S_k-S_{k-1})^2\rangle^c]^{1/2} \tag{3.8}$$

measures the effect of the truncation. This quantity is listed in the last column of Tables II–V and is in all cases much smaller than the square root of the corresponding volume.

To conclude this section we will comment on other determinations of the renormalized couplings. All our previous results were obtained by use of formula (2.15) with $a=b$. For $a\neq b$ consistent values of the couplings are obtained but the error is larger by several orders of magnitude, getting worse for larger values of $a-b$. Another possibility is to use formula (2.16) to determine the renormalized couplings in terms of the original ones. Table VI shows the values obtained for region III and 32^3 lattice. The results agree qualitatively with those shown in Table IV. On a more quantitative basis is seems that the agreement is mostly within two standard deviations, rather than one. There are several explanations for this fact. First, we recall that the values of the couplings are highly correlated and therefore the errors quoted are not independent. Second, for $a\neq 0$ two truncations are in-

TABLE V. The same as Table IV but now using a 16^3 lattice (region IIIb). The corresponding volumes $V^{(a)}$ are 4096, 512, 64, and 8. The seventh operator does not fit into the lattice of the third blocking.

T	BL	ρ	x_2	x_3	x_4	x_5	x_6	x_7	x_8	M^2/β_1	V	Δ
3	0	0.9928	2.769	0.176						0.231	4077	19.5
	1	1.454	2.846	0.113						0.154	508.3	6.32
	2	1.672	2.805	0.129						0.195	63.3	2.76
	3	2.379	2.798	0.085						0.202	8.2	1.32
4	0	0.9928	2.778	0.169	0.0015					0.223	4083	0.25
	1	1.455	2.866	0.0996	0.0025					0.133	510.2	0.25
	2	1.674	2.865	0.093	0.0063					0.135	64.1	0.25
	3	2.380	2.852	0.064	0.0023					0.148	8.35	0.09
5	0	0.9983	2.764	0.167	0.0015	0.0023				0.222	4083	0.098
	1	1.889	2.331	0.067	0.56-3	0.0902				0.128	511.3	1.78
	2	2.378	2.197	0.054	0.00162	0.113				0.123	64.5	0.85
	3	2.767	2.559	0.049	0.0014	0.051				0.134	8.35	0.15
6	0	1.001	2.756	0.166	0.0015	0.0091	−0.0078			0.221	4083	0.19
		(5)	(17)	(9)	(15)	(39)	(38)			(14)	(16)	
	1	1.816	2.424	0.074	0.50-3	0.0250	0.0720			0.138	511	0.72
		(2)	(6)	(2)	(25)	(16)	(17)			(4)	(1)	
	2	2.342	2.228	0.056	0.0017	0.0944	0.0197			0.126	64.5	0.08
		(3)	(8)	(2)	(3)	(13)	(14)			(5)	(2)	
	3	2.769	2.558	0.049	0.0014	0.0517	−0.55-3			0.134	8.35	0.0013
		(2)	(10)	(4)	(4)	(8)	(91)			(10)	(3)	
7	0	1	2.759	0.167	0.0015	0.0099	−0.0062	−0.0045		0.221	4083	0.13
	1	1.851	2.423	0.075	0.2-3	0.0014	0.0152	0.121		0.146	514.6	1.62
	2	2.632	2.113	0.0455	0.0011	0.0411	0.0062	0.166		0.117	65.9	0.88
	3											
8	0	1.000	2.742	0.187	−0.0076	0.0099	−0.0062	−0.0045	0.0013	0.237	4076	0.35
	1	1.851	2.422	0.075	0.14-3	0.0014	0.0152	0.121	0.5-5	0.146	514.6	0.0025
	2	2.632	2.121	0.0412	0.0021	0.0411	0.0062	0.166	−0.7-4	0.109	66.1	0.035
	3											

volved rather than one and the method should be considered less reliable.

C. Critical exponents

For completeness we will present our preliminary results concerning the eigenvalues and eigenvectors of the $\underline{T}$ matrix. The main difficulty we encounter is the determination of λ in formula (2.3) associated with the critical exponent η as follows:

$$\lambda = 2^{1/2+\eta/2} . \tag{3.9}$$

A good determination of this quantity would demand measuring expectation values and correlations of odd operators. These operators were not included in our analysis and therefore we can only obtain a rough estimate of λ. One possibility would be to determine λ in such a way so that $\langle A_i^{(a)} \rangle = \langle A_i^{(a+1)} \rangle$ at the fixed point. From our data at the second and third blocking level we

TABLE VI. A different determination of the values of ρ and x for region IIIa (to be compared with Table IV, $T=8$). This determination uses formula (2.16), and a and b have the same meaning as in this formula. In particular b has to be identified with the BL of Table IV.

(b,a)	ρ	x_2	x_3	x_4	x_5	x_6	x_7	x_8
10	1.82(2)	2.35(4)	0.11(2)	−0.007(6)	−0.005(8)	−0.003(10)	0.13(1)	0.0006(6)
20	2.44(8)	1.92(9)	0.18(5)	−0.02(1)	0.07(3)	−0.09(3)	0.17(2)	0.0015(10)
21	2.42(2)	2.13(2)	0.07(1)	−0.003(3)	0.034(6)	−0.011(5)	0.155(6)	0.0003(2)
30	3.28(28)	1.66(28)	0.27(12)	−0.03(3)	−0.03(7)	−0.05(7)	0.44(6)	0.002(2)
31	2.87(7)	2.03(5)	0.06(2)	0.002(4)	0.03(2)	0.007(17)	0.22(2)	6×10^{-4}(35)
32	3.01(3)	2.14(2)	0.021(7)	0.008(1)	0.060(5)	−0.024(5)	0.240(5)	-4×10^{-4}(1)

estimate $\lambda=1.5\pm0.2$ consistent with all different runs and operators. This value is also consistent with the behavior of the coupling constants under renormalization. Notice that for $\lambda=1.5$, β_1 does in fact behave as approaching a fixed point.

Unfortunately, the determination of λ affects the value of all critical exponents. The largest eigenvalue μ_0 of the $\underline{T}$ matrix is related to the correlation length critical exponent ν as

$$\mu_0=2^{1/\nu} . \tag{3.10}$$

The relevant operator associated to this eigenvalue is basically given by O_2 (95–99%). This explains why the truncation in the space of operators has little influence on the determination of this eigenvalue.

Although the value of ν itself depends on the value of η, it is possible to determine from the data the value of the combination

$$\Sigma=\frac{1}{\nu}+1.03(\pm0.01)\eta . \tag{3.11}$$

The numerical constant is found empirically; the value 1 is just half of the dimension of the O_2 operator, and the decimals come from contamination of higher dimensional operators.

In Table VII(a) the value of Σ is given as determined in the space of six operators for regions IIIa, IIa, IIIb, and Ia computed from the largest eigenvalue of all $\underline{T}^{(a,b)}$ matrices. In Table VII(b) the effect of different truncations is shown for region IIIa. A common feature of all determinations is that all data lie well above the value of $\Sigma=1.6218$ which corresponds to the Ising model as determined by other authors.[8] It is hard to draw any conclusion, however, without a detailed analysis of the effect of truncation, finite-size, transient motion towards the fixed point and departure from the critical surface, following the guidelines of Ref. 8. We did not attempt any such analysis since the main purpose of this paper is to illustrate the method of determining the renormalized couplings and computing the fixed point for the scalar field theory in three dimensions.

We would like to comment on the possibility of getting rid of our lack of knowledge on the value of η, by replacing the linear renormalization-group transformation by a nonlinear one. One can define the transformation in such a way as to keep $\beta_1^{(a)}=-1$. We did not choose this possibility since in this case the value of μ_0 depends strongly on the particular $\underline{T}^{(a,b)}$ matrix chosen. However, this variation is very small as seen in Table VII(a) when a constant value of λ is chosen. Thus a nonlinear transformation of this type would obscure the interpretation of the data.

A final remark that we want to make concerns the smaller eigenvalues of the $\underline{T}$ matrices. In the space of four operators all eigenvalues are real for all runs and $\underline{T}$ matrices. For $\lambda=1.5$ two eigenvalues become close to one but slightly higher (1.2 ± 0.2) and the smallest one is of the order of 0.4. One of the two operators which are close to marginal is associated with the invariance under rescaling of the field. As an example we show the values of x_2^*, x_3^*, and x_4^* determined by T^{21} and T^{32} of region IIIa by use of formula (2.24),

$$\begin{aligned} &T^{(2,1)}:\ x_2^*=2.18,\ x_3^*=0.23,\ x_4^*=0.007 , \\ &T^{(3,2)}:\ x_2^*=2.953,\ x_3^*=0.041,\ x_4^*=0.006 , \end{aligned} \tag{3.12}$$

obtained by normalizing to $\beta_1^{(1)}$ and $\beta_1^{(2)}$, respectively. The qualitative features are consistent with our determination of the fixed point of the previous section but errors are larger and these results cannot be used to improve on the former.

If we include up to six operators the results are

$$\begin{aligned} T^{(2,1)}:\ & x_2^*=2.48,\ x_3^*=-0.08 , \\ & x_4^*=-0.001,\ x_5^*=0.138,\ x_6^*=0.0112 , \\ T^{(3,2)}:\ & x_2^*=2.07,\ x_3^*=0.09 , \\ & x_4^*=0.006,\ x_5^*=0.115,\ x_6^*=-0.026 , \end{aligned} \tag{3.13}$$

TABLE VII. (a) Values of Σ [formula (3.11)] as obtained for several regions and using different $\underline{T}^{(a,b)}$ matrices [the value of (a,b) is indicated in the first line]. Six operators are used in all cases. (b) Σ as determined for region IIIa as a function of the truncation (T) in the number of operators. (a,b) have the same meaning as in (a). Errors are typically 0.001–0.002.

(a)

Region	(1,0)	(2,0)	(3,0)	(2,1)	(3,1)	(3,2)
IIIa	2.0845	2.0855	2.08	2.0869	2.0727	2.056
IIa	2.068	2.077	2.075	2.08	2.068	2.054
IIIb	2.071	2.0777	2.0814	2.0895	2.0845	2.0888
Ia	1.906	1.9431	1.9527	1.9842	1.9781	1.97

(b)

T	(1,0)	(2,1)	(3,2)
3	2.0915	2.0917	2.0627
4	2.0908	2.0919	2.0630
5	2.0854	2.0874	2.0568
6	2.0845	2.0869	2.0560
7	2.0836	2.0857	2.0495
8	2.0853	2.0857	2.0495

showing a sizable dependence on the truncation but still consistent with the qualitative features. Unfortunately after the seventh operator is included this eigenvalue and eigenvector disappear and a pair of complex-conjugate eigenvalues occur.

Concerning the other close to marginal operator we point out that it is basically associated with a combination of ϕ^2 and ϕ^4 with opposite signs and remains stable under different truncations or choices of $\underline{T}^{(a,b)}$ matrices.

IV. CONCLUSIONS

In this paper we have presented a new method to determine the coupling constants in terms of expectation values. It can be applied to block-spin averages to obtain the evolution of the coupling constants under the renormalization group. Compared to other methods, our proposal has several advantages. First of all, the determination of all renormalized couplings is done in one single Monte Carlo simulation and essentially requires no more information than the one used to compute critical exponents. Therefore, there is considerable gain in computer time and systematic errors compared to methods involving different simulations. Furthermore, the coupling constants are determined by solving a set of linear equations and there is no need to tune any parameters during the simulation. This feature contrasts with other methods based on one simulation which involve nonlinear equations to be solved by an iteration within the Monte Carlo run.

The main observation underlying our proposal is the fact that the Schwinger-Dyson equations are (except for discrete-spin theories) linear in the couplings. Thus it is possible to determine the couplings from the equations. In practice, one must restrict oneself to a finite number of couplings. The limitation is present in any other method, but in our case the validity of the truncation can be checked since there are more equations than unknowns. Furthermore, our data suggest that the subsequent approximations to the action obtained by truncating to the space of a finite number of couplings differ from the exact action by "small operators," i.e., operators with small fluctuations relative to the volume. This can be considered a reasonable "definition" of the restriction of the renormalization group to the space of a finite number of couplings.

There are some interesting theoretical properties which follow from our set of equations. For the case of scalar field theories some of them have been presented in the text. In particular a left and right eigenvector of the $\underline{T}$ matrix at the critical point are identified as the normal to the critical surface and some simple function of the fixed-point couplings, respectively.

An important advantage of our method is its generality. The main idea is to determine the couplings so that Schwinger-Dyson equations are satisfied. However, its practical application is not as simple in some theories as it is in other. The simplest case is the one of scalar field theories with or without internal degrees of freedom, which has been considered in the text. The worse situation applies for theories with discrete spins. In this case the Schwinger-Dyson equations are not any more linear in the couplings. It might well be that Swendsen's method[11] is superior to ours in this case, although the equations look very much alike. It would be desirable to check both methods for the Ising model to see which one gives smaller errors and requires less time.

There are field theories where our method does not apply: namely, when the expectation values are not enough to determine the couplings. The best known example is that of spin theories in the large-N limit for which many different actions lead to the same expectation values. The only way out is to restrict oneself to a minimal form of the action. Of course, this problem occurs for any method of the type.

In this paper we have illustrated the method of computing renormalized couplings for the case of scalar field theory in three dimensions. We proceeded in subsequent approximations towards the fixed point on a 32^3 lattice. The last two runs converged towards the same region in coupling-constant space most remarkably after two iterations of the renormalization group. Our best estimate for the couplings is given in formula (3.5). There are theoretical difficulties in being too precise about the location of the fixed point. Since all numerical estimations are necessarily limited to a finite volume and number of couplings, there need not be any fixed point at all. Once in the neighborhood of the true fixed point, the motion of the couplings may crucially depend on finite-size and truncation effects. In fact, the truncated finite volume $\underline{T}$ matrix has complex eigenvalues when many operators are included.

We have performed a number of cross-checks on the results. A different set of Schwinger-Dyson equations to the ones used in determining the couplings is seen to be reasonably well satisfied, although purely statistical errors seem too small. In any case the qualitative features of the fixed point remain unchanged. The effect of truncation and finite size has been studied in the text. The latter is obtained by comparison with a 16^3 lattice and remains small although increasing with the blocking level. The former is much larger especially for couplings involving bilocal operators other than nearest neighbor. It seems on the basis of our data that the different approximations to the action differ by small operators in the sense explained in the text.

Finally, to conclude this paper, we discuss how to generalize our method to include gauge interactions. Then we consider the SU(N) gauge theory and write the action as

$$S = \mathrm{Tr}(U_l F_l) + \mathrm{Tr}(U_l^\dagger F_l^\dagger) + \Delta S \ , \tag{4.1}$$

where U_l is the SU(N) gauge variable at link l and ΔS does not contain U_l. In general F_l depends on the coupling constants β^α linearly and can be written as

$$F_l = \sum_\alpha \beta^\alpha G_l^\alpha \ . \tag{4.2}$$

For simplicity, we assume that G_l^α do not involve U_l. (If G_l^α involve U_l, the final equation is modified by contact

terms.) Then we consider the quantity

$$\int \prod_l dU_l \mathrm{Tr}(\lambda^a U_l G_l^\alpha) e^{-S} \tag{4.3}$$

and make a change of variable $U_l \to U_l' = (1+i\epsilon\lambda^a)U_l$ in the path integral. Here λ^a is the generator of SU(N). By equating to zero the terms linear in ϵ and summing over a we obtain

$$\frac{1}{N}(N^2-1)\langle \mathrm{Tr}(U_l G_l^\alpha)\rangle = \sum_\gamma \langle \mathrm{Tr}(U_l G_l^\alpha U_l G_l^\gamma) - \mathrm{Tr}(G_l^\alpha G_l^{\gamma\dagger}) - \frac{1}{N}\mathrm{Tr}(U_l G_l^\alpha)\mathrm{Tr}(U_l G_l^\gamma) + \frac{1}{N}\mathrm{Tr}(U_l G_l^\alpha)\mathrm{Tr}(U_l^\dagger G_l^{\gamma\dagger})\rangle \beta^\gamma , \tag{4.4}$$

where use has been made of the identity

$$\sum_a \lambda_{ij}^a \lambda_{kl}^a = \tfrac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N}\delta_{ij}\delta_{kl} . \tag{4.5}$$

Equation (4.4) is the desired Schwinger-Dyson equation.[18] By solving (4.4) for β, we obtain the coupling constant.

For the SU(2) case this equation has a very simple form since $\mathrm{Tr}(U_l F_l) = \mathrm{Tr}(U_l^\dagger F_l^\dagger)$. In fact, (4.4) becomes

$$\tfrac{3}{4}\langle \mathrm{Tr}(U_l G_l^\alpha)\rangle = -\sum_\gamma \langle \mathrm{Tr}(G_l^\alpha G_l^{\gamma\dagger}) - \tfrac{1}{2}\mathrm{Tr}(U_l G_l^\alpha)\mathrm{Tr}(U_l G_l^\gamma)\rangle \beta^\gamma . \tag{4.6}$$

We have checked (4.6) by including four coupling constants β^1–β^4 corresponding to the simple plaquette, the 6-link planar, L-shaped, and twisted loops in four dimensions. Starting from the given coupling constants $\beta^1 = 1$, $\beta^2 = -0.2$, $\beta^3 = -0.4$, and $\beta^4 = -0.1$, we used (4.6) to reproduce these coupling constants on the 8^4 lattice. After system reaches to equilibrium, we took 50 gauge configurations to evaluate the vacuum expectation values appearing in (4.6). To increase statistics, (4.6) is averaged over the position and direction of link l. The calculated coupling constants are

$$\beta^1 = 1.004 \pm 0.005, \quad \beta^2 = -0.2007 \pm 0.0017 , \quad \beta^3 = -0.4005 \pm 0.0018, \quad \beta^4 = -0.1002 \pm 0.0009 , \tag{4.7}$$

showing nice agreement.

The above example shows that our method is really simple and general. Now it is also easy to combine (2.14) and (4.4) to write down the Schwinger-Dyson equations for SU(N) gauge-Higgs theory. These equations should be useful to study the nature of fixed points in four-dimensional real systems.

After completion of this paper we became aware of a paper [M. Falcioni *et al.*, Nucl. Phys. **B265** [FS15], 187 (1986)] in which the idea of using Schwinger-Dyson equations to determine the renormalized coupling constants has also been considered. Their practical implementation of this idea is, however, quite different from ours.

ACKNOWLEDGMENTS

A. Gonzalez-Arroyo wants to thank KEK for the hospitality offered to him. We are grateful to the DD division for continuous support.

*Permanent address: Departamento de Fisica Teorica C-XI, Universidad Autonoma de Madrid, Cantoblanco, Madrid, Spain.

[1]M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954); E. C. G. Stuckelberg and A. Peterman, Helv. Phys. Acta **26**, 499 (1953); K. G. Wilson, Phys. Rev. B **4**, 3174 (1971); **4**, 3184 (1971).

[2]K. G. Wilson and J. B. Kogut, Phys. Rep. **12C**, 75 (1974); S.-K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, MA, 1976).

[3]K. G. Wilson, Phys. Rev. D **10**, 2445 (1974); A. M. Polyakov (unpublished).

[4]N. Metropolis *et al.*, J. Chem. Phys. **21**, 1087 (1953); K. Binder, in *Monte Carlo Methods in Statistical Physics,* edited by K. Binder (Springer, Berlin, 1979).

[5]M. Creutz, L. Jacobs, and C. Rebbi, Phys. Rep. **95**, 2029 (1983).

[6]S.-K. Ma, Phys. Rev. Lett. **37**, 461 (1976); R. H. Swendsen, *ibid.* **42**, 859 (1979).

[7]R. H. Swendsen, in *Real Space Renormalization,* edited by T. W. Burkhardt and J. M. J. Van Leeuwen (Springer, Berlin, 1982).

[8]G. S. Pawley, D. J. Wallace, R. H. Swendsen, and K. G. Wilson, Phys. Rev. B **29**, 4030 (1984).

[9]A. Hasenfratz and A. Margaritis, Phys. Lett. **133B**, 211 (1983); Y. Iwasaki, Nucl. Phys. **B258**, 141 (1985); University of Tsukuba Report No. UTHEP-118 (unpublished); A. Guha, M. Okawa, and J. B. Zuber, Nucl. Phys. **B240**, 566 (1984); M. Okawa, Phys. Rev. Lett. **54**, 963 (1985).

[10]K. G. Wilson, in *Recent Developments in Gauge Theories,* edited by G. 't Hooft *et al.* (Plenum, New York, 1980).

[11]R. H. Swendsen, Phys. Rev. B **30**, 3866 (1984); Phys. Rev. Lett. **52**, 1165 (1984).

[12]H. B. Callen, Phys. Lett. **4B**, 161 (1963).

[13]C. B. Lang, Phys. Lett. **155B**, 399 (1985).

[14]M. Creutz, A. Gocksch, M. Ogilvie, and M. Okawa, Phys. Rev. Lett. **53**, 875 (1984); D. I. E. Callaway and R. Petronzio, Phys. Lett. **139B**, 189 (1984); R. Gupta and R. Cordery, *ibid.* **105A**, 415 (1984).

[15]T. L. Bell and K. G. Wilson, Phys. Rev. B **11**, 3431 (1975).

[16]R. Shankar, R. Gupta, and G. Murthy, Phys. Rev. Lett. **55**, 1812 (1985).

[17]Actually, the coupling β_1 also changes in such a way that the change in the action is proportional to $-\frac{2}{3}O_1 + O_5 - O_6$ which has very small fluctuations.

[18]Y. M. Makeenko and A. A. Migdal, Phys. Lett. **88B**, 135 (1979).